

Al-Nayrīzī’s Mysterious Determination of the Azimuth of the Qibla at Baghdād

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I Introduction

All muslims are required to perform their daily prayers towards the *qibla*, that is the direction of the ka‘ba in Mecca. The determination of the *azimuth* (direction) of the qibla at any given locality was a problem which received much attention in the medieval Islamic mathematical tradition. Already in the second century of the Hijra (eighth century CE), approximate solutions to this problem were known. The earliest correct method was called the “method of the *Zījes*” (astronomical handbooks) because it was widely used in the medieval Islamic astronomical tradition. This method was probably discovered in the third century of the Hijra (ninth century CE) and it was rendered by the astronomer Ḥabash al-Ḥāsib¹ and others [2]. The present paper is concerned with another correct method which was discovered by Abū'l-‘Abbās al-Fadl ibn Ḥātim al-Nayrīzī,² a mathematician and astronomer who originated from the city of Nayrīz in Iran and who worked in Baghdađ. His date is uncertain but he probably flourished around 900 CE.

Al-Nayrīzī’s treatise on the determination of the azimuth of the qibla is historically interesting for two reasons. First, it is the oldest known treatise containing a geometrical proof of a method for the determination of the azimuth of the qibla. This aspect of the treatise is well known in the modern historical literature, and various summaries of the proof have been published [11], [6], [1, pp. 62–63]. Secondly, al-Nayrīzī also presents a corresponding method of computation of the azimuth of the qibla at Baghdađ. This computation has not yet received due attention in the modern literature.

In the medieval Islamic mathematical tradition, the azimuth of the qibla is measured by the angle q on the horizon circle between the direction of Mecca and the Southern direction. In order to compute q at Baghdađ, we have to know the geographical latitude of Mecca ϕ_M , the geographical latitude of Baghdađ ϕ , and the difference $\Delta\lambda$ between the geographical longitudes of Baghdađ and Mecca.

¹On Ḥabash see [12, vol. 5, pp. 275–276; vol. 6, pp. 173–175] and [5, pp. 8–11].

²On al-Nayrīzī see [12, vol. 5, pp. 283–285; vol. 6, pp. 191–192].

Al-Nayrīzī assumes $\phi_M = 21^\circ 40'$, $\phi = 33^\circ 25'$, $\Delta\lambda = 3^\circ$, and he obtains $q = 29^\circ 7'$. For these values of the parameters ϕ_M , ϕ , $\Delta\lambda$ the mathematical correct value of q is $13^\circ 29'$, correct to minutes, and the four easy ninth-century approximation methods discussed in [7, p. 92] give $q \approx 14^\circ$. So how could al-Nayrīzī's correct method produce such a hopelessly erroneous result?

The answer to this question is essentially as follows (details will be given in the next section). Al-Nayrīzī's method is based on the solution of an auxiliary equation $\cos y = c$ for a number c , which in the case of Baghdād is very close to 1. The graph of $\cos y$ reaches its maximum value 1 at $y = 0$ and it is very flat near this maximum. Al-Nayrīzī makes a small error in c and thus he obtains a value of y which is more than two times as large as the correct value. In al-Nayrīzī's method, the relative error in y is approximately the same as the relative error in q , hence his value $q = 29^\circ 7'$ is also more than two times as large as the correct value $q = 13^\circ 29'$.

Al-Nayrīzī nevertheless believed that his computation of the azimuth of the qibla at Baghdad was correct, and at the end of the treatise he proudly states that the values found by Ḥabash and others are wrong. Al-Nayrīzī's contemporaries must have realized that the result of his computation was very incorrect, and this explains why his method did not become popular. One wonders whether they were able to explain the cause of the error.

Al-Nayrīzī's treatise on the azimuth of the qibla was translated into German by Schoy in 1922 [11], and the last passage and the figure, which were left out by Schoy, were provided by Debarnot [1, p. 62]. Section IV of this paper contains an edited Arabic text and an English translation of the treatise of al-Nayrīzī. In Section II of this paper I continue the mathematical analysis, and in Section III I discuss the possible historical, influence of al-Nayrīzī's method.

II Mathematical analysis of al-Nayrīzī's method.

Al-Nayrīzī says in the beginning that he will give the method for the example of Baghdađ. The figure in his geometrical proof is drawn for localities North-East of Mecca, so the method can be used without any change for a large area, including the whole of Iran. With simple modifications the method is also valid for localities West and South of Mecca.

Al-Nayrīzī's method is based on four applications of the spherical transversal theorem of Menelaus for arcs of great circles. This theorem was proved in Menelaus' *Spherics* III:1 [9, pp. 194–197] but also in Ptolemy's *Almagest* I:13 [14, pp. 68–69]. These two works had been translated into Arabic in the early third century of the Hijra (ninth century CE) [12, vol. 5, pp. 161–163; vol. 6, pp. 88–94]. Note that al-Nayrīzī authored a commentary to the *Almagest*, now lost [12, vol. 6, p. 192 no. 4].

As above we assume that the following parameters are known: the latitude ϕ_M

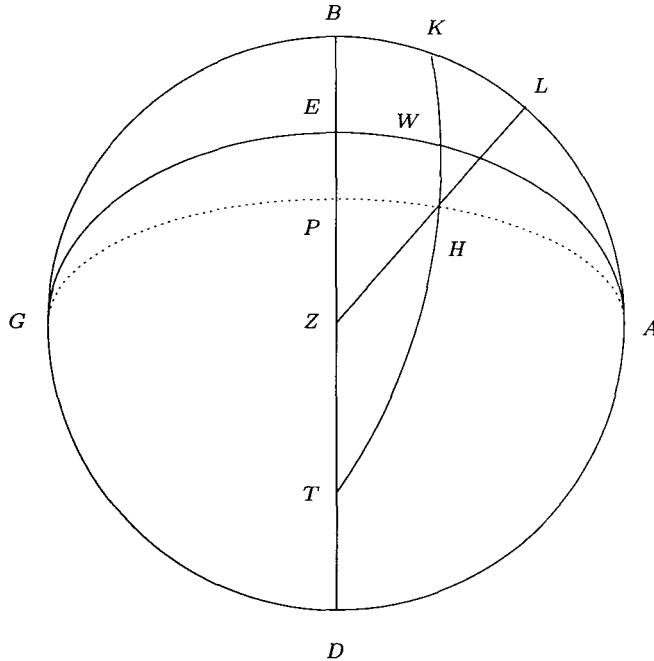


Figure 1

of Mecca, the latitude ϕ of the locality for which we want to compute the azimuth of the qibla, and the difference in longitude $\Delta\lambda$ between Mecca and the locality.

In Figure 1 the upper half of the celestial sphere at the locality has been drawn, and all circles on the sphere have been represented by their perpendicular projections on the horizon plane. $ABGD$ is the horizon, with A, B, G, D the West, South, East and North points respectively. $BEZD$ is the meridian, Z the zenith at the locality, AEG is the celestial equator, T the celestial North pole, H the zenith of Mecca. We draw the great circles THK and ZHL to intersect the southern horizon at K and L . Let THK intersect the celestial equator at W .

Then the following arcs are known: $TD = ZE = \phi$, $TZ = EB = 90^\circ - \phi$, $HW = \phi_M$, $HT = 90^\circ - \phi_M$, $EW = \Delta\lambda$, $WA = 90^\circ - \Delta\lambda$. Also $TW = ZL = 90^\circ$. Call d the great circle distance between Mecca and the locality, and q the azimuth of the qibla, then $d = ZH$, $q = BL$. In his computation, Al-Nayrīzī uses two auxiliary quantities $x = WK$, $y = KB$. We have $KA = 90^\circ - y$.

