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Abū Sahl al-Kūhī on Rising Times

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Abū Sahl al-Kūhī (fl. ca. AD 980), one of the great mathematicians of medieval Islam, was rare among his colleagues for seldom venturing into mathematical astronomy. Almost all of his works are devoted to practicing geometry in the Islamic style within the tradition of Euclid, Archimedes and Apollonius. Those that do relate to astronomy (for example, "On The Distance to the Shooting Stars"[1] and "What is Seen of Sky and Sea"[2]) further reveal his geometric preferences; they propose geometric means of solving astronomical problems, but give no computations and report no observations.[3]

The current treatise, which we shall call "Rising Times of a Known Arc of the Ecliptic" (it is untitled in the manuscript), is no exception. Astronomers of al-Kūhī's time often compiled their work in *zīj*es, large manuals which contained reports of observations, trigonometric computations to derive parameters, and many mathematically-computed tables to allow readers to determine (among other things) the positions of celestial bodies, events of astrological significance, and rising times. These works followed the tradition established by Ptolemy in his *Almagest*, sometimes improving the parameters, shortening the exposition of the mathematical underpinnings, and making the astronomical tables more useful. Although this treatise deals with a topic usually handled by the "the experts in the *zīj* operations", as al-Kūhī calls them, he does not actually compute any rising times, much less compile a table; but simply provides a trigonometric method by which one could compute them. He recognizes that this work is a departure for him, and indeed, expresses some uncertainty concerning its usefulness in practice.

Al-Kūhī's treatise is of particular interest for the history of spherical trigonometry. Until the tenth century AD, the workhorse of this discipline was Menelaus's

[1]This treatise is listed by Fuat Sezgin in his *Geschichte des Arabischen Schrifttums*, vol. VI, *Astronomie*, Leiden: Brill, 1978, as the second entry under al-Kūhī, p. 219. See also J. L. Berggren and Glen Van Brummelen, "Abū Sahl al-Kūhī on the Distance to the Shooting Stars", *Journal for the History of Astronomy* **32** (2001), 137–151.

[2]This treatise, which we are preparing for publication, is listed by Fuat Sezgin in *Geschichte des Arabischen Schrifttums*, vol. VI, *Astronomie*, as the fourth entry under al-Kūhī, p. 219.

[3]It is worth noting that near the end of the treatise on rising times, al-Kūhī lists among his research interests centers of gravity, optics, and conic sections. Although centers of gravity and conic sections occur within his extant writings, any work he did on optics is lost.

Theorem, both versions of which assert relations between sines of six arcs in a diagram that al-Kūhī and his contemporaries called the Transversal Theorem (see below). Since the time of Ptolemy's *Almagest*, this figure was almost the only means by which new results could be generated, and with clever application, it can be made to solve all the standard problems of spherical astronomy. In the last decade of the tenth century, however, two new theorems, the "Rule of Four Quantities" and the "Law of Sines", simplified considerably the practice of spherical trigonometry. It appears to be one of these two theorems (probably the former, as we shall see) to which al-Kūhī refers in his introduction.

Al-Kūhī's mission here is to demonstrate that the Transversal Theorem alone (to be precise, only one of the Transversal Theorem's two relations) is sufficient to compute rising times. He does this in response to an unnamed challenger, who claimed that a certain single new theorem was more efficient than methods employing the Transversal Theorem, thus rendering the latter obsolete. Once al-Kūhī has done this, he leaves the question of efficiency and superiority to future investigation. In fact, he wonders explicitly whether the experts in the *zīj* operations might have found some even better method.

The Theorems of Menelaus

Although Menelaus's Theorem is extant in Arabic works,[4] al-Kūhī's several references to the *Almagest* make it plausible that he learned it there. Two major differences exist between *Almagest* and modern trigonometry: the *Almagest* uses as its fundamental function the **chord** of a circular arc (the straight-line distance between its endpoints) rather than its sine; and it uses a base circle of radius $R = 60$ rather than the modern $R = 1$. The relation between the chord and the modern sine is $\operatorname{Crd} 2\theta = 2R \sin \theta$. Now, both statements in Menelaus's Theorem refer to the same diagram (Figure 1), where all arcs are portions of great circles on a sphere. Theorem I, in Ptolemy's terms, states that

$$\frac{\operatorname{Crd} 2m}{\operatorname{Crd} 2m_1} = \frac{\operatorname{Crd} 2r}{\operatorname{Crd} 2r_1} \cdot \frac{\operatorname{Crd} 2s_2}{\operatorname{Crd} 2s}$$

(using Neugebauer's notation[5]; $m = m_1 + m_2$, etc.), while Theorem II states that

$$\frac{\operatorname{Crd} 2r_2}{\operatorname{Crd} 2r_1} = \frac{\operatorname{Crd} 2m_2}{\operatorname{Crd} 2m_1} \cdot \frac{\operatorname{Crd} 2n}{\operatorname{Crd} 2n_2}.$$

[4]In fact, the Transversal Theorem was the subject of several Arabic treatises. For a work by one of al-Kūhī's contemporaries, see J. L. Berggren, "Al-Sijzī on the Transversal Figure", *Journal for the History of Arabic Science* **5** (1981), 23–36.

