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The Geometrical Works of Abū Saʿīd al-Ḍarīr al-Jurjānī

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For Abū Max

I Abū Saʿīd al-Ḍarīr al-Jurjānī

Almost nothing is known about the life of the geometer Abū Saʿīd al-Ḍarīr ("the blind") al-Jurjānī. His name indicates that he originated from the town of Jurjān, that is present-day Gorgān near the Caspian Sea in Iran. The most complete form of his name appears in one version of al-Bīrūnī's *Extraction of Chords* [20, p. 13][1] as Abū Saʿīd Muḥammad ibn ʿAlī al-Ḍarīr al-Jurjānī. In [19, p. 27], Suter confused the geometer, Abū Saʿīd al-Ḍarīr al-Jurjānī, with the philologist, Abū Saʿīd al-Ḍarīr Aḥmad ibn Khālid, who died in 895 CE.[2] Consequently, al-Jurjānī is listed as a ninth-century mathematician by Sezgin [17, vol. 5, pp. 263–264, vol. 6, p. 159] and by Matvievskaya and Rozenfeld [13, vol. 2, p. 76, no. 48]. In the *Geometrical Problems*, published below, al-Jurjānī mentions Abū ʿAbdallāh al-Shannī,[3] who lived in the second half of the 10th century CE. This reference confirms that the geometer Muḥammad ibn ʿAlī al-Jurjānī is not identical to the philologist Aḥmad ibn Khālid. We will see below that al-Jurjānī must have flourished around 1000 CE.[4]

Two treatises by al-Jurjānī have come down to us, both of considerable historical interest.

1. His *Geometrical Problems* begin with a construction of a *neusis* by means of a parabola and hyperbola. This construction is a variation of a construction in Book IV of the *Mathematical Collection* of Pappus of Alexandria. Then follow solutions of two problems related to the trisection of the angle, and a theorem about the altitude of a triangle. The *Geometrical Problems* is addressed to an anonymous mathematician, who will be identified below as al-Bīrūnī. One of

[1] On al-Bīrūnī (973–1048) see for example [17, vol. 6, pp. 261–276].

[2] On Abū Saʿīd al-Ḍarīr Aḥmad ibn Khālid see [17, vol. 8, pp. 167–168].

[3] On al-Shannī see [17, vol. 5, p. 352]. Around 970 CE he quarreled with Abū'l-Jūd in connection with the regular heptagon, see [7, pp. 243–244].

[4] In [5, p. 331] Hermelink stated that al-Jurjānī lived in the second half of the 10th century CE.

al-Jurjānī's constructions is related to the "Problem of Alhazen" in the *Optics* of Ibn al-Haytham. Thus the *Geometrical Problems* of al-Jurjānī are connected to some of the highlights of Arabic-Islamic geometry.

2. Most of al-Jurjānī's *Extraction of the Meridian Line* is based on a lost work *Analemma* by Diodorus of Alexandria, a Greek expert on sundial theory who lived in the first century BCE. The only other extant trace of Diodorus' *Analemma* is Chapter 20 in al-Bīrūnī's *Exhaustive Treatise on Shadows*.

Al-Jurjānī's treatise on the meridian line was published in 1922 in a free German translation by Schoy [18], but the Arabic text has not hitherto been edited. The *Geometrical Problems* seem to be completely unknown to the literature. The only other traces of geometrical work by al-Jurjānī are two short proofs which are cited by al-Bīrūnī in the *Extraction of Chords*.[5]

The purpose of this paper is to publish and translate the *Geometrical Problems* and the *Extraction of the Meridian Line*, in order to make the entire extant work of al-Jurjānī available for further historical analysis. I begin with a brief description of the contents of the *Geometrical Problems*, and a brief summary of the *Extraction of the Meridian Line*. This is followed by a section on manuscripts and editorial procedures, and edited Arabic texts and English translations of the two treatises.

II The *Geometrical Problems* of al-Jurjānī

Al-Jurjānī begins his *Geometrical Problems* with a preliminary construction which I render in modernized notation (Figure 1). The text and translation can be found below.

Given: three points A, B, G on a circle, and a segment HZ. Required: to construct a straight line AED which intersects segment BG at E and the circle at D in such a way that $ED = HZ$.[6] In ancient Greek geometry, this type of construction was called a *neusis*, that is the insertion of a straight segment (DE) of given length (HZ) between two given lines which may be straight or curved (in this case segment BG and arc BG), in such a way that a given point (A) is on the segment or on its rectilinear extension.

Al-Jurjānī bisects BG at T and he draws a perpendicular to BG through T. He draws AL parallel to BG to meet the perpendicular at L, he drops perpendicular

[5] The Arabic text is available, on the basis of the Patna manuscript, in [1, no. 1, pp. 8, 23] and [2, pp. 40, 57–48], for a facsimile of the Patna manuscript of the first proof see [2, p. 30]. A German translation of the two proofs, on the basis of the Leiden manuscript, can be found in [20, pp. 13, 15].

[6] In this paper I use a notation such as ED in the ancient Greek way. Thus ED may indicate a straight line segment with endpoints D and E, but also the length of this segment.

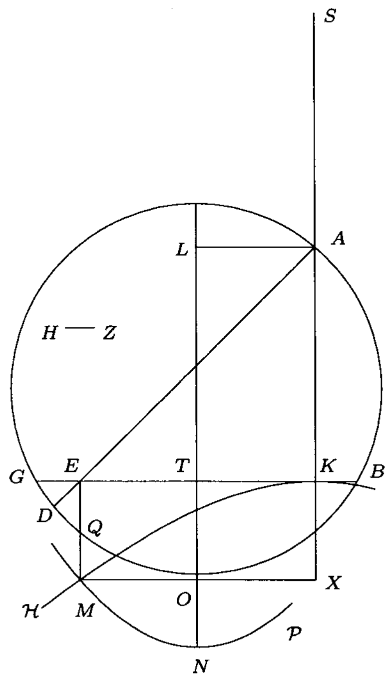

Figure 1

AK onto BG, and he chooses point S on KA extended such that $AS = AK$.

He then defines point N on line LT such that $LN \cdot HZ = \frac{1}{4}BG^2$. He assumes $HZ \cdot AK < \frac{1}{4}BG^2$, whence $LN > LT$.