Al-Nayrīzī applies the theorem of Menelaus for great circle arcs on the sphere four times. Thus he finds x, y, d, q respectively. In the following summary I present for each of the four steps (1) the geometrical identity (using the modern sine function), (2) the corresponding formula in modern notation, and (3) the corresponding step in al-Nayrīzī's computation for Baghdađ. Al-Nayrīzī used the medieval sine and cosine functions which I will indicate by the abbreviations Sin and Cos (with capital letters), and which are defined as the lengths of line segments in circles with radius

60. Thus $\sin x = 60 \sin x$ and $\cos x = \sin(90^\circ - x) = 60 \cos x$. Al-Nayrīzī expressed all fractions in the sexagesimal system. From now on, I will use the standard transcription 59; 32, 43, 54 for $59 + 32/60 + 43/3600 + 54/216000$ (al-Nayrīzī would say: 59 degrees, 32 minutes, 43 seconds and 54 thirds). Al-Nayrīzī used the sexagesimal system not only for angles and arcs but also for Sines and Cosines.

Al-Nayrīzī says that $\Delta\lambda = 3^\circ$ and that the complement of the latitude of Mecca is $68;20^\circ$, so $\phi_M = 21;40^\circ$. From his statements $\sin\phi = 33;2,38$ and $\cos\phi = 50;4,54$ it follows that he used $\phi = 33;25^\circ$ for the latitude of Baghdađ. All these values are attested in medieval sources [4, pp. 55–56, 225–226].

Step 1: Determination of x

We have by Menelaus' theorem

$$\frac{\sin TB}{\sin BE} = \frac{\sin TK}{\sin KW} \cdot \frac{\sin WA}{\sin AE},$$

or

$$\frac{\sin\phi}{\cos\phi} = \frac{\cos x}{\sin x} \cdot \frac{\cos\Delta\lambda}{1},$$

so

$$\cot x = \frac{\tan\phi}{\cos\Delta\lambda}.$$

Al-Nayrīzī first computed $(\sin\phi/\cos\phi) : (\cos\Delta\lambda/60) \approx (142517/60^2) : (215704/60^2) = 0;39,51,14 = \cot x$. He made a computational mistake in the division because actually $142517 : 215704 = 0;39,38,33$ correct to three sexagesimals. Because al-Nayrīzī did not possess tables of the tangent and cotangent functions, he converted $\cot x$ to $\sin x$ using $\sin x = 60/\sqrt{1 + (\cot x)^2}$. He found $\sin x = 49;58,37$ and $x = 56;24,8^\circ$. The correct values are $\sin x = 50;3,37$ and $x = 56;32,49^\circ$, and the errors of only a few minutes seem quite harmless up to this point.

Step 2: Determination of y

We have:

$$\frac{\sin TE}{\sin EB} = \frac{\sin TW}{\sin WK} \cdot \frac{\sin KA}{\sin AB},$$

or:

$$\frac{1}{\cos\phi} = \frac{1}{\sin x} \cdot \frac{\cos y}{1},$$

that is to say

$$\cos y = \sin x / \cos\phi.$$

Using $\cos \phi = 50; 4, 54$ and $\sin x = 49; 58, 37$ al-Nayrizi obtained $\cos y = (60 \sin x) / \cos \phi = 59; 52, 6$ and $y = 3; 47, 19^\circ$. Unfortunately, $\cos y$ is very close to its maximal value 60, so small differences in $\cos y$ correspond to large differences in y . Using the correct value $\sin x = 50; 3, 37$ one obtains $y \approx 1; 40, 29^\circ$.³ Thus the error of only 5 minutes in $\sin x$ caused an error of more than two degrees in y .

Step 3: Determination of d

We have:

$$\frac{\sin TB}{\sin BZ} = \frac{\sin TK}{\sin KH} \cdot \frac{\sin HL}{\sin LZ},$$

or

$$\frac{\sin \phi}{1} = \frac{\cos x}{\sin(x + \phi_M)} \cdot \frac{\cos d}{1},$$

so

$$\cos d = \frac{\sin(x + \phi_M) \cdot \sin \phi}{\cos x}.$$

Al-Nayrizi computed $\sin(x + \phi_M) \cdot \sin \phi \approx 6983215/60^2$ and he divided this by $\cos x = 119530/60^2$ to obtain $\cos d = 58; 25, 20$. He concluded $d = 13; 10^\circ$.

Using the correct value $\sin x = 50; 3, 37$ and using the same values for ϕ_M and $\sin \phi$ as al-Nayrizi, we obtain $\cos d = 58; 40, 46$, whence $d = 12; 3^\circ$.

Step 4: Determination of q

We have:

$$\frac{\sin LB}{\sin BK} = \frac{\sin LZ}{\sin ZH} \cdot \frac{\sin HT}{\sin TK},$$

or

$$\frac{\sin q}{\sin y} = \frac{1}{\sin d} \cdot \frac{\cos \phi_M}{\cos x},$$

so

$$\sin q = \frac{\sin y}{\sin d} \cdot \frac{\cos \phi_M}{\cos x}. \quad (1)$$

Al-Nayrizi first computed $60 \sin y / \sin d = 17; 24, 35$ and he then multiplied this by $\cos \phi_M = 55; 42, 18$ and divided it by $\cos x$. He did not state the value of $\sin q$ which he obtained, but he mentioned the final result $q = 29; 7^\circ$.

Using my value $y = 1; 40, 29^\circ$ and al-Nayrizi's values for $\sin d (13; 40, 1)$, $\cos \phi_M$

³To be more precise, if $\sin x = 50; 3, 37$ correct to two sexagesimals, we know that $50; 3, 36, 30 \leq \sin x \leq 50; 3, 37, 30$, therefore $1; 40, 9^\circ \leq y \leq 1; 40, 48^\circ$.

and $\cos x$, we obtain $q = 12; 25^\circ$, which is much closer to the correct value $q = 13; 29^\circ$. Thus the error in y is the cause of most of the error in q .

Formula (1) can be written as $\sin q = k \sin y$ with $k = \cos \phi_M / \sin d \cos x$. Because al-Nayrīzī makes only small errors in the factors of k , and sines are proportional to angles if the angles are small, the relative error in q is nearly the same as the relative error in y .

Some general remarks can be made on al-Nayrīzī's computations. He performed his multiplications in the sexagesimal system, but in his divisions he first reduced the dividend and the divisor to an integer number of seconds or "thirds" (60^{-3}), which he wrote in the decimal system. It may seem odd that he expressed the result of the division in sexagesimals, but we should bear in mind that decimal fractions had not yet been invented in his time. Most (but not all) of the divisions in the treatise are correct to the last sexagesimal (examples: $118958 : 180294 = 0;39,35,17$ and $6983215 : 119530 = 58;25,20$). Al-Nayrīzī was sloppier in his multiplications. Examples:⁴ $58;42,12 \times 33;2,28 = 1939;46,55$ according to him, but the correct product is $1939;47,11,7,36$. Again, $55;42,18 \times 17;24,35 = 969;42,15$ according to him, but the correct product is $969;48,30,52,30$. Al-Nayrīzī must have had a good table of Sines, because the errors in his Sine values are usually at most 2 units of the last sexagesimal digit. Some examples: he says $\cos 3^\circ = 59;55,4$, and $\sin 33;25^\circ = 33;2,38$, while the correct values are $\cos 3^\circ = 59;55,3,59 \dots$ and $\sin 33;25^\circ = 33;2,36,18 \dots$

Al-Nayrīzī's treated ratios as real numbers in an unproblematic way. In step 1 of his computation, he divided the number $0;39,35,17$, which he called "ratio of the first to the second" (quantity), by another number $0;59,55,4$, the "ratio of the fifth to the sixth," and he then obtained what he called "the ratio of the third to the fourth". He then added 1 to the square of this number and extracted the square root of the sum. Modern historians of mathematics have admired 'Umar al-Khayyām because he interpreted ratios as numbers [3, p. 254], but the example of al-Nayrīzī shows that al-Khayyām's interpretation was the reflection of an age-old practice.