[5]See Otto Neugebauer, *A History of Ancient Mathematical Astronomy*, Part 1, New York: Springer-Verlag, 1975, 27–29.

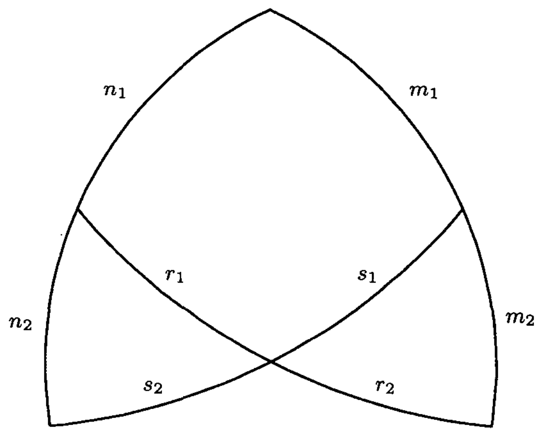

Figure 1

Converted to modern functions, they are

$$\frac{\sin m}{\sin m_1} = \frac{\sin r}{\sin r_1} \cdot \frac{\sin s_2}{\sin s} \tag{1}$$

and

$$\frac{\sin r_2}{\sin r_1} = \frac{\sin m_2}{\sin m_1} \cdot \frac{\sin n}{\sin n_2}. \tag{2}$$

Al-Kūhī and other Islamic astronomers use the medieval trigonometric functions, which differ from their modern equivalents only in their use of a circle of radius $R = 60$; thus, denoting by "Sin" the medieval sine function, $\mathrm{Sin}\,\theta = R \sin\theta$. This has one impact on the reading of the translation: for example, when al-Kūhī describes the use of the formula $\sin\delta = \sin\lambda \sin\varepsilon$ to find the solar declination δ, he must divide the right hand side by 60 before taking the arc sine; assuming the standard sexagesimal number system was used, this means shifting the "sexagesimal point" one place to the left. He describes the result of the calculation of $\sin\lambda\sin\varepsilon$ before the place shift as "the result in minutes". In our exposition, for ease of comparison between methods, we follow Neugebauer's use of modern trigonometric functions and the notation for spherical astronomy in his *History of Ancient Mathematical Astronomy.*[6] Readers who wish to reconstruct the medieval trigonometric expressions need only multiply each modern function by R.

The Problem of Rising Times

In the Ptolemaic cosmos, the Earth is located at the center of a large sphere containing on its surface the fixed stars. This celestial sphere (observed from the outside in

[6]Neugebauer's account of the spherical astronomy relevant to this paper may be found in *A History of Mathematical Astronomy*, Part 1, 30–39.

Figure 2; E is the East point) rotates uniformly from east to west about its poles (N is the north pole), making one revolution per day, or 15° per hour. Thus, at least