He then draws one branch $\mathcal{H}$ of the equilateral (orthogonal) hyperbola through K with centre A and transverse axis SK, and the parabola $\mathcal{P}$ with vertex N, axis NL and parameter (latus rectum) HZ.

Al-Jurjānī supposes that $\mathcal{P}$ and $\mathcal{H}$ intersect at point M, and he drops perpendiculars ME onto BG and MX onto AK extended. Let MX intersect LN at O.

Since M is on $\mathcal{H}$, we have by Apollonius, *Conics* I:21,[7] $MX^2 = SX \cdot KX$. But $MX = EK$, hence $EA^2 = EK^2 + AK^2 = SX \cdot KX + AK^2 = (AX^2 - AK^2) + AK^2 = AX^2$, so $EA = AX$.

Since M is on the parabola $\mathcal{P}$ with *latus rectum* HZ, we have by *Conics* I:11[8] $MO^2 = ON \cdot HZ$. But $MO = ET$, and by definition $GT^2 = LN \cdot HZ$. Thus by subtraction $BE \cdot GE = GT^2 - ET^2 = LO \cdot HZ = AX \cdot HZ = AE \cdot HZ$. From the geometry of the circle, $BE \cdot GE = AE \cdot ED$. Hence $ED = HZ$ as required.

Al-Jurjānī's construction can be seen as a variation on a construction in the end of Book IV of the *Mathematical Collection* of Pappus of Alexandria (ca. 300 CE).[9] Pappus solves the same problem by means of two conic sections $\mathcal{P}'$ and $\mathcal{H}'$, which

[7]Putting $XM = y$, $KX = x$, $SK = c$ we obtain the modern equation $y^2 = x(x+c)$ of the hyperbola.

[8]Putting $OM = y_1$, $NO = x_1$, $HZ = p$ we obtain the modern equation $y_1^2 = px_1$ of the parabola.

[9]Pappus divides the construction into three propositions, see [23, vol. 1, pp. 231–235], [15, vol. 1,

are the images of $\mathcal{P}$ and $\mathcal{H}$ under the translation of the plane which maps A to K.[10] The perpendicular projection of the intersection M' of $\mathcal{P}'$ and $\mathcal{H}'$ on line BG is of course point E in Figure 1. The similarity with Pappus suggests that al-Jurjānī's construction is ultimately of ancient Greek origin.

We now turn to the *diorismos* of the problem, that is to say, the necessary and sufficient condition for the existence of a solution.[11] Al-Jurjānī wrongly states that the diorismos is $HZ \cdot AK \leq \frac{1}{4}BG^2$. Actually, this condition is necessary but not sufficient. If $HZ \cdot AK = \frac{1}{4}BG^2$, the problem can only be solved if K is the midpoint of BG. Figure 2 is drawn for the case where K is not the midpoint of BG and $HZ \cdot AK$ is slightly less than $\frac{1}{4}BG^2$, so that N is a little below T. In this case $\mathcal{H}$ and $\mathcal{P}$ do not intersect, and the problem has no solution.

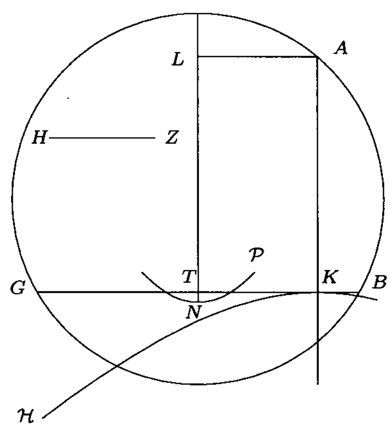

Figure 2

If K is not the midpoint of BG, the correct diorismos has the form $HZ \leq C$ for some segment C with $C \cdot AK < \frac{1}{4}BG^2$. If $HZ = C$, the curves $\mathcal{P}$ and $\mathcal{H}$ are tangent, and by analyzing this case one can in principle find a construction of C from AK, BK and BG. This problem is by no means easy: the construction of line AED itself is equivalent to the solution of an irreducible algebraic equation of degree 4, and the length of C is the root of an irreducible cubic equation. By means of a different construction, Ibn al-Haytham and al-Mu'taman ibn Hūd (died 1085) found C as a normal from a point to a certain hyperbola.[12]

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pp. 298–303, corrected in vol. 3, pp. 1231–33].

[10] I use the concept of "translation" for sake of brevity. Of course I do not want to suggest that Pappus or al-Jurjānī had the modern concept of a translation of the plane. Note that point K is the centre of $\mathcal{H}'$, and that $\mathcal{P}'$ passes through points B and T because $BT^2 = LN \cdot HZ$.

[11] Pappus does not concern himself with any diorismos to this problem. On the concept of diorismos in general see [4, vol. 1, pp. 130–131], ancient examples are e.g. *Elements* I:22 [4, vol. 1, p. 239] and the ancient additions to Archimedes, *Sphere and Cylinder* II:4, see [3, p. 197].

[12] Ibn al-Haytham considered the case where BG is a diameter [16, p. 318], and al-Mu'taman found the generalisation to arbitrary positions of BG [8, p. 77].

Al-Jurjānī says that the person to whom he addresses these problems had said, on the authority of al-Shannī, that the problem does not require the condition $HZ \cdot AK \leq \frac{1}{4}BG^2$. Al-Shannī may have thought about solutions ADE for which point E is on BG extended outside the circle. These solutions can be found for any choice of the segment HZ, by means of the intersections of $\mathcal{P}$ with the other branch of the hyperbola $\mathcal{H}$.

In the second problem, al-Jurjānī considers points B and G on the arms of a given angle BAG such that $AB = AG$ (Figure 3). Required: to construct point Z on segment AB such that $GZ \cdot AB + AZ^2 = AB^2$. The solution is easy: Al-Jurjānī draws the circle with centre A and radius AB, he extends BA to meet the circle again at point D, and he constructs, by the method of his first problem, the straight line GZE meeting BA at point Z and the circle at point E such that $EZ = AB$. Then $GZ \cdot AB + AZ^2 = GZ \cdot EZ + AZ^2 = BZ \cdot DZ + AZ^2 = AB^2$, as required. Al-Jurjānī uses the theorems $GZ \cdot EZ = BZ \cdot DZ$ and $AB^2 - AZ^2 = BZ \cdot DZ$ from Euclid's *Elements*.