I conclude this section by an explanation of the "method of the Zījes," in order to show that al-Nayrīzī's method can be seen as a variation of this method (compare [1, pp. 50–51]).⁵ Draw a great circle through A, H and G and let this great circle meet the meridian at P (dotted line in Figure 1). Now call $x' = HP, y' = EP$. Then x', y', d and q are determined in four steps:

⁴In these examples obvious scribal errors have been corrected.

⁵Al-Bīrūnī's proof in [1, pp. 252–253] is mathematically equivalent to the following proof, but he uses the sine theorem, which can be seen as a special cases of the theorem of Menelaus.

Step 1:

$$\frac{\sin HP}{\sin PA} = \frac{\sin HT}{\sin TW} \cdot \frac{\sin WE}{\sin EA},$$

or:

$$\sin x' = \cos \phi_M \cdot \sin \Delta\lambda.$$

Step 2:

$$\frac{\sin PE}{\sin ET} = \frac{\sin PA}{\sin AH} \cdot \frac{\sin HW}{\sin WT},$$

or:

$$\sin y' = \frac{\sin \phi_M}{\cos x'}.$$

Then we have $ZP = \phi - y'$.

Step 3:

$$\frac{\sin HL}{\sin LZ} = \frac{\sin HA}{\sin AP} \cdot \frac{\sin PB}{\sin BZ},$$

or:

$$\cos d = \cos x' \cdot \cos(\phi - y').$$

Step 4:

$$\frac{\sin LB}{\sin BA} = \frac{\sin LZ}{\sin ZH} \cdot \frac{\sin HP}{\sin PA},$$

or:

$$\sin q = \frac{\sin x'}{\sin d}.$$

A historical relation between the two methods is suggested by the existence of a technical vocabulary for both. Al-Nayrīzī called arc $WK = x$ the “first connected arc,” or simply the “connected arc,” arc $HL = 90^\circ - d$ the “second connected arc,” arc $KB = y$ the “first separated arc” and arc $ZH = d$ the “second separated arc.” The various authors who discussed the “method of the *Zījes*”, including Ḥabash, also used technical terms for their auxiliary quantities x' and y' (see [2, p. 6]).

Al-Nayrīzī does not need the auxiliary circle AHP so he may have viewed his own method as a simplification of the “method of the *Zījes*.” Unfortunately for al-Nayrīzī, the “method of the *Zījes*” was better suited to numerical computations for localities such as Baghdād, because the small auxiliary quantities x' and y' are computed from their sines, not their cosines.

III Influence of al-Nayrīzī's method

Al-Nayrīzī's method is geometrically correct and yet his computation of the qibla for Baghdād produced a highly erroneous result. Thus the question arises how his treatise and method were received by his contemporaries and successors.

Al-Nayrīzī's treatise was known to a number of 10th-century mathematicians. The extant manuscript of the treatise is a copy of a manuscript copied in the year 358 H./A.D. 970 in Shirāz by the mathematician al-Sijzī (see Section IV and [12, vol. 5, pp. 329–334]). According to a remark at the end, al-Sijzī copied the text from a manuscript in the possession of the Christian physician Naṣīf ibn Yunn [12, vol. 5, pp. 313–314]. Therefore al-Sijzī and Naṣīf ibn Yunn knew al-Nayrīzī's method.

It seems to me that there is an implicit reference to al-Nayrīzī's method for the computation of the qibla in the trigonometrical work *Keys to Astronomy* of al-Bīrūnī. Al-Bīrūnī says that Abū Naṣr ibn Ḥarāq proved in his *Book on Azimuths*, now lost, a theorem “at the end of the proof of a procedure of al-Nayrīzī for (determining) the azimuth of the qibla in his *Zīj* (i.e. astronomical handbook)” [1, p. 133]. The theorem in question is the Sine theorem, which is as follows in the notation of Figure 1: if on a sphere we have two great circle arcs AB, AE equal to a quadrant and two great circle arcs BE, KW perpendicular to AE , then $\text{Sin } WK : \text{Sin } EB = \text{Sin } KA : \text{Sin } AB$.

Al-Bīrūnī informs us that al-Sijzī had collected a number of procedures by different mathematicians for computing the azimuth of the qibla, and that these procedures “led to different results” and were not accompanied by proofs [1, p. 96]. According to al-Bīrūnī, Abū Naṣr wrote his *Book on Azimuths* to give the proofs for the procedures.

The question arises whether al-Nayrīzī gave the same method in his treatise and in his *Zīj*. I believe that this is likely for two reasons. First, the Sine theorem was related to the method in the *Zīj* but also to the treatise, for it corresponds to Step 2 in al-Nayrīzī's computation mentioned above. Note that he states the Sine theorem explicitly in the present treatise (see footnote 18 below). Secondly, al-Bīrūnī's statement that these procedures led to “different results” may have been caused by the result $q = 29; 7^\circ$ of al-Nayrīzī's computation, which was very different from the usual values $q \approx 13; 30^\circ$ for Baghdād.⁶

I conclude that there probably was some discussion of al-Nayrīzī's method in the late tenth century. For more information on the contents of this discussion we will

⁶Debarnot argues in [1, p. 132] that al-Sijzī knew al-Nayrīzī's treatise and hence his proof, so she argues that al-Nayrīzī's method cannot have belonged to al-Sijzī's collection of methods without proofs. She concludes that in the *Zīj*, al-Nayrīzī gave another method for the computation of the azimuth of the qibla, probably the “method of the *Zījes*.” However, I believe that al-Sijzī had every reason to include al-Nayrīzī's method as long as the error in his computation for Baghdād was unexplained.

have to await the discovery of new sources.

IV Introduction to the edition and translation.

The following edition and translation are based on the Arabic manuscript Paris, Bibliothèque Nationale, Fonds Arabe 2457, ff. 78b–80b [13, p. 432]. This is the only manuscript of al-Nayrīzī's treatise which is known to be extant.

The Paris manuscript to which al-Nayrīzī's treatise belongs is a collection of more than fifty mathematical treatises. At the end of several treatises in this collection one finds statements to the effect that the treatise was copied on a date between A.H. 358 and 361 / A.D. 969–972 by Ahmad ibn Muhammad ibn 'Abdaljalil in Shīrāz. The scribe is the famous Iranian mathematician al-Sijzī. Whether the Paris manuscript was really copied by al-Sijzī has been a matter of controversy between modern authors (see [10]). I believe that the Paris manuscript is a later copy of a manuscript written by al-Sijzī and that the scribe simply copied the statements by al-Sijzī at the end of the treatises. Like many other treatises in the manuscript, the treatise by al-Nayrīzī contains silly scribal errors which make it unlikely that the scribe of the manuscript was a competent mathematician. Thus the scribe wrote instead of the sentence “this circle (*LHZ*) passes through the two poles of the two horizons so it is perpendicular to the two horizons” the sentence “this circle (*LHZ*) passes through the two points of the two horizons so it is perpendicular to the two horizons,” which is completely meaningless (footnote 9 below). The scribe also confused al-Nayrīzī's technical terminology “connected” and “separated” arc for the quantities x and y in Section II (see footnotes 19–20 below).

In my edition of the text I have left most grammatical errors in the text uncorrected, but I have used some modern orthography without notice. There are some emendations in the text but the manuscript readings can be found in the apparatus in the end. I have tried to reconstruct the original of al-Nayrīzī and to this effect I have emended some obvious scribal errors in the numbers. There are probably more scribal errors which cannot be identified since it is not always possible to distinguish an error due to a scribe from a computational error made by al-Nayrīzī. The numbers in the text are written either in words or in Hindu-Arabic numbers, as in the following quotation: “thirty-three degrees and two minutes and thirty-eight seconds, that is 118958 seconds, or 7137480 thirds.” For sake of brevity, numbers in words in the manuscript appear as Hindu-Arabic numbers in normal print in the translation, and Hindu-Arabic numbers in the manuscript appear in boldface in the translation. Therefore, the quotation appears in the translation as: “33;2,38, that is **118958** seconds, that is **7137480** thirds.”