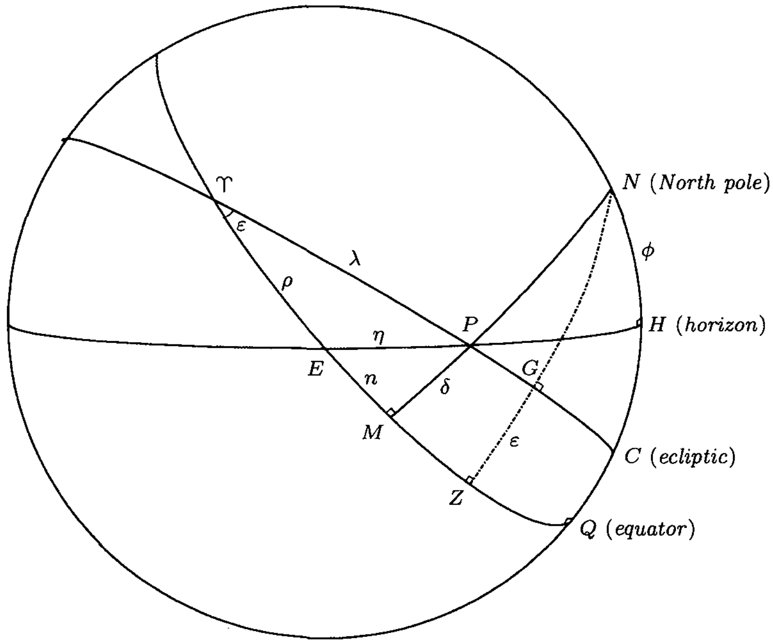

Figure 2

for arcs of the equator, arcs and time intervals are in a sense the same thing, once the conversion factor is applied. However, the Sun, P, is on the ecliptic, a great circle inclined to the equator by $\varepsilon \approx 23.6°$. Knowing, for instance, the length of time before sunrise that the vernal equinox ♈ rises, requires the ability to calculate the length of the arc on the equator that rises during the same period of time as arc ♈P of the ecliptic. Thus we define the **rising time** (or **oblique ascension**) ρ of any point P on the ecliptic, with longitude λ measured eastward from ♈), as the arc on the equator ♈E that rises simultaneously with ♈P.[7] ρ is a function of both λ and $\phi = NH$, the local terrestrial latitude. In the case where $\phi = 0$ (*sphaera recta*), N coincides with H and ♈M coincides with ♈E; we refer to ♈M as α, the **right ascension** of P. Arc $\delta = PM$ is the **declination** of P; together with the right ascension it forms the equatorial coordinate system. The difference between ρ and α, $n = EM$, is the **equation of daylight** (or **ascensional difference**); the arc $\eta = EP$ along the horizon is known as the **ortive amplitude** of P. η and n are functions of both λ and ϕ. The task is: given λ and ϕ, find ρ.

An ability to compute rising times was necessary for both the astronomers and the astrologers. An astronomer might use rising times, for instance, to find the

[7] P does not have to represent the Sun; in horoscope problems, for instance, it was common to find the point P on the ecliptic corresponding to a given rising time.

length of daylight on a given day in a given location. An astrologer might need a table of rising times to compute, say, the cusps of the astrological houses or the ascendent.[8]

Rising Times in the Almagest

Ptolemy gives two different methods to calculate rising times in the *Almagest*, both relying on Menelaus II. The first method applies it to figure $EPNQ$, and concludes from the fact $EQ = 90^\circ$ the relation

$$\frac{\cos\delta}{\sin\delta} = \frac{\sin\phi}{\cos\phi} \cdot \frac{1}{\sin n},$$

which may be simplified to

$$\sin n = \tan\phi \tan\delta.^9 \tag{3}$$

Since $n = \alpha - \rho$, and Ptolemy had previously tabulated the right ascensions α and declinations δ,[10] he is finished.

Ptolemy's second method[11] streamlines the process of computing rising times for his particular situation. He is working with eleven localities with terrestrial latitudes determined by equally spaced lengths of longest daylight $T\,(12, 12^1/_2, \ldots, 17$ hours). He thus converts T (or rather, an arc derived from it) into an arc on the celestial sphere and finds the rising time directly from it and λ, thereby bypassing the need to find ϕ first.

Al-Kūhī on Rising Times

Al-Kūhī's purpose in this work is to show that the rising time and various associated quantities may be found using only Menelaus's Theorem ("and no other theorems"), and he uses only Menelaus I, not the Menelaus II used by Ptolemy. His description

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[8]For an example of the use of rising times to find the cusps of the astrological houses, see E. S. Kennedy, "The Astrological Houses as Defined by Medieval Islamic Astronomers", in *From Baghdad to Barcelona: Studies in the Islamic Exact Sciences in Honour of Prof. Juan Vernet* (Barcelona: University of Barcelona, 1996), esp. 541–543; for the ascendent see E. S. Kennedy, "Treatise V of Kāshī's Khāqānī Zīj: Determination of the Ascendent", *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften* **10** (1996), 123–145.

[9]Since Ptolemy (unlike the Arabic astronomers) did not have access to a table of tangents, or even a tangent function, he would have had to compute the chord equivalents to the division of sine and cosine of the same arc directly.

[10]For declinations see Claudius Ptolemy, *Ptolemy's Almagest*, trans. Gerald J. Toomer, London: Duckworth / New York: Springer-Verlag, 1984, I.14, 69–70; for right ascensions, see I.16, 71–74; for the first method of finding rising times, see II.7, 92–94.

[11]See Claudius Ptolemy, *Ptolemy's Almagest*, II.7, 94–99.

is brief, giving only the relations in the numbered equations below, and there is no diagram in the manuscript; we give here a reconstruction of his method.