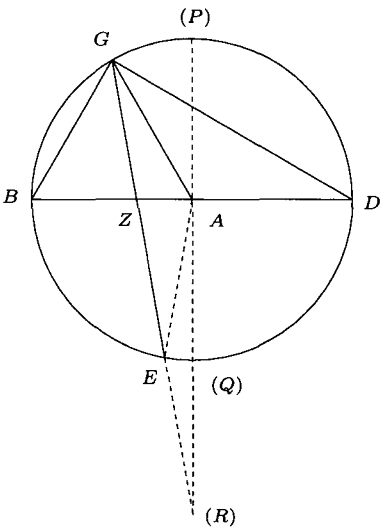

Figure 3

This problem makes it possible to identify the mathematician to whom al-Jurjānī sent this treatise. In a work on the trisection of the angle by the late tenth-century geometer al-Sijzī, the problem is mentioned as a "lemma by al-Bīrūnī" for the trisection of the angle [24, p. 119].[13] Al-Bīrūnī posed this problem to Abū'l-Jūd, who then gave a solution by means of a parabola and a hyperbola which does not in-

[13]To see the connection, consider in Figure 3 $PAQR$ perpendicular to BAD, and extend GZ to meet PQ at R. By al-Jurjānī's construction, $GA = AE = ZE$, and because $\angle BAR = 90°$, $AE = ZE = ER$. Put $\angle ARZ = x$. Then $\angle RAE = x$, hence $\angle AGZ = \angle AEZ = 2x$, so $\angle GAP = 3x$. Thus $\angle GAP = 90° - \angle GAB$ can be trisected by means of this lemma.

volve al-Jurjānī's first problem [24, pp. 114–115]. Thus the mathematician to whom al-Jurjānī sent his *Geometrical Problems* was probably al-Bīrūnī. Abū'l-Jūd flourished around 970 CE, so he was at least one generation older than al-Bīrūnī, who was born in 973 CE, and therefore their correspondence probably took place by the end of the 10th century or around 1000 CE. It is likely that al-Bīrūnī corresponded at the same time with various people about similar problems, so a date around 1000 CE seems to be most probable for the *Geometrical Problems* of al-Jurjānī as well.

In his third problem (Figure 4), al-Jurjānī considers a given triangle ABG with right angle A, and a given point D on the hypotenuse. Required: to construct point Z on AD in such a way that if GZE is drawn and extended to meet AB at point E, $GE:AZ = GB:AD$.

Al-Jurjānī does not give any motivation for this odd problem. If $AD:BG = 1:2$, the solution of the problem can be used for the trisection of an angle. See Figure 4, where $BG = 2AD$, $EG = 2AZ$, C is the midpoint of EG, and $\angle EGA = \frac{1}{3}\angle BGP$.[14] In his work on the trisection of the angle, al-Sijzī mentions this problem, with $AD:BG = 1:2$, as one of five problems to which al-Bīrūnī had related the trisection of the angle [24, p. 124]. This confirms my conclusion that al-Jurjānī sent his *Geometrical Problems* to al-Bīrūnī.

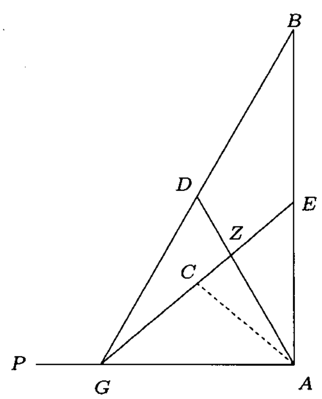

Figure 4

I now turn to the connection between al-Jurjānī's solution to the third problem, and the "Problem of Alhazen" in the *Optics* of Ibn al-Haytham. First it is necessary to summarize al-Jurjānī's proof. Assume that the ratio $AD:BG$ is arbitrary (Figure 5).

Al-Jurjānī constructs the circumscribed circle of triangle ABG and he extends AD to meet this circle at point T. Using his first construction, he draws the straight line TLK to intersect line BG at point L and the circle at K such that $LK = AD$. He draws KB and KG. From the geometry of the circle he deduces $\angle BKL =$

[14]Since $\angle EAG$ in Figure 4 is a right angle, $EC = CG = CA$, so triangle CZA is isosceles. If we put $\angle BGA = \alpha$, $\angle EGA = \beta$, we can express the three angles of triangle EZA in terms of α and β, and thus arrive at the conclusion $\alpha + 3\beta = 180°$.

$\angle BKT = \angle BAT = \angle BAD$ and similarly $\angle GKL = \angle GAD$. Al-Jurjānī now draws line MDS such that $\angle MDA = \angle BLK$. Since $KL = AD$, the figures $BKLG$ and $MADS$ are congruent, so $MS = BG$. He then draws GZE parallel to SDM. By similar triangles $EG : AZ = MS : AD$ so line GZE is the required solution.

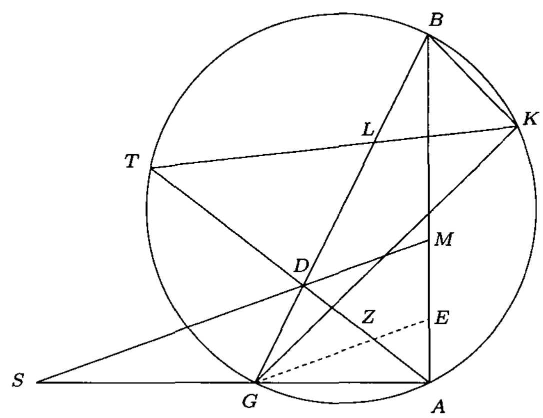

Figure 5

In the course of his proof, al-Jurjānī notes that he has now solved the following problem: to construct through a given point D a straight line (MDS) which intersects two given lines (AB and AG) at points M and S in such a way that the segment MS has a given length (equal to BG). This problem is a *neusis* between two straight lines (AB and AG). Al-Jurjānī's first problem was a *neusis* between an arc and a chord of a circle.[15] What al-Jurjānī has shown is how the *neusis* between the two straight lines can be reduced to the *neusis* between the chord and the arc of a circle.

The "problem of Alhazen" is a problem about spherical, cylindrical and conical mirrors, which was solved by al-Ḥasan ibn al-Haytham (ca. 965–1041)[16] in Book V of his *Optics*. The *Optics* were translated into Latin in the twelfth century and thus the problem became famous in the West. In this problem one assumes that the positions of the eye and the image are given, and one is required to construct the points of reflection in the mirror. Ibn al-Haytham knew that there is a maximum of four points of reflection. The problem is equivalent to an algebraic equation of degree four.