Edition

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ
 كَوْرِسَةُ الْفَضْلِ بْنِ حَاتِمَ النَّيْرِيزِيِّ فِي سِمتِ الْقَبْلَةِ
 أَوْ لِيْكَنَ الْمَثَالُ لِمَدِينَةِ السَّلَامِ وَنَجْعَلُ دَائِرَةَ الْأَفْقَنِ بِمَدِينَةِ السَّلَامِ أَبْجَدَ وَالْمَرْكَزُ نَقْطَةُ هَـ وَلِيْكَنَ
 نَقْطَةً آمَغِيبَ أَوْلَى الْحَمْلِ وَأَوْلَى الْمِيزَانِ وَنَقْطَةً جَـ مَطْلُومَهَا (۱) وَلِتَكُنَ دَائِرَةُ مَعْدُلِ النَّهَارِ جَـهَا
 وَنَصْفُ دَائِرَةِ نَصْفِ النَّهَارِ بِمَدِينَةِ السَّلَامِ بَهَـ وَلِتَكُنَ نَقْطَةُ زَـ سِمتُ الرَّأْسِ بِمَدِينَةِ السَّلَامِ
 وَنَقْطَةُ حَـ سِمتُ الرَّأْسِ عَلَى أَهْلِ مَكَّةَ وَالْقَطْبِ الشَّمَالِيِّ نَقْطَةً طَـ وَنَجِيزُ عَلَيْهَا وَعَلَى نَقْطَةِ حَـ
 قَوْسُ طَحْوَكَ وَقَوْسُ هَـ هُوَ فَضْلُ الطَّولِ وَهُوَ ثَلَاثَ درَجَاتٍ بِمَدِينَةِ السَّلَامِ فِيمَا بَيْنَ
 الْمَدِينَتَيْنِ فِي الطَّوْلِ وَهُوَ مَعْلُومٌ وَقَوْسٌ وَـ تَبْقَى مَعْلُومَةُ لَاهِئَةِ تَعَامٍ لِقَوْسٍ هَـ (۲) تَسْعِينَ
 دَرْجَةً وَقَوْسُ طَحْوَكَ هُوَ مِنْ دَائِرَةِ نَصْفِ النَّهَارِ أَهْلِ مَكَّةَ وَهِيَ مَجْهُولَةٌ وَنَرِيدُ أَنْ نَعْلَمُهَا.
 وَنَخْرُجُ عَلَى نَقْطَتِي حَـ زَـ رَبِيعُ دَائِرَةِ لَهَـزِّ فِيهِنَّ هُوَ أَنَّ الصَّلَوةَ بِمَدِينَةِ السَّلَامِ تَكُونُ تَحْتَ
 رَبِيعِ دَائِرَةِ لَهَـزِّ وَذَلِكَ يَجُوزُ عَلَى قَطْبِي (۳) الْأَفْقَيْنِ فَهِيَ قَائِمَةٌ عَلَى الْأَفْقَيْنِ عَلَى زَوَافِيَا قَائِمَةٌ
 فَرِيدٌ أَنْ نَبَيِّنَ كَيْفَ نَعْلَمُ قَوْسَ بَلَـ مِنْ دَائِرَةِ الْأَفْقَنِ الَّتِي إِذَا عَلَمْنَاهَا عَلَمْنَا سِمتَ الْقَبْلَةِ.
 وَلَانَّ فِيمَا بَيْنَ قَوْسِي طَبَـ وَابَـ قَوْسَا طَحْوَكَ وَهَـ تَقَاطِعَانِ عَلَى نَقْطَةٍ وَـ فَانَّ نَسْبَةُ جَيْبِ
 (۴) قَوْسِ طَبَـ إِلَى جَيْبِ قَوْسٍ بَـ كَنْسَبَةٌ جَيْبِ قَوْسِ طَكَـ إِلَى جَيْبِ قَوْسٍ كَـوْ مُؤَلَّفَةٌ بِنَسْبَةٍ
 جَيْبِ قَوْسٍ وَـ إِلَى جَيْبِ قَوْسٍ هَـ وَقَوْسُ طَبَـ مَعْلُومَةٌ لَانَّ طَـ رَبِيعُ دَائِرَةٍ وَبَـ هِيَ مَقْدَارُ
 ارْتِفَاعِ أَوْلَى الْحَمْلِ وَأَوْلَى الْمِيزَانِ فَقَوْسَا طَبَـ وَبَـ مَعْلُومَتَانِ وَقَوْسُ وَـ (۵) مَعْلُومَةٌ وَهَـ ۹۰
 دَرْجَةٌ وَامَّا كُلُّ وَاحِدَةٍ مِنْ قَوْسِي طَكَـ وَـ فَمَجْهُولَةٌ لَكِنَّ فَضْلَ قَوْسِ طَكَـ عَلَى قَوْسِ وَـ
 مَعْلُومَةٌ وَهِيَ تَسْعِونَ دَرْجَةً فَتَصِيرُ قَوْسُ كَـ مَعْلُومَةٌ عَلَى مَا اَصْفَـ.

بَابٌ فِي حَسَابِ قَوْسٍ وَـ وَنَسْمِيهَا الْقَوْسَ الْمَتَّصِلَةُ. نَقْسُمُ جَيْبِ قَوْسِ طَبَـ (۶) الْمَسَاوِيِّ
 جَيْبِ قَوْسِ عَرْضِ مَدِينَةِ السَّلَامِ عَلَى جَيْبِ قَوْسٍ بَـ الَّتِي هِيَ جَيْبِ تَعَامٍ عَرْضِ مَدِينَةِ السَّلَامِ
 فَالَّذِي يَخْرُجُ مِنَ الْقَسْمَةِ سَمِّيَّنَا نَسْبَةَ الْأَوَّلِ إِلَى الْثَّانِيِّ. ثُمَّ نَقْسُمُ جَيْبِ تَعَامِ النَّفْضِ فِيمَا بَيْنَ
 الْطَّوْلَيْنِ اعْنَى فَضْلُ طَوْلِ مَدِينَةِ السَّلَامِ مِنَ الْمَغْرِبِ عَلَى طَوْلِ مَكَّةَ عَلَى الْجَيْبِ الْأَعْظَمِ
 فَالَّذِي يَصْحَّ مِنَ الْقَسْمَةِ سَمِّيَّنَا نَسْبَةَ الْخَامِسِ إِلَى السَّادِسِ. ثُمَّ نَقْسُمُ نَسْبَةَ الْأَوَّلِ إِلَى الْثَّانِيِّ
 عَلَى نَسْبَةِ الْخَامِسِ إِلَى السَّادِسِ فَالَّذِي يَصْحَّ مِنَ الْقَسْمَةِ سَمِّيَّنَا نَسْبَةَ الْثَّالِثِ إِلَى الْرَّابِعِ. ثُمَّ
 نَصْرُبُ نَسْبَةَ الْثَّالِثِ إِلَى الْرَّابِعِ فِي نَفْسِهِ فَمَا يَجْتَمِعُ نَزِيدُ عَلَيْهِ وَاحِدًا وَنَأْخُذُ جَذْرَ الْمَجْتَمِعِ

ونقسم على الجذر الذي وجدناه الجيب الاعظم فالذى يصح من القسمة هو جيب القوس المتصلة جعلناه قوسنا فالذى يصح من القوس سميناه القوس المتصلة التي هي قوس وك .