Let G be the summer solstice point on the ecliptic and draw arc NGZ; then $GZ = \varepsilon$. First, apply Menelaus I to the figure $NZ\Upsilon P$; this produces

$$\frac{\sin 90^\circ}{\sin\varepsilon} = \frac{\sin 90^\circ}{\sin\delta}\cdot\frac{\sin\lambda}{\sin 90^\circ},$$

which is rearranged to the standard formula to determine declinations,

$$\sin\delta = \sin\lambda\sin\varepsilon. \tag{4}$$

Next apply Menelaus I to $EPNQ$; we find

$$\frac{\sin 90^\circ}{\cos\phi} = \frac{\sin 90^\circ}{\sin\delta}\cdot\frac{\sin\eta}{\sin 90^\circ},$$

or

$$\sin\eta = \frac{\sin\delta}{\cos\phi}, \tag{5}$$

which allows al-Kūhī to compute the ortive amplitude η using his result from (4). Now apply Menelaus I, again to $EPNQ$ but assigning the quantities the other possible way; we get

$$\frac{\sin 90^\circ}{\sin MQ} = \frac{\sin 90^\circ}{\cos\eta}\cdot\frac{\cos\delta}{\sin 90^\circ},$$

or

$$\sin MQ = \frac{\cos\eta}{\cos\delta}. \tag{6}$$

Similarly apply Menelaus I, again to $NZ\Upsilon P$ but again assigning the quantities the other way, producing

$$\frac{\sin 90^\circ}{\sin MZ} = \frac{\sin 90^\circ}{\cos\lambda}\cdot\frac{\cos\delta}{\sin 90^\circ},$$

or

$$\sin MZ = \frac{\cos\lambda}{\cos\delta}. \tag{7}$$

The difference between the arcs determined in (6) and (7), $ZQ = MQ - MZ$, is equal to the rising time ρ (since ZQ and ρ are what remain when arc EZ is removed from the 90° arcs EQ and ΥZ respectively). In addition, the complement of MQ is $EM = n$, the equation of daylight, and the complement of MZ is $\Upsilon M = \alpha$, the right ascension. Thus we may express (6) and (7) respectively as

$$\cos n = \frac{\cos\eta}{\cos\delta} \qquad \text{and} \qquad \cos\alpha = \frac{\cos\lambda}{\cos\delta},$$

from which ρ is found as $\alpha - n$.

Compared with Ptolemy's (3), al-Kūhī's (4)–(7) may seem complicated. However, Ptolemy's method has already tabulated δ using (4), and the right ascensions α via

$$\sin\alpha = \cot\varepsilon\tan\delta; \tag{8}$$

and while both methods produce the equation of daylight n along the way (Ptolemy by (3)), al-Kūhī also finds the ortive amplitude η. Al-Kūhī uses (1) four times to generate five useful quantities (ρ, δ, α, n, and η), whereas Ptolemy uses (1) and (2) a total of three times to generate four (ρ, δ, α, and n).

None of al-Kūhī's relations (4)–(7) were new; in fact, all but (7) appear in the *Almagest.* Relation (7) may be found in works by Ḥabash al-Ḥāsib, Ibn Yūnus, and al-Nayrīzī, who gives a direct proof of it.[12] We have no evidence to conclude whether or not al-Kūhī found it independently.

Comparisons with al-Kūhī's Contemporaries

Both the Rule of Four Quantities and the Law of Sines obviate the need to search within diagrams for, or construct, instances of Menelaus's Theorem. The Rule of Four Quantities states that for the configuration in Figure 3,

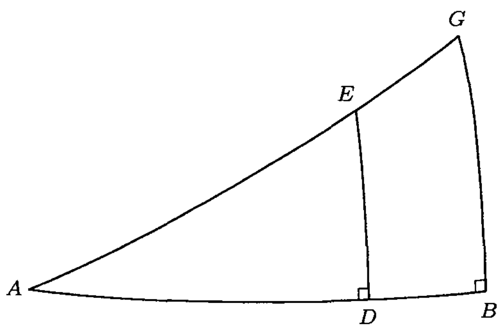

Figure 3

$$\frac{\sin BG}{\sin GA} = \frac{\sin DE}{\sin EA},$$

while the Law of Sines states that for any spherical triangle (Figure 4),

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin g}{\sin G},$$

where a, b, and g are the sides, and A, B, and G are the opposite angles. Abū 'l-Wafā'

[12]Marie-Thérèse Debarnot, in Abū 'l-Rayḥān al-Bīrūnī, *Kitāb Maqālīd ʿIlm al-Hayʾa: La Trigonométrie Sphérique chez les Arabes de l'Est à la Fin du Xᵉ Siècle*, edition and French translation by Marie-Thérèse Debarnot, Damas: Institut Français de Damas, 1985, p. 30 note 8; p. 150 note 1; and p. 200 note 4.