Ibn al-Haytham reduces the "problem of Alhazen," through a series of complicated preliminaries, to two *neusis*-constructions involving conic sections, namely *Optics* V:33 (Lemma 1) and *Optics* V:34 (Lemma 2), which have been translated by Sabra [16, pp. 303–309, 315–320]. In Lemmas 1 and 2, he considers a given point A on a circle with given diameter BG, and he constructs line AED which intersects

[15] Note that the given point should be on the same circle.

[16] On Ibn al-Haytham see e.g. [17, vol. 5, pp. 251–261; vol. 6, pp. 358–374].

the circle in D and BG at point E, such that DE is equal to a given segment HZ, and such that E is outside the circle in Lemma 1, and inside the circle in Lemma 2 (for which see Figure 6). I have pointed out previously that his construction of Lemma 1 is similar to a *neusis*-construction by means of a hyperbola and a circle, which is related to the trisection of an angle, and which is found in Book IV of the *Collection* of Pappus of Alexandria and in treatises by the Banū Mūsā (ca. 850) and Thābit ibn Qurra (836–901) [8, p. 62]. Ibn al-Haytham's solution of Lemma 2 is essentially different from Lemma 1, and this is odd because the problems are so similar.[17] However, I will now show that Lemma 2 is closely related to al-Jurjānī's solution of his third geometrical problem.

Al-Jurjānī shows how a *neusis* between two given straight lines can be reduced to a *neusis* between a chord and an arc of a circle. Ibn al-Haytham uses this relationship, but he argues the other way around. Thus he shows how the *neusis* between the chord and arc of a circle can be reduced to the *neusis* between the two straight lines. To illustrate the relationship, I have redrawn al-Jurjānī's figure as Figure 6, with notations used by Ibn al-Haytham in the translation by Sabra [16, pp. 318–320]. The situation is complicated by the fact that Ibn al-Haytham uses two figures. The letters A, B, G, D, E occur in his original figure, and $[Z]$, $[H]$, $[L]$ and $[M]$ in his auxiliary figure. The auxiliary figure is congruent with part of the original figure and I have added the square brackets to avoid confusion.[18]

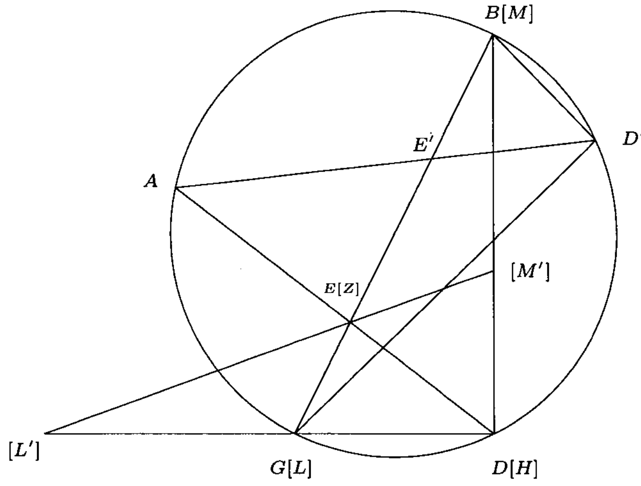

Figure 6

Ibn al-Haytham argues as follows (Figure 6). Suppose that point A is a given

[17]In his simplified solution of the "problem of Alhazen," the Andalusī mathematician al-Mu'taman ibn Hūd solved Ibn al-Haytham's Lemma 2 by the same method as Lemma 1, cf. [8].

[18]The reader who is still confused by the original and auxiliary figure should redraw the auxiliary figure on a separate sheet.

point on a circle with given diameter BG. Let the segment HZ be given. Required: to construct a straight line AED which intersects the diameter at E and the circle at D in such a way that $ED = HZ$. Ibn al-Haytham notes that, although the position of AED is not given, we have by an elementary theorem on the geometry of the circle $\angle ADG = \angle ABG$ and $\angle ADB = \angle AGB$, so $\angle ADG$ and $\angle ADB$ are known.

He then draws in an auxiliary figure the segment HZ and two lines HL and HM such that $\angle LHZ = \angle ABG$ and $\angle MHZ = \angle AGB$ (hence $\angle LHM = 90°$). Note that the precise positions of point L on line HL and point M on line HM are still to be determined.

Ibn al-Haytham now constructs in his auxiliary figure a straight line MZL, with point M on line HM and L on line HL, in such a way that $LM = BG$, where BG is a segment in the original figure. In this construction, which does not concern us here, he uses the intersection of another circle and a hyperbola. Thus he has now solved the *neusis* between two straight lines.

In his original figure, he now constructs point D on the circle such that $\angle GBD = \angle LMH$ and he draws DEA and DG. Then figure $LZMH$ in the auxiliary figure is congruent to figure $GEBD$ in the original figure, whence $ED = ZH$ as required. Thus he has now reduced the *neusis* between the arc and chord BG to the *neusis* between the straight lines MH and HL.

Ibn al-Haytham renders only one solution in his figure but he is well aware of the fact that there may be a second solution. I have indicated the second solution by E', D', M', L'.

The reasonings of Ibn al-Haytham and al-Jurjānī differ to the extent that al-Jurjānī wants to construct a second solution of the *neusis* when one solution is already given. This is why Ibn al-Haytham needs an auxiliary figure, in which his given segment HZ corresponds to the given line AD in al-Jurjānī's Figure 5. For the rest, the reasonings by al-Jurjānī and Ibn al-Haytham are so closely related that they can hardly be independent, and thus the question arises who influenced whom. We have seen that al-Jurjānī's diorismos for the *neusis* between the chord and arc of a circle is incorrect, but Ibn al-Haytham was able to find the correct diorismos by means of the circle and hyperbola in Lemma 2 in his construction of the *neusis* between two given straight lines. If al-Jurjānī had seen this construction, he would not have given a wrong *diorismos* for the *neusis* between the chord and arc of a circle. I conclude that al-Jurjānī influenced Ibn al-Haytham and not the other way around.

Ibn al-Haytham's solution of the "Problem of Alhazen" is one of the high points of mathematics in Islamic civilization, and it now turns out that al-Jurjānī was an important source of inspiration.