باب في معرفة قوس ك وك . نسبة جيب قوس ط الى جيب قوس هب كنسبة جيب قوس ط الى جيب قوس وك (ص ١٧٩) مؤلفة من نسبة جيب قوس اك الى جيب قوس اب وكل واحدة من قوسي ط و ط تسعون درجة تصير نسبة (٧) جيب قوس وك الى جيب قوس هب كنسبة جيب قوس كا الى جيب قوس اب فنضرب جيب قوس كوك الذي هو جيب القوس المتصلة في جيب قوس اب الذي هو الجيب الاعظم مقسوم على جيب قوس هب الذي هو جيب تمام عرض البلد فان الذي يصح من القسمة جيب قوس اك فقوس اك معلومة فتبقى قوس كب وتنقصه من تسعين درجة فيكون القوس المنفصلة (٨) الاولى.

باب في حساب القوس المنفصلة الاولى. نضرب جيب القوس المتصلة (٩) في الجيب الاعظم ونقسم المجتمع على جيب تمام عرض البلد ونجعل ما يصح من القسمة قوسنا فتكون تلك القوس تمام القوس المنفصلة الاولى.

باب معرفة قوسي زح ، لخ (١٠) . فنسبة جيب قوس بط الى جيب قوس زب كنسبة جيب قوس طك الى جيب قوس كح مؤلفة (١١) بنسبة جيب قوس لخ الى جيب قوس لز . فلان القوس الثانية مساوية للسادسةتين هما زب ولز فانه تصير نسبة الاول الذي هو جيب قوس طب الى الخامس الذي هو جيب قوس لخ كنسبة جيب قوس طك الى جيب قوس كح الرابع. فنضرب جيب طب الاول في جيب قوس كح الذي هو جيب القوس المتصلة بمجموع اليها قوس هو التي (١٢) عرض مكة مقسوم على جيب قوس كط المساوية لجيب تمام القوس المتصلة التي هي قوس وك فان الذي يصح من القسمة يكون جيب قوس لخ فقوس لخ تصير معلومة وكذلك قوس زح تصير معلومة ونسميتها اما قوس لخ فالمتصلة الثانية واما قوس زح فالمفصلة الثانية.

باب في حساب القوس المتصلة الثانية والمنفصلة الثانية. انا نضرب جيب عرض (١٣) البلد في جيب مجموع القوس المتصلة الاولى مع قوس عرض مكة فما يحصل من الضرب سميناه على جيب تمام القوس المتصلة الاولى فالذى يصح من القسمة يجعله قوسنا فالذى يحصل من القوس هي القوس المتصلة الثانية تنقصها من تسعين درجة فيكون الباقي القوس المنفصلة الثانية.

باب معرفة السمت الذي هو قوس لب . نسبة جيب قوس لب الى جيب قوس كب

كنسبة جيب قوس لـ π الى جيب قوس زـ θ مؤلفة (١٤) بنسبة جيب قوس طـ θ الى جيب قوس كـ θ والقسي كلـها معلومة غير قوس لـ π الاولى فتعلم من سائر القسي:

باب حسابه اعني قوس السمت فيما بين طرف خطـ θ نصف الجنوب الى ما يلي المغرب بمدينة السلام وكذلك كلـ مدينة يكون طولها زائداً على طول مـ π من المغرب. اتا نضرب جيب قوس المنفصلة الاولى في الجيب الاعظم ونقسم المجتمع على جيب قوس المنفصلة الثانية فالذى يصح من القسمة ضربناه في جيب (ص ٧٩ ب) تمام عرض مـ π فالذى يجتمع من الضرب قسمناه على جيب تمام القوس المتصلة الاولى فالذى يصح من القسمة جعلناه قوسنا فالذى يحصل من القوس فتكون قوس السمت الذى ذكرناه.

مثال ذلك ولاتي الى هذه الغاية ما امكننى ان ارصد شيئاً من الارصاد البتة فاكون مع ذلك رصدت مقدار الفضل الذي فيما بين نصف نهار مـ π ونصف نهار مدينة السلام لأنـ هذا الرصد يحتاج ان يكون من قبل الكسوفات القمرية ويكون الراصد لذلك راصدين احدهما بمدينة السلام والباقي بمـ π ليعلم كلـ واحد منها الماضي من الليل عند مبدأ الكسوف واما عند تمامه واما عند تمام الخلافة . ثمـ يوجد الفضل فيما بين الوقتين في الموضعين اعني بالوقتين الماضي من نصف الليل واما الباقى الى نصف الليل لتكون الفضلة الباقية بعد فيما بين دائرتى نصف النهار ومن قبل ان هذه الفضلة لمدينة السلام على مـ π ثلاثة درجات كما وجدته مكتوبـاً على ما اصنف وذلك ان امير المؤمنين المأمون رضى الله عنه احب ان يصحح سمت القبلة فوجد دائرة نصف النهار بمـ π غربى عن نصف نهار مدينة السلام ثلاثة درجات بالتقريب فيـن هو ان هذه الدائرة درجات هي من دائرة معدل النهار وهي مقدار قوس هو .

جعلنا عرض مدينة السلام جـ π فكان ثلاثة وثلاثين درجة ودقيقتين وثمانين ثانية ثـ θ ثـ θ ١١٨٩٥٨ ثـ θ ٢١٣٢٤٨٠ وجيب تمام عرض البلد خمسون درجة واربع دقائق واربع وخمسون ثانية يكون ثـ θ ١٨٠٢٩٤ قسمت جيب عرض مدينة السلام على جيب تماماً فخرج من القسمة تسعة وثلاثون دقيقة وخمس وثلاثون ثانية وسبعين عشرة ثلاثة يكون ثـ θ ١٤٢٥١٧ سـ θ نسبة الاول الى الثاني. جيب تمام الثلاث درجات الذي هو جيب تمام الفضل فيما بين الدائرين تسعة وخمسون درجة وخمس وخمسون دقيقة واربع ثـ θ ثـ θ ١١٨٩٥٨ على الجيب الاعظم فخرج من القسمة تسعة وخمسون دقيقة وخمس وخمسون ثانية واربع ثـ θ ثـ θ ٢١٥٧٠٤ وهو نسبة الخامس الى السادس تكون ثـ θ ثـ θ ٢١٥٧٠٤ ثمـ قسمت نسبة الاول الى الثاني على نسبة الخامس الى السادس فتخرج من القسمة نسبة الثالث الى الرابع

تسع وثلاثون دقيقة وأحد وخمسون ثانية واربع عشرة ثلاثة ضربته في مثله فما اجتمع ستة وعشرون دقيقة وثمان وعشرون ثانية وعشرون ثلاثة زدت عليه واحداً اعني درجة واحدة واخذت جذر المجتمع فكان الجذر ثانوي ٤٢٢ يكون درجة واثنى عشرة دقيقة وثانيتين قسمت عليه الجيب الاعظم فخرج من القسمة تسعة واربعون درجة وثمان وخمسون دقيقة وسبعين وثلاثون ثانية وهو جيب القوس المتصلة الاولى التي هي قوس \bar{K} من الشكل المتقدم فيكون ثانوي ١٢٩٩١٧ ضربناه في الجيب الاعظم وقسمنا المجتمع على جيب تمام عرض البلد الذي هو خمسون درجة واربع (ص ٨٠) دقائق واربع وخمسون ثانية فالذى يخرج من القسمة تسعة وخمسون درجة واثنان وخمسون دقيقة وست ثانوي وهو جيب قوس تمام المفصلة الاولى التي هي قوس \bar{K} . القوس المتصلة الاولى ستة وخمسون واربع وعشرون دقيقة وثمان ثانوي: عرض مكة احادي وعشرون درجة (١٦) واربعون دقيقة زدناه على القوس المتصلة فصار الجميع ثمان وسبعين درجة واربع دقائق وثمان ثانوي يكون جيبه ثمانية وخمسين درجة واثنين واربعين دقيقة واثنى عشر ثانية ضربته في جيب عرض مدينة السلام الذي هو ثلاث وثلاثون درجة ودققتان وثمان وثلاثون ثانية فاجتمع من الضرب الف وتسع مائة وتسعة وثلاثون درجة وست واربعين دقيقة وخمس وخمسين ثانية. تمام القوس المتصلة ثلاث وثلاثين درجة وخمس وثلاثون دقيقة واثنان وخمسون ثانية جيبه ثلاثة (١٢) وثلاثون درجة واثنا عشر دقيقة وعشرين ثانوي قسمت عليه الذي اجتمع من الضرب الاول ثانوي ٦٩٨٣٢١٥، الثاني ١١٩٥٣٠، فخرج من القسمة ثمان وخمسون درجة وخمس وعشرون دقيقة وعشرون ثانية وهو جيب قوس المتصلة الثانية فتكون القوس المتصلة الثانية سبع وسبعين درجة وخمسين دقيقة نقصته من تسعين درجة يكون الباقي قوس المفصلة الثانية ثلاثة عشرة درجة وعشرين دقائق وهي قوس \bar{R} .

