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Reexamination of the Susa Mathematical Text No. 12: A System of Quartic Equations

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I Introduction

The Susa mathematical text No. 12, which was published by E. M. Bruins and M. Rutten, deals with a certain system of simultaneous equations.[1] The problem treated there clearly shows one of the characteristic features of Babylonian mathematics, that is, the algebraic character. In the text a complicated system of quartic equations is solved without using any mathematical symbols. It is regrettable, however, that the statement of the problem has not been completely understood as yet, although it is well preserved on the tablet. Moreover, the mathematical interpretation proposed by Bruins and followed by J. Friberg[2] does not agree with the text itself. In addition, there are several mistakes in the transliteration by Bruins and Rutten. Therefore I shall present in this paper a new interpretation of the problem with my own transliteration and translation.

II The technical term, *manâtum*

The Akkadian word, *manâtum*, is the plural form of *manītum* meaning "amount, number, length, accounting"[3] and is used in the sense of "numbers" in the mathematical text IM 52301, obverse, lines 19 and 20:[4]

5 *ša e-te-ru* / 10 *ša tu-iṣ-bu* 40 *ši-ni-pé-tim* a-ra$^{\text{sic}}$ *ma-ni-a-ti-a lu-pu-ut-ma*

[1] E. M. Bruins and M. Rutten, *Textes mathématiques de Suse* [= TMS], 1961, pp. 78–81. Cf. The review of TMS by W. von Soden, *Bibliotheca Orientalis* 21 (1964), pp. 44–50. See further, K. Muroi, Quadratic Equations in the Susa Mathematical Text No. 21, *SCIAMVS*, vol. 1, 2000, pp. 3–10.

[2] J. Friberg, *A Survey of Publications on Sumero-Akkadian Mathematics, Metrology and Related Matters (1854–1982)*, 1982, p. 89. D. O. Edzard, ed., *Reallexikon der Assyriologie und Vorderasiatischen Archäologie*, Band 7 (1990), p. 576.

[3] A. L. Oppenheim and E. Reiner, *The Assyrian Dictionary* [= CAD], vol. 10, MII, 1977, p. 86.

[4] Taha Baqir, Another Important Mathematical Text from Tell Harmal, *Sumer* 6 (1950), pp. 130–148. Cf. CAD, vol. 1, AI, p. 227; *aramanītu* (a mathematical term).

> "5 by which (the upper width) exceeds (the lower width). 10 which you added. 0;40 of two thirds. Write down the factors of my numbers, and ..."

In our text the *manâtum* occurs in the following technical expression:

> *ma-na-at* uš *a-na* sag *i-la-ku ma-na-at* sag *a-na* uš *i-la-ku*
>
> "The number with which the length goes to the width, (and) the number with which the width goes to the length".

As will be shown in the next section, we can interpret this passage as:

> "The ratio of the length to the width (and) the ratio of the width to the length" or "x/y (and) y/x" in modern symbols (x = the length, y = the width).

It should be emphasized that a similar expression is found in the so-called series texts, which are ancient drill books in mathematics. I cite here only one example from YBC 4668, namely, C 20, lines 24–28:[5]

24. a-šà 1 (èše)$^{\text{iku}}$	The area is 1 (èše).
25. igi-te-en níg sag uš-šè	The reciprocal ratio of the width to the length
26. ù igi-te-en níg uš sag-šè	and the reciprocal ratio of the length to the width
27. gar-gar-*ma* 2,10	are added together, and 2;10.
28. uš sag-bi en-nam	What are the length (and) its width?

The system of simultaneous quadratic equations treated here is:

$$xy = 1\ (\text{èše}) = 10{,}0 \qquad (\text{square nindan; 1 nindan} \fallingdotseq 6\text{m}),$$
$$x/y + y/x = 2;10.$$

As is usual with the series texts, neither the process of calculation nor the answer ($x = 30$ and $y = 20$) is given in the text.

We will see in the following section that the problem in our text, which is similar to, but more complicated than, the above cited problem, is actually solved.

[5]O. Neugebauer, *Mathematische Keilschrift-Texte*, Erster Teil, 1935, p. 430.

III Susa mathematical text No. 12

Transliteration

Obverse

1. [*m*]*a-na-at* uš *a-na* sag *i-la-ku ma-na-at* [sag]
2. *a-na* uš *i-la-ku* ul-gar *a-na* ul-gar uš ù sag *aš-ší*
3. *šà i-la it-ti* ul-gar uš ù sag íl(!)-*ma*
4. 1,30:16,40 *a-tu-úr ma-na-at* uš
5. *a-na* sag *i-la-ku ma-na-at* sag *a-na* uš *i-la-ku* ul-gar
6. *a-na li-ib-bi* 2 a-šà ù [a-šà]
7. *šà* uš ugu sag *i-te-ru* dah̬ 2,3[1,40 u]š ù sag *mi-nu*
8. za-e 2 *ka-aiia-ma-na a-na* [2,31,40 dah̬]
9. 5$^{\mathrm{sic}}$,31,40 *ta-mar* 1/2 4,31,40 *h̬*[*e-pe*]
10. 2,15,50 *ta-mar* 2,15,50 du$_8$ gar nigin
11. 5,7,30:41,40 *ta-mar i-na* 5,[7,30:4]1,40
12. 1,30:16,40 zi 3,37,[14,1,40 *ta*]-*mar*
13. *mi-na* íb-si 1,54,10 [íb-si 1,54,10]
14. *a-na* 2,15,50 *ta-ki-*[*il-tim* dah̬ 4,10 *ta-mar*]
15. 2 *ka-aiia-ma-na* [...

(the rest of the tablet is lost)

Translation

1, 2. I added together the ratio of the length to the width (and) the ratio of the width to the length. I multiplied (the result) by the sum of the length and the width.