The fourth proposition in al-Jurjānī's *Geometrical Problems* concerns an arbitrary triangle ABG with its altitude AD. Al-Jurjānī proves an identity which can be used to determine BD in terms of the sides AB, AG, BG. The proof is elementary.

III The *Extraction of the Meridian Line*

The *Book of the Extraction of the Meridian Line From the Book Anālīmā, and the Proof For It by Abū Saʿīd al-Ḍarīr* was first published in 1922 in a German translation by Schoy [18]. I have divided the text into four sections, numbered (1), (2), (3) and (4). Sections (1)–(3) deal with the construction of the north-south line on a horizontal plane, by means of three unequal shadows cast by a vertical stick at three different moments during the same day. Section (4) presents a method for finding the cardinal directions by means of one shadow measurement at the right moment. In the northern hemisphere, the method can only be used during the spring and the summer. Section (4) of the text seems to be an addition by al-Jurjānī, whereas Sections (1) to (3) appear to be based on the *Book Anālīmā* mentioned in the title.[19]

Although Schoy could not identify this book, he pointed out that the method in Sections (1)–(3) is of ancient origin and that it was also used by the Roman surveyor Hyginus (ca. A.D. 100). In 1959, Kennedy published a construction of the meridian line by means of three shadows in Chapter 20 of the *Exhaustive Treatise on Shadows* by al-Bīrūnī [9]. Al-Bīrūnī says that the method was taken from the "Book Anālimā" of Diodorus of Alexandria, a Greek mathematician from the first century BCE. Heinrich Hermelink [6] then pointed out that the methods in [18] and [9] were the same, so the "book Anālīmā" in the title of al-Jurjānī's treatise is the *Analemma* of Diodorus, now lost. Thus al-Bīrūnī and al-Jurjānī preserved two versions of a construction by Diodorus. The two versions have been further explained and compared by Neugebauer [14, vol. 2, pp. 840–843] and Kennedy [10, vol. 2, pp. 91–93], and I see no need to repeat what has been said in these publications. I will only compare the structure of Sections (1)–(3) of al-Jurjānī's text, in his notations, with Chapter 20 of the *Exhaustive Treatise on Shadows* by al-Bīrūnī. The Arabic text of this chapter is available in [1, no. 2, pp. 116–119, 121], for an English translation see [10, vol. 1, pp. 162–166].

Al-Jurjānī considers a gnomon AB with tip B and foot A, which casts three unequal shadows $AG > AD > AE$ in the horizontal plane (see Figure 11 below). He defines points H and T on lines BD and BG such that $BH = BT = BE$, and he drops perpendiculars HL and TK onto the horizontal plane. He then defines S as the point of intersection of KL and GD. In (1) he proves that ES is parallel to the East-West line. He then shows in (2) that GD and KL are never parallel, so the intersection S always exists. In (3) he presents the practical construction of S from the positions of G, D and E and the lengths of BG, BD and BE.

Al-Bīrūnī presents these elements in a different order. He begins with the practical

[19] In [13], the treatise by al-Jurjānī is incorrectly listed as a commentary on the *Analemma* of Ptolemy.

construction (3) and he also explains how the lengths of BG, BD and BE are to be found from those of AB, AG, AD and AE. In the proof of the correctness of the construction, al-Bīrūnī presents the argument (2) in the same way as al-Jurjānī but before the definition of point S.

Al-Bīrūnī uses a different notation, but there are some striking linguistic similarities between his explanations and those of al-Jurjānī. So it is clear that the two authors were excerpting the same work.

The gist of (4) is as follows in modern notation. For solar declination $\delta > 0$, geographical latitude ϕ and a gnomon of length g, al-Jurjānī first constructs the altitude h of the sun in the prime vertical from $\sin h = \sin\delta \sin\phi$. He then draws a circle with centre the foot of the gnomon and radius $l = g \cot h$. If the shadow of the gnomon crosses this circle, the sun is due East or due West, so the cardinal directions can be found. In [5], Hermelink compared al-Jurjānī's method with a similar but more elegant construction in Chapter 21 of al-Bīrūnī's *Exhaustive Treatise on Shadows* [1, no. 2, p. 121], [10, vol. 1, pp. 168–169].

IV Manuscripts and Editorial Procedures

The two extant texts by al-Jurjānī have come down to us in a single Arabic manuscript, namely Cairo, Dār al-Kutub, Muṣṭafā Fāḍil Riyāḍa 41m. The *Geometrical Problems* are in ff. 69b–71a and the *Extraction of the Meridian Line* is in ff. 153b–154b, see [11, p. 36, no. B24] and [12, vol. 1, pp. 443, 445; vol. 2, pp. 833–834]. The codex Muṣṭafā Fāḍil Riyāḍa 41m was copied around 1153 H./1740 CE by Muṣṭafā Ṣidqī ibn Ṣāliḥ. In [13, vol. 2, pp. 76–77], Rosenfeld and Matvievskaya mention a Berlin manuscript of the two treatises by al-Jurjānī. This "Berlin manuscript" was a modern copy, now lost, of the Cairo manuscript.

For easy reference I have divided each of the two treatises into four sections. In my edition of the Arabic texts I have added the interpunction, and I have made some changes in orthography without noticing this in the apparatus. Thus I have changed مسئلة , يلقا and مهنا to مسألة, يلقى and هاهنا. In his translation of the second text [18], Schoy made a number of corrections to make mathematical sense. In my apparatus to the Arabic text I have indicated these corrections by the notation Sch. There are a few passages which Muṣṭafā Ṣidqī did not understand, but on the whole the quality of his manuscript is very good. I have redrawn the figures but have tried to stay close to the figures in the manuscript. My own additions to the text or translation are in angular brackets < > or in parentheses ( ).

V The *Geometrical Problems*: Edition

بسم الله الرحمن الرحيم

مسائل هندسية لابي سعيد الضرير الجرجاني رحمة الله تعالى عليه

< ا > ليكن دائرة ابدج ووتر بج معلوم ونريد ان نخرج خطا مثل خط اهد حتى يكون خط هد مثل خط مفروض وليكن الخط المفروض خط زح.