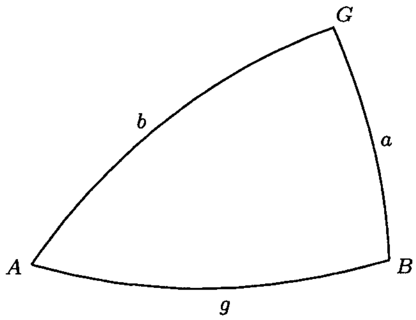

Figure 4

al-Būzjānī and his student Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq (who in turn taught al-Bīrūnī) made competing claims to the discovery of these new theorems; their chronology was outlined in al-Bīrūnī's *Keys of Astronomy*. (See the excellent edition, French translation, and commentary on this work by Marie-Thérèse Debarnot.[13])

It seems certain that al-Kūhī was responding to the use of one of the new formulas to compute rising times. He was likely faced with the challenge of using only one of (1) or (2) to compute ρ and n, by someone who claimed to be able to compute them using only one of the new theorems. Either the Law of Sines or the Rule of Four Quantities is sufficient, and in fact the Rule of Four Quantities produces al-Kūhī's method quite naturally: formula (4) emerges from applying the Rule to $\Upsilon PGZM$; (5) from $EPHQM$; (6) from $NPMQH$; and (7) from $NGZMP$. (The Law of Sines is more cumbersome in this situation, ironically, because of its innovation of using angles in addition to arcs.)

Was al-Kūhī responding to someone using the Rule of Four Quantities? Although Abū Maḥmūd al-Khujandī seems to have been the first to use it, a conjecture in favour of the Rule of Four Quantities would be better supported by ascribing al-Kūhī's references in his introductory passage to competing claimant Abū Naṣr or one of his followers. In his *Book of the Azimuth* (lost but quoted in al-Bīrūnī's *Keys of Astronomy*), Abū Naṣr states two theorems based on his "Figure that Frees" (referring to its superceding of the Transversal Theorem),[14] using a word with the same Arabic root as that possibly used by al-Kūhī (gh-n-y)[15] when referring to the theorem used by his interlocutor. These theorems state the following (Figure 5):

$$\frac{\sin DE}{\sin BG} = \frac{\sin AE}{\sin AG} \quad \text{and} \quad \frac{\sin EF}{\sin GF} = \frac{\sin AE}{\sin ED}.$$

[13] Abū 'l-Rayḥān al-Bīrūnī, *ibid.*

[14] Abū 'l-Rayḥān al-Bīrūnī, *ibid.*, 132–137.

[15] The word in the manuscript at this point is clearly wrong, but it can be restored to *al-mustaghnun*, with root gh-n-y.

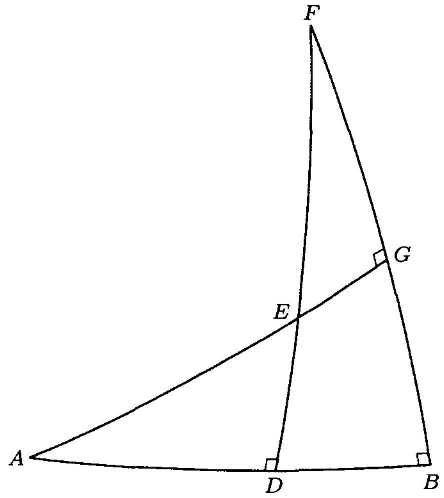

Figure 5

The former is the Rule of Four Quantities, although here with the additional restriction of requiring a right angle at G; the latter is related to it, but different. Abū Naṣr trumpeted the "Figure that Frees" as a superior replacement to the Transversal Theorem,[16] and used it to derive a new method of computing right ascensions and rising times.[17]

Although Abū Naṣr's *Book of the Azimuth* is lost, his extant *Treatise on the Determination of Spherical Arcs* contains a treatment of right ascensions and rising times which uses only the "Figure that Frees", as follows. Abū Naṣr applies the Rule of Four Quantities to establish (4) which gives δ; next, the second theorem to establish

$$\sin\alpha = \frac{\sin\lambda\cos\varepsilon}{\cos\delta} \tag{9}$$

which gives α; then the Rule of Four Quantities to establish (5) which gives η; and finally, the second theorem to establish

$$\sin n = \frac{\sin\eta\sin\phi}{\cos\delta}$$

which gives n. The rising times are found, as usual, by $\rho = \alpha - n$.[18]

The Arabic word *al-shakl* can refer both to "figure" and "theorem"; hence there is some confusion regarding whether Abū Naṣr's above use of one figure but two relations would have met al-Kūhī's challenge. However, if al-Kūhī had meant "figure",

[16]Marie-Thérèse Debarnot, *ibid.*, 13.

[17]Abū 'l-Rayḥān al-Bīrūnī, *ibid.*, 136.

[18]See Paul Luckey, "Zur Entstehung der Kugeldreiecksrechnung", *Deutsche Mathematik* **5** (1941), 438–440, for an exposition of Abū Naṣr's method with a comparison to Ptolemy's.

it is hard to understand why he felt he had to improve on Ptolemy's use of (1) and (2), which refer to one and the same geometric figure; and in fact, al-Kūhī uses only (1). Possibly then, al-Kūhī's colleague was thinking of something like our use of the Rule of Four Quantities to determine rising times, given above, which applies only Abū Naṣr's restricted version of the Rule in all four of its applications.