3. I multiplied the result which came out and the sum of the length and the width together, and (the result is)

4, 5. 1;30,16,40.[6] I returned. I added together the ratio of the length to the width (and) the ratio of the width to the length.

6, 7. I added (the result) to the "inside" of 2 areas ($= 2xy$) and [of the square] of the amount by which the length exceeded the width ($= (x-y)^2$), (and the result is) 2;3[1,40]. What are [the l]ength and the width?

8. You, [add] the normal (number) 2 to [2;31,40],

9. (and) you see 4;31,40. Ha[lve] 4;31,40,

10. (and) you see 2;15,50. Put down 2;15,50 which was separated (from

[6] In our text the cuneiform sign GAM occurs as a separation mark. It is denoted by a colon in my transliteration. It seems to be used here in order to make the reading of numbers easy.

4;31,40).[7] Square (it),

11, 12. (and) you see 5;7,30,41,40. Subtract 1;30,16,40 from 5;[7,30,4]1,40, (and) [you] see 3;37,[14,1,40].

13. What is the square root (of 3;37,14,1,40)? 1;54,10 is [the square root].

14. [Add 1;54,10] to 2;15,50 which was [used in squaring, (and) you see 4;10].

15. The normal number 2 [...

Mathematical Commentary

In the statement of the problem (lines 1–7), the following system of simultaneous quartic equations is presented:

$$\left(\frac{x}{y}+\frac{y}{x}\right)(x+y)^2 = 1;30,16,40 \tag{1}$$

$$2xy+(x-y)^2+\left(\frac{x}{y}+\frac{y}{x}\right) = 2;31,40. \tag{2}$$

Adding 2 to both sides of (2), we obtain

$$x^2+y^2+\frac{(x+y)^2}{xy} = 4;31,40 \qquad \text{(lines 8 and 9).} \tag{3}$$

If we denote x^2+y^2 by X and $(x+y)^2/xy$ by Y at our convenience, equations (1) and (3) turn out to be a typical "normal form" of equations in Babylonian mathematics:

$$\begin{aligned} XY &= 1;30,16,40 \\ X+Y &= 4;31,40. \end{aligned}$$

Next begins the usual Babylonian method to obtain X and Y:

$$\begin{aligned} \frac{X+Y}{2} &= \frac{4;31,40}{2} \\ &= 2;15,50 \qquad \text{(lines 9 and 10),} \\ \left(\frac{X+Y}{2}\right)^2 &= (2;15,50)^2 \\ &= 5;7,30,41,40 \qquad \text{(lines 10 and 11)} \end{aligned}$$

[7]In line 10 the verbal adjective du$_8$ (= *paṭru*) "to be separated" occurs. Cf. K. Muroi, Reexamination of the Susa Mathematical Text No. 11: A Primitive Indeterminate Equation and a Complicated System of Quadratic Equations, to appear in *Acta Sumerologica*, vol. 21, 1999.

$$\begin{aligned}
\left(\frac{X+Y}{2}\right)^2 - XY &= 5;7,30,41,40 - 1;30,16,40 \\
&= 3;37,14,1,40 \qquad \text{(lines 11 and 12)}, \\
\sqrt{\left(\frac{X+Y}{2}\right)^2 - XY} &= \sqrt{\left(\frac{Y-X}{2}\right)^2} \\
&= \sqrt{3;37,14,1,40} \\
&= 1;54,10 \qquad \text{(line 13)}. \\
\therefore \frac{Y-X}{2} &= 1;54,10 \qquad (Y > X \text{ is implicitly assumed}). \\
\therefore Y &= \frac{X+Y}{2} + \frac{Y-X}{2} \\
&= 2;15,50 + 1;54,10 \\
&= 4;10 \qquad \text{(line 14)}.
\end{aligned}$$

The next step, that is, the subtraction in order to obtain X:

$$\begin{aligned}
X &= \frac{X+Y}{2} - \frac{Y-X}{2} \\
&= 2;15,50 - 1;54,10 \\
&= 0;21,40,
\end{aligned}$$

which is expected between lines 14 and 15, seems to be omitted. Although the text after line 15 has been lost forever, we can reconstruct the calculations as follows.

$$\begin{aligned}
Y &= \frac{(x+y)^2}{xy} \\
&= \frac{x^2+y^2}{xy} + 2 \\
&= 4;10, \\
\frac{x^2+y^2}{xy} &= 4;10 - 2 \\
&= 2;10 \qquad \text{(line 15)}, \\
2;10\, xy &= x^2 + y^2 \\
&= X \\
&= 0;21,40, \\
\therefore xy &= 0;10 \qquad \text{(obtained by trial and error)}. \\
(x+y)^2 &= 4;10 \cdot 0;10 \\
&= 0;41,40,
\end{aligned}$$

$$\begin{aligned}
\therefore x + y &= \sqrt{0;41,40} \\
&= 0;50. \\
\frac{x+y}{2} &= 0;25, \\
\sqrt{\left(\frac{x+y}{2}\right)^2 - xy} &= \sqrt{0;10,25 - 0;10} \\
&= \sqrt{0;0,25} \\
&= 0;5. \\
\therefore x &= \frac{x+y}{2} + \frac{x-y}{2} \\
&= 0;25 + 0;5 \\
&= 0;30, \\
y &= \frac{x+y}{2} - \frac{x-y}{2} \\
&= 0;25 - 0;5 \\
&= 0;20.
\end{aligned}$$

Thus our text substantially includes the system of quadratic equations;

$$xy = 0;10, \qquad \frac{x}{y} + \frac{y}{x} = 2;10,$$

which is the same as the one of YBC 4668, C 20.

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