فنخرج خط بج في الجهتين جميعًا ونخرج من نقطة ا عمودًا على الخط وهو اك. فان كان ضرب اك في حز اعظم من ربع مربع بج فالمسألة غير مفتوحة وان كان مساويًا له او اصغر منه فهي مفتوحة. وذلك ان ضرب اه في هد ليس باعظم من ربع مربع بج لانّه مساو لضرب به في هج وضرب اه في هد اعظم من ضرب اك في هد المساوي لخط زح. وانّما اوردت هذا لانّك ذكرت عن ابي عبد الله الشني ايّده الله ان هذه الشريطة لا يحتاج اليها. فارجع الآن الى ايراد ما بقي في المسألة.

فاقول نقسم خط بج بنصفين على نقطة ط ونقيم على نقطة ط خطا يكون عمودًا على خط بج ونخرجه في الجهتين جميعًا وهو خط نطل. ونخرج من نقطة ا خطا موازيًا لخط بج وهو خط ال. ونفصل من خط لطن خط لن وليكن لن في زح مثل مربع خط بط[20] فظاهر ان نقطة ن تقع[21] تحت خط بج لان ضرب اك في زح اصغر من مربع بط[22] وطل مساو ل اك ف لن اعظم من لط.

ونخرج اك من جهة ا الى س ونجعل اس مثل اك ونخرج خط ساك من جهة ك غير محدود ونعمل قطعًا زائدًا يكون رأسه نقطة ك وسهمه المجانب سك وضلعه القائم ايضًا مثل سك. ونعمل قطعًا مكافيًا يكون رأسه نقطة ن وقطره خط لطن وضلعه القائم خط زح فيتقاطع هذان[23] القطعان على نقطة م. ونخرج من م خطا موازيًا لخط بج وهو خط معخ ومن نقطة م ايضًا خطا موازيًا لخط سك وهو مقه.

فعلى موجب الشكل الحادي والعشرين من المقالة الاولى من المخروطات يكون ضرب سخ في خك مثل مربع مخ ونجعل مربع كا مشتركًا فضرب سخ في خك مع مربع اك مثل مربع مخ ومربع اك معًا ومخ مثل هك فمربع اه مثل مربع اك وضرب سخ في خك ومربع اه مثل مربع اخ ف اخ مثل اه.

ولان قطع من قطع مكافي وقطره نل وضلعه القائم خط زح يكون ضرب زح في عن مثل مربع هط المساوي ل مع. وضرب زح في لن مثل مربع بط وضرب زح في نع بيّن انّها مثل مربع هط. فضرب زح في لع الباقي مثل به في هج لان خط بج مقسوم بنصفين على

[20] بط : نط MS. [21] تقع : يقع MS. [22] بط : نط MS. [23] هذان : هاتان MS.

ط وبمختلفين على ه فضرب به في هج مع مربع هط مساوٍ لمربع طج المساوي ا طب. لكن ضرب زح في نع مساو لمربع هط المساوي لمربع عم، وقد كان لن في حز مساويًا لمربع طج. فاذا القينا من ضرب زح في لن ضرب زح في عن ومن ضرب به في هج ومربع هط مربع هط يبقى ضرب لع في زح مساو لضرب به في هج. ولع مثل اه لانّه مثل اخ ايضًا فضرب اه في زح مثل ضرب به في هج لكن ضرب به في هج مثل ضرب اه في هد و هد مثل زح المفروض وذلك ما اردنا ان نبين.

< ب > خطا جا اب مفروضان يحيطان بزاوية وهي زاوية باج ونريد ان نخرج من نقطة ج خطا كخط جز يكون ضرب جز في اب مع مربع از مثل مربع اب.

فلنخرج اب الى د و نجعل اد مثل اب ونصل دج ونعمل على مثلث بدج دائرة وهي دائرة بهدج فدائرة بهدج معلومة الوضع ونقطة ج معلومة عليها وقطعة بهد مفروضة من هذه الدائرة. فنصل من نقطة ج خطا مستقيمًا الى قوس بهد يكون الخط المستقيم الذي يقع بين هذا القوس وبين وترها وهو بد مثل اب. فليخرج وليكن ذلك جزه. فاقول ان جز في اب مع مربع از مثل مربع اب.

برهان ذلك انّ جز في زه مثل ضرب بز في زد و نجعل مربع زا مشتركًا فضرب جز في زه مع مربع زا مثل ضرب بز في زد مع مربع زا اعني مثل مربع اب فضرب جز في زه مع مربع زا مثل مربع اب وزه مثل اب فضرب جز في اب مع مربع از مثل مربع اب وذلك ما اردنا بيانه.

< ج > مثلث ابج قائم الزاوية وزاوية ا منه قائمة وهو[24] معلوم القدر والصورة وقد اخرج فيه كخط اد ونسبة بج الى اد نسبة معلومة كيف نخرج من نقطة ج كخط جزه حتى يكون نسبة جه الى از مثل نسبة بج الى اد. [25]

وهذه المسألة تحتاج[26] الى تلك المقدمة التي ذكرناها لانّها تحتاج[27] ان نخرج من نقطة د خطا يكون ما يقع منه بين خطي اب اج مثل خط بج فبالمقدمة التي ذكرناها يمكن اخراج هذا وغيرها وفي مثلث قائم الزاوية وغير قائم الزاوية. وذلك ان الدائرة المرسومة على مثلث ابج دائرة معلومة.

فلنعمل على هذا المثلث دائرة وهي دائرة بطجا ولنخرج خط اد الى نقطة ط فنقطة ط معلومة على الدائرة وقطعة باج قطعة معلومة فلنخرج من نقطة ط خطا مستقيمًا الى قوس باج يكون ما يقع منه بين قوس باج وبين وتر بج مثل اد. فان خرج غير طد فالمسألة

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[24] وهو : وهي MS. [25] Here the scribe wrote in the margin: اد above ٦ and بج above ١٤. [26] تحتاج : يحتاج MS. [27] تحتاج : يحتاج MS.

مفتوحة وان لم يخرج فليس يمكن وذلك سهل لمن يتأمل[28] ويستعمل فيه المقدمة التي ذكرناها.

فلنخرج هذا الخط وهو طلك فليكن لك مثل اد فنصل بك جك فظاهر ان زاوية بكط مثل زاوية باد لانّهما[29] في قطعة واحدة. فنعمل على خط اد مثلثًا شبيهًا بمثلث بكل فظاهر ان احد اضلاعه من خط با وليكن مثلث امد. ولنخرج خط مد حتى يلقى خط اج على نقطة س فاقول انا قد اخرجنا على نقطة د خطا مستقيمًا يكون ما يقع منه بين خطي اب اج مثل بج وهو خط مس.