The originality of al-Kūhī's method of computing rising times may be addressed by comparing it with the various means of calculating the quantities δ, α, η, n, and ρ that al-Bīrūnī lists (but does not attribute to particular people) in his *Keys of Astronomy*. All use (4) to find δ, and the difference of α and n to find ρ. Al-Bīrūnī gives three methods to compute α: al-Kūhī's (and others' — see above) (7), Ptolemy's (8), and Abū Naṣr's (9).[19] The computation of n via the intermediate of η is given using first al-Kūhī/Ptolemy's (5), and then either al-Kūhī/Ptolemy's (6) or another formula.[20] Finally, al-Bīrūnī alludes to, but does not state explicitly, the use of Ptolemy's (3) to find n without first finding η.[21] Thus al-Kūhī's method is entirely contained within the several alternatives provided in the *Keys of Astronomy*.

Of course, al-Kūhī's point is not to provide new means of computing rising times, but to show that the formulas can be established using only (1). Nevertheless, he wonders aloud whether some computation used by the experts in the *zīj* operations might be more efficient; he may have been answered in the affirmative by Ptolemy's bypassing of the calculation of the ortive amplitude.

Conclusion

This short treatise by al-Kūhī provides a brief glimpse into the mathematical sciences at a time when the part that was to become modern spherical trigonometry was undergoing fundamental changes. (One is reminded of 15$^{\text{th}}$ and 16$^{\text{th}}$ century Europe, when similarly important changes in algebra resulted in public contests between mathematicians.) Abū Naṣr's unabashed promotion of his new methods and the priority disputes between him and Abū 'l-Wafā' show us the challenges issued by the discoveries of the new methods; al-Kūhī's reply, the subject of this paper, shows us that the challenges did not go unanswered.

[19]See Abū 'l-Rayḥān al-Bīrūnī, *ibid.*, 200–201.

[20]Two methods of finding η once n is known, one of them a re-ordered (6), are also given. See Abū 'l-Rayḥān al-Bīrūnī, *ibid.*, 202–205.

[21]Abū 'l-Rayḥān al-Bīrūnī, *ibid.*, 204–205.

Translation

Our edition of the text is based on the unique known copy of this work, dating from 628 A.H., found in the Istanbul (Süleymaniye Library) manuscript AS 4830, 181^{a}–182^{a}. In our translation, parentheses are used as punctuation, which (like other modern punctuation — periods, commas, etc.) is entirely lacking in the Arabic original. Square brackets enclose explanatory remarks that we have inserted into the text. Dots within square brackets indicate text too badly effaced to restore. In our edition of the text we have used superscript Arabic numerals to refer to our critical notes. In the opening lines of the translation we have chosen to let the copyist's spelling of al-Kūhī's name (al-Qūhī) stand. In editing the text we have not noted the few places where we have changed the orthography to standard classical orthography, nor the several places where we have corrected what seemed to be trivial scribal errors, such as misreadings of *yā'* for *tā'* (and conversely). In general, vowelings and other auxiliary marks are not indicated in the original manuscript. (For example, only the skeleton of the verb *yataḥayya'u* appears, and even that without the two dots over the *tā'*.) In particular the shadda is not indicated in the manuscript, nor is the hamza (except for one medial hamza), and we have supplied these where necessary or customary.

In the name of God the Merciful, the Compassionate
I ask for help from God

Treatise of
Abū Sahl Wījan b. Rustam al-Qūhī

He said: Some of our colleagues who are well-advanced in this art of ours asked us at the Royal Palace, in the presence of some of the honourable members of this art attached to the Noble [i.e., the King's] Service about finding the rising time of a known arc of the ecliptic in a town of known latitude, or its equation of daylight in that [town], if nothing is known except the maximum declination of the ecliptic from the equator and the sines of arcs. And he requested us to do that for him using [only] our knowledge of the Transversal Theorem, which is in Ptolemy's *Almagest*, and no other theorems. And he claimed that he can derive that by a way that is shorter, easier and involves less work than that of the people who know [just] the Transversal Theorem, and that that is not only because of his acuity in this art, but because of another theorem not known as "The Transversal". And his support is it alone, nothing else. And he claimed that he and others were freed by it [the new theorem] from knowing the "Transversal Theorem" in these operations, and from looking into it. But it is my opinion that, although his judgment may be allowable for himself, it is not so for others. But [here is] an investigation of it.