وكتنا حسبنا جيب تمام المفصلة الاولى الذي هو جيب قوس \bar{K} تسع وخمسين درجة واثنتين (١٨) وخمسين دقيقة وست ثانوي تكون قوسه التي هي تمام قوس المفصلة الاولى سبع وثمانين درجة واثنتي عشرة دقيقة وأحد واربعين ثانية ويكون تامها التي هي المفصلة الاولى ثلاثة درجات وسبعين واربعين دقيقة وتسعة عشرة ثانية التي هي قوس \bar{K} يكون جيبه ثلاثة درجات وسبعين وخمسين دقيقة وست وخمسين ثانية ضربته في الجيب الاعظم فاجتمع مائتان وسبعين وثلاثون درجة وستة وخمسون دقيقة قسمناه على جيب قوس المفصلة الثانية الذي هو ثلاثة عشرة درجة واربعون دقيقة وثانية واحدة فيخرج من القسمة سبع عشرة

درجة واربع وعشرون دقيقة وخمس وثلاثون ثانية. تمام عرض مكة ثمان وستون درجة وعشرون دقيقة جيـه خـس وخمـسون درـجـة واثـنـتـان وارـبـعون دقـيقـة وثمانـعـشرـة ثـانـيـة ضـربـتـهـ فيما خـرـجـ منـ القـسـمـةـ فـاجـتـمـعـ تـسـعـ مـائـةـ وـتسـعـ وـسـتـوـنـ (١٩) درـجـةـ وـاثـنـتـانـ وـارـبـعونـ دقـيقـةـ وـخـمـسـ عـشـرـةـ ثـانـيـةـ قـسـمـتـهـ عـلـىـ جـيـبـ تـامـ المـتـصـلـةـ الـأـولـىـ الـذـيـ هوـ ثـلـاثـ وـثـلـاثـوـنـ درـجـةـ وـخـمـسـ عـشـرـةـ دقـيقـةـ وـخـمـسـ عـشـرـةـ ثـانـيـةـ تـكـوـنـ قـوـسـهـ تـسـعـ وـعـشـرـونـ درـجـةـ وـسـبـعـ دقـائقـ وـهـوـ مـقـدـارـ قـوـسـ بـلـ الـذـيـ هوـ مـقـدـارـ السـمـتـ اـعـنـيـ الـبـعـدـ فـيـماـ بـيـنـ طـرـفـ خـطـ نـصـفـ النـهـارـ الـجـنـوـبـيـ إـلـىـ ماـ يـلـيـ الـمـغـرـبـ عـلـىـ دـائـرـةـ اـفـقـ مـدـيـنـةـ السـلـامـ فـيـماـ بـيـنـ النـقـطـةـ الـتـيـ إـلـيـهاـ تـكـوـنـ الـصـلـوةـ تـسـعـ وـعـشـرـونـ درـجـةـ وـسـبـعـ دقـائقـ وـذـكـ ماـ اـرـدـنـاـ انـ نـيـنـ.

(ص ٨٠ ب) قال النيرزي لم يسبقني الى هذا الباب احد ولذلك صار ما حسب حبس
وغيره من المهندسين والحساب خطئاً
تمت الرسالة والحمد لله رب العالمين وصلى الله على محمد وآلـهـ كـتـبـتـ منـ نـسـخـةـ نـظـيفـ
في شهر رجب

(١) مـطـلـعـهـماـ :ـ مـطـلـعـهـاـ .ـ (٢) هـوـاـ :ـ هـوـمـ (٣) قـطـبـيـ :ـ نقطـيـ .ـ (٤) جـيـبـ :ـ فيـ الحـاشـيـةـ
فـقـطـ .ـ (٥) وـاهـ .ـ (٦) طـبـ :ـ كـذـاـ فـيـ النـسـخـةـ ،ـ فـيـ الحـاشـيـةـ هـزـ .ـ (٧) نـسـبةـ :ـ فيـ الحـاشـيـةـ
فـقـطـ .ـ (٨) المـنـفـصـلـةـ :ـ المـنـفـصـلـةـ .ـ (٩) المـتـصـلـةـ :ـ المـنـفـصـلـةـ .ـ (١٠) قـوـسـ زـحـ ،ـ لـخـ :ـ فـيـ النـسـخـةـ
قوـسـ دـهـ ،ـ فـيـ الحـاشـيـةـ قـوـسـيـ .ـ (١١) مـؤـلـفـةـ :ـ مؤـلـفـ .ـ (١٢) الـتـيـ :ـ فـيـ النـسـخـةـ إـلـىـ ؟ـ فـيـ
الـحـاشـيـةـ :ـ الـتـيـ .ـ (١٣) عـرـضـ :ـ فـيـ المـخـطـوـطـ عـرـضـ .ـ (١٤) مـؤـلـفـةـ :ـ مؤـلـفـ .ـ (١٥) ثـوـالـثـ
:ـ ثـالـثـ .ـ (١٦) درـجـةـ :ـ فـيـ المـخـطـوـطـ وـاحـدـ .ـ (١٧) ثـلـاثـةـ :ـ مـكـتـرـ فـيـ المـخـطـوـطـ .ـ (١٨) وـاثـنـيـنـ
فـيـ المـخـطـوـطـ وـارـسـنـ .ـ (١٩) وـسـتـوـنـ :ـ فـيـ المـخـطـوـطـ وـخـمـسـونـ .ـ

Translation

In the name of God, the Merciful, the Compassionate

26.⁷ Treatise of al-Fadl ibn Ḥātim al-Nayrīzī on the Azimuth of the Qibla

1. Let the example be for the City of Peace (= Baghādād). (Figure 2) We make the horizon at the City of Peace $ABGD$ and the centre point E . Let point A be the setting point of the beginning of Aries and Libra, and let point G be their rising point.⁸ Let the (celestial) equator be GEA , and half of the meridian at the City of Peace BED , and let point Z be the zenith at the City of Peace, and point H the zenith for the people of Mecca, and let the (celestial) north pole be point T . We draw through it (T) and point H arc $THWK$. Arc EW is the difference in longitude between the two cities, that is three degrees at the City of Peace, and this (difference) is known. By subtraction, arc WA is known because it is the complement (of arc EW) to arc EWA which is ninety degrees. Arc $THWK$ belongs to the meridian for the people of Mecca; it is unknown and we want to determine it.