برهانه ان مثلث دام شبيه بمثلث بكل وخط كل مثل خط اد فخط مد مثل بل وكذلك نبين ايضًا ان دس مثل جل فـ مس مثل بج فقد اخرجنا من نقطة د خطا كخط مدس مساويًا لـ بج.

فلنخرج الآن من نقطة ج خطا موازيًا لـ مس وهو جزه. فاقول ان جزه هو المطلوب. برهانه انا نخرج من نقطة ز خطا موازيًا لخط بج وهو فزق فنسبة بج الى اد كنسبة فق الى از ونسبة مس الى اد كنسبة جه الى از لكن مس مثل بج فنسبة بج الى اد كنسبة جه الى از وذلك ما اردنا ان نبين.

< د > مثلث ابج اذا اخرج عموده وهو اد واخرج با على استقامته الى ه حتى صار اه مثل اج انّ زيادة ضعف سطح هب في با على ضعف سطح جب في بد مثل زيادة مربع به على مربع بج.

فان البرهان عليه ان ضعف سطح به في با هو مثل ضعف مربع با وضعف سطح با في اه وان ضعف سطح بج في بد هو ضعف مربع بد وضعف سطح بد في دج. لكن زيادة ضعف مربع اب على ضعف مربع بد هو ضعف مربع اد فاذن زيادة ضعف مربع اب وضعف سطح با في اه على ضعف مربع بد وضعف سطح بد في دج هو ضعف مربع اد وزيادة ضعف سطح با في اه على ضعف سطح بد في دج. ومربع به مثل مربعي با اه وضعف سطح با في اه ومربع بج مثل مربعي بد دج وضعف سطح بد في دج وزيادة مربع اب على مربع بد هو مثل مربع اد وزيادة مربع اه المساوي لـ اج على مربع جد هو مثل مربع اد فاذن زيادة مربع به على مربع بج هو مثل ضعف مربع اد وضعف زيادة سطح با في اه على سطح بد في دج وهو مثل الزيادة التي خرجت لنا قبل هذا وذلك ما اردنا ان نبين. تمت المقالة.

في يوم الاثنين التاسع والعشرين من جمادى الاولى لسنة ثلاث وخمسين ومائة والف.

[28] تأمل : يتأمل MS. [29] لانّها : لانّهما MS.

VI The *Extraction of the Meridian Line*: Edition

بسم الله الرحمن الرحيم

كتاب استخراج خط نصف النهار من كتاب اناليما والبرهان عليه لابي سعيد الضرير رحمة الله عليه يخرج مع تسهيل العمل

< آ > يرصد للمقياس ثلاثة اظلال متوالية وليكن احدها وهو اطولها اج واوسطها اد واقصرها اه والمقياس اب. ونصل بج بد به وندير على مركز ب وببعد به قوس هحط. ونخرج عمود طك على اج[30] وعمود حل على اد[31] ونصل كل وجد ونخرجهما حتى يلتقيا على نقطة س. ثم نصل سه فيكون موازيًا لخط الاعتدال والعمود النازل عليه من نقطة آ هو من خط نصف النهار.

برهان ذلك ان من البين ان الفصل المشترك بين سطح دائرة هحط وبين سطح الخط الذي يرسمه الظل في سطح الافق هو موازٍ لمعدل النهار وأن خط حط هو فصل مشترك لثلاثة سطوح احدها سطح دائرة هحط والثاني سطح حطل والثالث سطح حطجد. وخط كل هو في سطح حطكل وخط جد هو في سطح حطجد والتقاؤهما على نقطة س. فنقطة س على الفصل المشترك للثلاثة السطوح التي ذكرناها. فبيّن أن خط طحس خط مستقيم وهو الفصل المشترك للسطوح المذكورة.

وايضًا فنقطة س على الفصل المشترك لسطحي حطكل حطجد وهذا الفصل المشترك هو ايضًا فصل مشترك بين هذين السطحين المذكورين وبين سطح دائرة هحط. فنقطة س في سطح هذه الدائرة وفي سطح الافق فهو على الفصل المشترك بين سطح الافق وسطح هذه الدائرة ونقطة ه في سطح الافق وفي سطح دائرة هحط ايضًا ونقطة س قد بيّنّا انّها في سطحيهما ايضًا فخط سه هو الفصل المشترك بين سطح الافق وبين الدائرة التي ذكرنا وهو موازٍ لخط الاعتدال.

< ب > وأما البرهان على ان خطي جد لك يلتقيان في جهة نقطتي ل[32] د فلانّ خط بج اطول من خط بد وخط بط مثل بج يكون[33] طج اطول من دح فنسبة جط الى طب اعظم من نسبة دح الى حب واذا ركبنا كانت نسبة جب الى بط اعظم من نسبة دب الى بح لكن نسبة جب الى بط كنسبة جا[34] الى اك ونسبة دب الى بح كنسبة دا الى ال[35] فنسبة جا الى اك اعظم من نسبة دا الى ال[36]. فاذا اخرجنا من نقطة ك خطًا موازيًا لخط جد كخط كع وقعت نقطة ع بين نقطتي آ ل. فزاويتا عكج دجك[37] مساويتان[38] لزاويتين قائمتين وهما

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[30] اج Sch. : بج MS. [31] اد Sch. : بد MS. [32] ل Sch. : ك MS. [33] يكون : تكون MS. [34] جا Sch. : جك MS. [35] ال Sch. : اك MS. [36] ال Sch. : اك MS. [37] عكج دجك : علح حكع MS. [38] مساويتان : مساويتين MS.

اعظم من زاويتي دجك جكل[39] فزاويتا دجك جكل[40] اقلّ من قائمتين فخطا جد لك سيلقيان في الجهة التي قلنا، وذلك ما اردنا ان نبيّن.

< ج > وايضًا فانّا نعيد الاظلال الثلاثة كهيئتها وندير في سطح الافق على مركز ج وببعد جب قوس زم وكذلك ندير فيه على مركز د وببعد دب قوس زع فيتقاطعان على ز ونصل جز دز ونفصل من كل واحد من خطي زج زد مثل خط به وهما زط زح. ونصل طح جد ونخرجهما حتى يلتقيا على س. فاقول ان هذه النقطة هي نظيرة نقطة س في الصورة الاولى بعينها.