Solution of the Problem Using the Transversal Theorem

If we multiply the sine of a known arc of the ecliptic by the sine of the whole declination, the result is the sine, expressed in minutes, of the inclination of that degree. And if that is divided by the cosine of the latitude of the town there results the sine of the ortive amplitude of that degree in that town. And if the cosines, expressed in minutes, of each of the ortive amplitude and of the known arc of the ecliptic are divided by the cosine of the latitude of that degree there result two sines. The difference between the arc of one of them and the other is the rising time of that degree in that town. The complement of one of the two arcs is the rising time in sphaera recta for that degree, and the complement of the other is the equation of daylight for that degree in that town. That is what we wanted to prove.

Now, we found by this number of operations all these things, i.e., the declination of the degree, the ortive amplitude in that town, its ascension, its equation of daylight, and its right ascension — all from our knowledge of the Transversal Theorem, which is in the *Almagest*, without anything else. Thus we know that to abandon these things which follow from this theorem and depend on anything else, and praising one of them and blaming the other, is impossible until we have investigated the matter completely, and have realized the superiority of one of them over the other and the distinction between the two, if there is between them any distinction at all — as he [our interrogator] claims there is.

Notwithstanding this condition, I am not sure that, for solving this question, there is an easier, nearer or more expeditious operation than the one we did, [one] used by the skilled mathematicians and the experts in *zīj* operations, with their tendency to use multiplication first rather than using division, to delay one until after the other, to substitute one for another, and other operations — until they get many methods, some of which are easier than others, and all of which are from the Transversal Theorem and nothing else. This is my own opinion, although I have not yet proved it. For, though I have fully investigated these kinds of sciences, I do not claim to have tackled a sufficient amount of the problems. Therefore, I have to admit that I do not have enough mastery of something which I have not persistently pursued, since one cannot master those practical matters except by diligent and continuous practice, as Ptolemy related in the *Almagest.*

The same applies to theoretical science, which cannot be mastered except by persistent investigation. I have not investigated the science which belongs to this matter [the derivation of the Transversal Theorem] in Ptolemy's book [the *Almagest*] more than I have other matters which are discussed in it, *e.g.*, the circumstances of the mean motions of the planets, their anomalous motions, the positions of their spheres, the magnitudes of the diameters of the orbs of their epicycles, their apogees, the distance of the centers of their orbs from the earth, the movements of their twist,[22] and circumstances [i.e., states] of that which is between their retrograde and direct motion — and the other sciences which are mentioned in the *Almagest* along with their proofs.

Notwithstanding the large number of these sciences, their majesty, their sublimity, their beauty and the difficulty in understanding them, we do not confine ourselves to their investigation and ignore the other sciences which scholars usually investigate, *e.g.*, the science of the centers of gravity, the [science of] optics, the science of the characteristics of the figures of the segments of cones, which is the most astonishing of the things, as [...] the ancients. Add to that the theorems of Archimedes and the derivation of the geometrical figures, each of which is an independent science, namely one established by itself. The pride of mathematicians in deriving things like these [i.e., the sciences mentioned above] is always greater than in deriving other sciences. We, likewise, [follow in the same path] without any boasting or belittling.

Now, if this discourse, which is additional to the [main] argument, is out of place, please excuse me. This was mentioned only because one statement led to another.

The treatise is completed. Praise and gratitude be to God, and may God bless our master Muhammad, his family and his companions.

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[22]This is defined by Naṣīr al-Dīn al-Ṭūsī in his *Memoir on Astronomy*, in his discussion of Mercury and Venus (Book II, Chapter 10, paragraph 5; see Jamil Ragep, *Naṣīr al-Dīn al-Ṭūsī's Memoir on Astronomy*, New York: Springer-Verlag, 1993, Vol. 1, p. 195), as "the size of the angle of intersection of the plane of the epicycle [of the planet] with a plane passing through its center and parallel to the ecliptic".

بسم الله الرحمن الرحيم استعنت بالله

رسالة لأبي سهل ويجن بن رستم[1] القوهي

قال: سألنا بعض أصحابنا المتقدمين في صناعتنا هذه في الدار المعمورة، بحضرة جماعة من فضلاء أهل هذه الصناعة الموسومين بالخدمة الشريفة، عن وجود مطالع قوس معلومة من فلك البروج في بلد معلوم العرض، أو تعديل نهارها في ذلك، إذا لم يكن شيئًا معلومًا سوى غاية الميل لفلك البروج عن معدل النهار وجيوب قسي معلومة، والتمس أن نعمل له ذلك من وجه معرفتنا بالشكل القطاع الذي في كتاب بطلميوس، أعني المجسطي، دون غيره من الأشكال. وزعم أنه يعمل ذلك بأقرب وأسهل وأقل عملًا مما يعمله القوم الذين معرفتهم بالشكل القطاع، وأنه ليس ذلك لحذقه في هذه الصناعة فقط لكن بشكل آخر غير الملقب بالقطاع، واعتماده عليه لا على غيره. وزعم أنه هو وغيره مستغنون[2] به عن معرفة شكل القطاع في هذه الأعمال، وعن[3] النظر فيه. وعندي أن حكمه، وإن كان جائزًا على نفسه، فعلى غيره لا. ولكن فيه نظر.