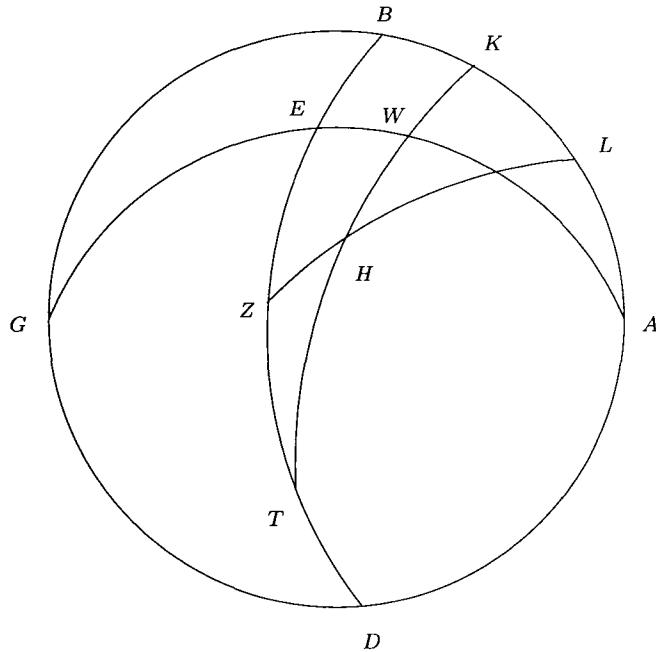


Figure 2

We draw through points H and Z quadrant LHZ . Then it is clear that the prayer

⁷The treatise was no. 26 in the manuscript in its original form. See [10].

⁸The text means that A is the West point of the horizon and G the East point.

at the City of Peace is under the quadrant *LHZ*. That (circle) passes through the two poles⁹ of the two horizons,¹⁰ so it is perpendicular to the two horizons. Thus we want to explain how we determine arc *BL* of the horizon, for if we know it, we know the azimuth of the qibla.

Since the two arcs *THWK, EWA* are between arcs *TB, AB* and intersect at point *W*, the ratio of the Sine¹¹ of arc *TB* to the Sine of arc *BE* is as the ratio of the Sine of arc *TK* to the Sine of arc *KW* compounded with the ratio of the Sine of arc *WA* to the Sine of arc *EA*. Arc *TB* is known since *TE* is a quarter of a circle and *BE* is the magnitude of the altitude of the beginning of Aries and the beginning of Libra,¹² so arcs *TB* and *BE* are known. Arc *WA* is known and *EA* is **90** degrees. The arcs *TK, WK* are both unknown, but the excess of arc *TK* over arc *WK* is known, namely ninety degrees. Thus arc *KW* can be determined, as I shall now describe.

Chapter on the computation of arc *WK*; we will call it the connected arc.¹³ We divide the Sine of arc *TB*, which is equal to the Sine of the latitude of the City of Peace, by the Sine of arc *BE*, that is the Cosine of the latitude of the City of Peace. We have called¹⁴ the quotient the ratio of the first to the second (quantities). Then we divide the Cosine of the difference between the two longitudes, I mean the excess of the longitude of the City of Peace from the West¹⁵ over the longitude of Mecca, by the greatest Sine.¹⁶ We have called the quotient the ratio between the fifth and sixth (quantities). Then we divide the ratio between the first and second (quantities) by the ratio between the fifth and sixth (quantities) and we have called the quotient the ratio between the third and fourth (quantities).¹⁷ Then we multiply

⁹The scribe wrote “the two points” instead of “the two poles.”

¹⁰The two horizons are the horizons of Mecca and Baghdād.

¹¹I write Sine and Cosine in capitals to remind the reader of the fact that al-Nayrīzī’s Sine and Cosine are defined in a circle with radius 60, thus Sine (x) = 60 sin x , Cosine (x) = 60 cos x .

¹²The text means the maximum altitude of the beginning of Aries and Libra, that is the altitude of the intersection between the meridian and the celestial equator. This altitude is the complement of the geographical latitude of Baghdād.

¹³Al-Nayrīzī considers arc *WK* to be the arc connected to arc *HW*, the latitude of Mecca; the two arcs are on the same great circle.

¹⁴Al-Nayrīzī may refer to another work, perhaps his lost commentary on the *Almagest* in which he discussed the transversal theorem of Menelaus, as al-Bīrūnī tells us in his *Keys to Astronomy* [1, p. 92–93].

¹⁵In medieval Islamic geography, terrestrial longitudes were often measured with respect to a meridian through the Canary Islands, which were believed to be the Western extreme of the inhabited world.

¹⁶For al-Nayrīzī, the “greatest Sine” is the maximum of his Sine function, that is, 60.

¹⁷Al-Nayrīzī has now obtained $\cot x$ with $x = WK$, as the ratio between the third and the fourth

the ratio between the third and fourth by itself, and we add one to the product, and we take the root of the sum, and we divide the greatest Sine by the root which we have found, and the quotient is the Sine of the connected arc. We convert it to an arc (by means of a Sine table), and we have called the resulting arc the connected arc, that is arc *WK*.

Chapter on the determination of arcs *KA* and *KB*. The ratio of the Sine of arc *TE* to the Sine of arc *EB* is as the ratio of the Sine of arc *TW* to the Sine of arc *WK* compounded with the ratio of the Sine of arc *AK* to the Sine of arc *AB*. Each of the arcs *TE*, *WT* is ninety degrees, so the ratio of the Sine of arc *WK* to the Sine of arc *EB* turns out to be equal to the ratio of the Sine of arc *KA* to the Sine of arc *AB*.¹⁸ Thus we multiply the Sine of arc *KW*, that is the Sine of the connected arc, with the Sine of arc *AB*, that is the greatest Sine, divided by the Sine of arc *EB*, that is the Cosine of the latitude of the locality. The quotient is the Sine of arc *AK*, so arc *AK* is known, so arc *KB* remains. We subtract it (*AK*) from ninety degrees, then it (*BK*) is the first separated¹⁹ arc.

Chapter on the computation of the first separated arc. We multiply the Sine of the connected²⁰ arc by the greatest Sine, and we divide the product by the Cosine of the latitude of the locality, and we convert the quotient to an arc (by means of a Sine table), then that arc is the complement of the first separated arc.

Chapter on the determination of arcs *ZH*, *LH*. The ratio of the Sine of arc *BT* to the Sine of arc *ZB* is as the ratio of the Sine of arc *TK* to the Sine of arc *KH* compounded with the ratio of the Sine of arc *LH* to the Sine of arc *LZ*. Since the second arc, *ZB*, is equal to the sixth arc, *LZ*, the ratio of the first (quantity), that is the Sine of arc *TB*, to the fifth quantity, that is the Sine of arc *LH*, is equal to the ratio of the Sine of arc *TK* to the Sine of arc *KH*, the fourth (quantity). So we multiply the Sine of *TB*, the first (quantity) by the Sine of arc *KH*, that is the Sine of the sum of the connected arc and arc *HW* which is the latitude of Mecca, divided by the Sine of arc *KT*, which is equal to the Cosine of the connected arc *WK*. The quotient is the Sine of arc *LH*. Thus *LH* becomes known, and in the same way *ZH* becomes known. We call arc *LH* the second connected arc and arc *ZH* the second separated arc.

Chapter on the computation of the second connected arc and the second separated arc. We multiply the Sine of the latitude of the locality with the Sine of the sum of the first connected arc plus the arc of the latitude of Mecca, and we divide the

quantity. Because he did not have tangent tables, he has to transform this to Sine *WK* first and then use a Sine table.

¹⁸Here al-Nayrīzī states the theorem which was later called the Sine theorem: $\text{Sin } WK : \text{Sin } EB = \text{Sin } KA : \text{Sin } AB$.

¹⁹The scribe wrote: connected.

²⁰The scribe wrote: separated, and therefore the manuscript text is nonsensical.

product by the Cosine of the first connected arc. We convert the quotient into an arc (by means of a Sine table), and the resulting arc is the second connected arc. We subtract it from ninety degrees and the result is the second separated arc.

Chapter on the determination of the azimuth, that is arc *LB*. The ratio of the Sine of arc *LB* to the Sine of arc *KB* is as the ratio of the Sine of arc *LZ* to the Sine of arc *ZH* compounded with the ratio of the Sine of arc *TH* to the Sine of arc *KT*. All arcs are known except arc *LB*, the first arc, so it is determined by the other arcs.