استخراج المسألة من الشكل القطاع

إذا ضرب جيب قوس معلومة من فلك البروج في جيب الميل كله فالخرج هو دقائق جيب ميل تلك الدرجة. وإذا قسم ذلك على جيب تمام عرض البلد خرج جيب سعة مشرق تلك الدرجة في ذلك البلد. وإذا قسم دقائق جيب تمام كل واحد من سعة المشرق والقوس المعلومة من فلك البروج على جيب تمام ميل تلك الدرجة خرج جيبان[4]، فضل قوس أحدهما على الآخر هو مطالع تلك الدرجة في ذلك البلد. وتمام إحدى القوسين هي مطالع الفلك المستقيم لتلك الدرجة، وتمام الأخرى تعديل النهار لتلك الدرجة في ذلك البلد. وذلك ما أردنا أن نبيّن.

فإذا وجدنا بهذا المقدار من الأعمال هذه الأمور كلها، أعني ميل الدرجة وسعة مشرقها في ذلك البلد ومطالعها فيه وتعديل نهارها والمطالع في الفلك المستقيم [١٨١ ب] من معرفتنا بشكل القطاع الذي في المجسطي، دون غيره، علمنا أن ترك هذه الأشياء التي تنتج من هذا الشكل وتعلُّقنا بغيره ومدحَ أحدهما والطعن على الآخر محال، إلا بعد نظرنا فيها نظرًا تامًّا ومعرفتنا بفضل أحدهما على الآخر والفرق بينهما، إن كان بينهما فرق كما زعم.

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[1] رستم — وستم [2] مستغنون — مستغنيون [3] عن — من [4] جيبان — جيبين

ومع هذه الحال كلها لست آمنًا أن يكون في وجوه[5] هذه المسألة عمل[6] هو[7] أسهل وأقرب وأكثر اختصارًا مما عملناه فيها عند الحذّاق من الحُسّاب والماهرين في أعمال الأزياج، بتقديمهم الضرب على القسمة، وتأخير أحدهما عن الآخر، وإبدالهم شيئًا بشيء، وغير ذلك من الأعمال، حتّى يتهيّأ لهم منها وجوه[8] كثيرة، بعضها أسهل من بعض، وكلها تكون من الشكل القطاع لا غيره. وهذا اعتقادي، وإن لم يصحّ لي ذلك. لأني وإن كنت نظرت في هذا الجنس من العلوم نظرًا شافيًا، فما تطلّعت من الأعمال مقدارًا كافيًا. ولذلك لست أدّعي غاية الحذق فيما ليس لي عليه مواظبة، لأن الأشياء العملية لا يبلغ إلى غايتها أحد إلا بكثرة العمل والمواظبة عليها أبدًا، كما ذكر بطلميوس في المجسطي. وكذلك في العلم النظري، إلا بالازدياد من النظر فيه. وليس نظري في علم هو من فن هذه المسألة من كتاب بطلميوس أكثر من نظري في غير ذلك من الفنون التي فيه، كأحوال حركات أوساط الكواكب، وحركات اختلافها، وأوضاع[9] أفلاكها، ومقادير أقطار أفلاك تداويرها، وأوجاتها، وبعد مراكز أفلاكها عن الأرض، وحركات التوائها، وأحوال ما بين رجوعها واستقامتها، وغير ذلك من العلوم التي هي في المجسطي بالبراهين.

ومع كثرة هذه العلوم وجلالها وشرفها وحسنها وبعدها عن الفهم، لسنا نقتصر بالنظر فيها دون غيرها من العلوم التي ينظر فيها أصحاب التعاليم، كعلم مراكز الأثقال، و<منها> المناظر وعلم خواصّ أشكال قطوع المخروطات، وهي أعجب الأشياء كلها <كما ... القدماء>. وكذلك أشكال أرشميدس، واستخراج الأشكال الهندسية التي كل واحد منها علم مفرد، وهو قائم بنفسه. وعجب أصحاب التعاليم باستخراج أمثالها أكثر أبدًا من عجبهم باستخراج غيرها من العلوم. وكذلك نحن بغير افتخار [١٨٢ أ] ولا تعريض. وإن كان هذا الكلام، الذي هو فضل على المسألة، شيئًا[10] في غير مكانه، فاعذر فإن الحديث ذو شجون.

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[5] وجوه — وجود [6] عمل — عملا [7] هو — مكتوبة في الهامش [8] وجوه — وجوها [9] وأوضاع — واوضلاع

[10] شيئًا — شيء

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