Chapter on the computation of it, that is, of the arc of the azimuth between the endpoint of the line of the middle of the south,²¹ towards the west, at the City of Peace, and in the same way for every city of which the longitude is greater than the longitude of Mecca from the West. We multiply the Sine of the first separated arc with the greatest Sine, and we divide the product by the Sine of the second separated arc. We multiply the quotient by the Cosine of the latitude of Mecca, and we divide the product by the Cosine of the first connected arc. We convert the quotient to an arc (by means of a Sine table), then the resulting arc is the arc of the azimuth which we mentioned.

Example of this. Until now it has not been possible for me to make any observations with which I can observe the magnitude of the difference between the meridian of Mecca and the meridian of the City of Peace. This observation requires lunar eclipses, which have to be observed by two observers, one of them in the City of Peace and the other in Mecca. Each of them determines the amount of time of night that has elapsed until the beginning of the eclipse, or its end, or the end of the (complete) occultation. Then one finds the difference between the two moments in the two localities. I mean by the two moments the amount of time that has passed since midnight or that remains until midnight. The remaining difference is the distance between the two meridians. As has been (mentioned) above, this excess for the City of Peace over Mecca is three degrees. This I have found written as I will now describe. The Commander of the Faithful (i.e. Caliph) al-Ma'mūn, may God be pleased with him, desired to verify the azimuth of the qibla, so he found the meridian of Mecca to the West of the meridian of the City of Peace by three degrees approximately. It is clear that these three degrees are (measured) on the equator, and they are the magnitude of arc *EW*.

We made the latitude of the City of Peace²² into a Sine, namely 33;2,38, that is **118958** seconds, that is **7137480** thirds. The Cosine of the latitude of the locality (i.e. the City of Peace) is 50;4,54, that is **180294** seconds. I divided the Sine of the

²¹The line of the middle of the south is the intersection of the meridian plane and the southern half of the horizon plane.

²²Al-Nayrīzī must have assumed the latitude of Baghdađ to be 33;25°, note that Sin 33;25° = 33;2,36,17

latitude of the City of Peace by the Cosine of it, and the quotient was 0;39,35,17, that is **142517** thirds. We have called this the ratio of the first to the second (quantity). The Cosine of the three degrees, that is the Cosine of the longitude difference between the two circles (i.e. meridians) is 59;55,4. I divided it by the greatest Sine and the quotient was 0;59,55,4, and this is the ratio of the fifth to the sixth (quantity), that is **215704** thirds. Then I divided the ratio of the first to the second by the ratio of the fifth to the sixth, and the quotient is the ratio of the third to the fourth,²³ 0;39,51,14. I multiplied it by itself and the product was 0;26,28,20. I added one to it, that is, one degree, and I took the root of the sum, the root was **4322** seconds, that is 1;12,2. I divided the greatest Sine by it, and the quotient was 49;58,37, and that is the Sine of the first connected arc *KW* from the previous proposition, namely **179917** seconds.²⁴ We multiplied it with the greatest Sine and we divided the product by the Cosine of the latitude of the locality, that is 50;4,54. The quotient is 59;52,6,²⁵ and this is the Cosine of the first separated arc,²⁶ which is the Sine of arc *KA*. The first connected arc is 56;24,8 and the latitude of Mecca is 21;40.²⁷ We add this to the first connected arc, and the sum is 78;4,8. The Sine of this is 58;42,12. I multiplied this with the Sine of the latitude of the City of Peace, which is 33;2,38, and the product was²⁸ 1939;46,55. The complement of the (first) connected arc is 33;35,52, and its Sine is 33;12,10.²⁹ I divided by it the product of the first multiplication, **6983215** seconds,³⁰ the second (number was) **119530**,³¹ and the quotient was 58;25,20 and that is the Sine of the second connected arc. Therefore the second connected arc is 76;50. I subtracted it from 90 degrees and the remainder is the second separated arc, 13;10, that is arc *ZH*.

We had (already) computed the Cosine of the first separated arc, that is the Sine of arc *KA*, namely 59;52,6.³² The corresponding arc, which is the complement of the first separated arc, is 86;12,41. Its complement, the first separated arc, is 3;47,19,

²³ Al-Nayrīzī must have made a mistake, because as a matter of fact $142517 : 215704 = 0;39,38,32$

....

²⁴ Schoy incorrectly reads 179417.

²⁵ Schoy incorrectly reads 59;5,6.

²⁶ Again there is an error in the computation. As a matter of fact $(179917 \cdot 60) : 180294 = 59;52,28,....$

²⁷ The number 21;40 is required by the mathematical context since $56;24,8 + 21;40 = 78;4,8$ and since al-Nayrīzī says that the complement of the latitude of Mecca is 68;20°. I have therefore emended the value 21;41 in the manuscript. Schoy reads 21;41.

²⁸ Here is another computational mistake. As a matter of fact, $58;42,12 \times 33;2,38 = 1939;47,11,7,36$.

²⁹ Schoy incorrectly reads 33;12,18. As a matter of fact, $\sin 33;35,52 = 33;12,5,36,....$

³⁰ 6983215 seconds equals 1939;46,55.

³¹ 119530 seconds equals 33;12,10. Schoy incorrectly reads 112530.

³² As above, Schoy incorrectly reads 59;5,6.

that is arc *KB*. Its Sine is 3;57,56.³³ I multiplied it with the greatest Sine, and the product is 237;56. We divided it by the Sine of the second separated arc, that is 13;40,1, and the quotient is 17;24,35.³⁴ The complement of the latitude of Mecca is 68;20, and its Sine is 55;42,18.³⁵ I multiplied that with the quotient, and the result is 969;42,15.³⁶ I divided this by the Cosine of the first connected (arc), that is 33;12,15.³⁷ The arc (such that its Sine is equal to the quotient) is 29;7 and that is the magnitude of arc *BL*, which is the magnitude of the azimuth; I mean that the distance between the endpoint of the southern meridian, towards the West, on the horizon of the City of Peace, and the point to which the prayer has to be made, is 29;7 and that is what we wanted to demonstrate.³⁸

Al-Nayrīzī said: Nobody preceded me in this subject, and thus the computations of Ḥabash and other geometers and calculators are false.

End of the treatise. Praise to God, the Lord of the Worlds, and may God bless Muḥammad and his family. I copied (this) from a text of Nazīf³⁹ in the month Rajab.⁴⁰

³³Schoy incorrectly reads 3;57,5. As a matter of fact, $\text{Sin}(3;47,19) = 3;57,52 \dots$

³⁴Schoy incorrectly reads 17;44,35. It is likely that al-Nayrīzī ignored the last digit 1 in the divisor, because, as a matter of fact, $237;56 : 13;40,1 = 17;24,33,51\dots$ and $237;56 : 13;40 = 17;24,35,7\dots$

³⁵Schoy reads 55;45,18. There is a problem here because $\text{Sin } 68;20 = 55;45,39 \dots$ However, Schoy's reading cannot be correct because $55;45,18 \times 17;24,35 = 970;40,45\dots$, compare the next footnote.

³⁶I have emended the number 959 in the manuscript to 969 to make mathematical sense of the rest of the computation. Schoy follows the reading of the manuscript. As a matter of fact $55;42,18 \times 17;24,35 = 969;48,30,52,30$.

³⁷Above this Cosine was said to be 33;12,10, compare footnotes 29 and 31.

³⁸Schoy's translation ends here.

³⁹The name appears more completely in the text following al-Nayrīzī's treatise:

هذا ما نقله نظيف بن يمن المطبي مما وجد في اليوناني من الزيادة في اشكال المقالة العاشرة
These are the additions to the propositions of the Tenth Book (of Euclid's *Elements*) which were found and which were transmitted by Nazīf ibn Yunn the Physician.

⁴⁰Here (on fol. 80b) the text probably meant Rajab of the year 359 H., because folio 75b contains the date: the last day of Jumādā II 359 H., that is May 9, A.D. 970. The month Rajab 359 H. corresponds to May 10 - June 8, A.D. 970.

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