برهان ذلك ان خط جز مساو لـ جب في الاولى ودز في هذه مثل دب في تلك وكل واحد من خطي زح زط هاهنا مثل كل واحد من خطي بح بط في تلك. فأضلاع مثلث جزد هاهنا مساوية لأضلاع مثلث جدب هناك كل واحد لنظيره. فزاوية جزد مثل زاوية جبد[41] هناك وخطا طز زح مثل خطي طب بح كل واحد لنظيره فـ حط هاهنا مثل حط هناك وطج هاهنا مثل طج هناك وكذلك حد فيهما متساويان، وزاويتا زجد زدج هاهنا مثل زاويتي بجد بدج هناك فـ حط هاهنا يلقى جد على س ويكون خط جس هاهنا مثل خط جس هناك لان جط فيهما متساويان وزاوية زجد هاهنا مثل زاوية بجد هناك وزاوية جطح هاهنا مثل زاوية[42] جطح هناك فخط طح يلقى جد على نقطة بعدها من ج كبعد نظيرتها من ج في الصورة الاولى وتلك نقطة س وذلك ما اردنا بيانه.

< د > ليكن لاستخراج خط نصف النهار دائرة ابجد على لوح مستوي مربعة بقطري اهج بهد ودورها مقسوم بثلاثمائة وستين جزءًا. ونأخذ قوس از بقدر عرض البلد ونصل هز ونخرج عمود اح على هز. ونفرض اط مثل ميل الشمس في ذلك اليوم ونخرج عمود طك على اه ونفصل من اح مثل كط وهو ال ونخرج من ل عمود لم على اح يلقى اه على نقطة م ونعمل على خط اه نصف دائرة وهو انه. ونخرج ان مثل ام ونصل نه ثم نفصل من خط ان خط اس بقدر مقياس الظل ونخرج عمود سع على ان يلقى خط اه على ع. ثم نخط في سطح الافق دائرة في الموضع الذي نريده بمقدار بعد سع ونركز في مركزها مقياسًا عمودًا على سطح الافق بمقدار اس ثم نرصد ظله قبل نصف النهار او بعده حتى يقع على حرف تلك الدائرة ونصل بين موقعه ومركز المقياس بخط مستقيم فيكون خط الاعتدال. ونخرج من المركز عمودًا على هذا الخط فيكون خط نصف النهار وذلك ما اردنا عمله والله تعالى اعلم. تم.

في يوم السبت الحادي والعشرين من شهر ربيع الآخر لسنة ثلاث وخمسين ومائة والف.

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[39] دجك جكل : حلك حكل MS. [40] دجك جكل : حلك حكل MS. [41] جبد Sch. : حبد MS. [42] The MS. repeats بجد هناك ... مثل زاوية .

VII The *Geometrical Problems*: Translation

In the name of God, the merciful, the compassionate.

Geometrical Problems by Abū Saʿīd al-Ḍarīr al-Jurjānī, may God Most High have mercy upon him.

< 1 > (Figure 7) Let there be a circle $ABDG$. Chord BG is known.[43] We want to draw a line such as line AED in such a way that line ED is equal to an assumed line, and let the assumed line be line ZH.

We extend line BG on both sides, and we drop from point A perpendicular AK to the line. If the product AK times HZ is greater than one quarter of the square of BG, the problem is impossible, but if it is equal to or less than one quarter of it, it is possible. That is, the product AE times ED is not greater than a quarter of the square of BG since it is equal to the product BE times EG, and the product AE times ED is greater than (or equal to) the product AK times ED, which is equal to line ZH.[44] I mention this because you said on the authority of Abū ʿAbdallāh al-Shannī, may God support him, that this condition is not necessary. Now I return to the presentation of the rest of the problem.

I say: We bisect line BG at point T. We erect at point T a line perpendicular to line BG and we extend it on both sides, and it is line NTL. We draw from point A line AL parallel to line BG. We cut off from line LTN line LN and let LN times ZH be equal to the square of BT.[45] Then it is clear that point N is located under line BG since the product AK times ZH is less than the square of BT and TL is equal to AK, so LN is greater than LT.

We extend AK on the side of A towards S and we make AS equal to AK. We extend line SAK indefinitely on the side of K. We construct a hyperbola with vertex point K, transverse axis SK, and latus rectum also equal to SK.[46] We construct a parabola with vertex point N, diameter line LTN and latus rectum line ZH.[47] Then these two (conic) sections intersect at point M.[48] We draw from M line MOX parallel to line BG and also from point M line MQE parallel to line SK.

Then, as a consequence of the twenty-first proposition of the first Book of the *Conics*,[49] the product SX times XK is equal to the square of MX. We add the square of KA, then the product SX times XK plus the square of AK is equal to the

[43] Al-Jurjānī also assumes that point A is known.

[44] This argument shows that the condition $AK \cdot ZH \leq \frac{1}{4}BG^2$ is a necessary condition for the existence of a solution. The condition is not sufficient, see the commentary above.

[45] For the construction of point N see Euclid, *Elements* I:44–45 [4, vol. 1, pp. 341–347].

[46] See Apollonius, *Conics* I:54 [22, p. 101].

[47] See Apollonius, *Conics* I:52 [22, p. 97]. Line LTN is the axis of the parabola.

[48] This is not necessarily true, see the commentary above.

[49] For *Conics* I:21 see [22, p. 43].

square of MX plus the square of AK. But MX is equal to EK. Thus the square of AE is equal to the square of AK and (i.e. plus) the product SX times XK, and (therefore)[50] the square of AE is equal to the square of AX. So AX is equal to AE.

But since (conic) section MN is a parabola with diameter NL and latus rectum line ZH, the product ZH times ON is equal to the square of ET,[51] which is equal to MO. The product ZH times LN is equal to the square of BT, but the product ZH times NO was shown to be equal to the square of ET. Therefore the product ZH times LO, which is the remainder, is equal to BE times EG,[52] since line BG is divided into halves at T and into unequal (parts) at E, so the product BE times EG together with the square of ET is equal to the square of TG,[53] which is equal to TB. But the product ZH times NO is equal to the square of ET, which is equal to the square of OM, and LN times HZ was equal to the square of TG. So if we subtract from the product ZH times LN the product ZH times ON, and from the product BE times EG plus the square of ET the square of ET, the remainder will be the product LO times ZH, equal to the product BE times EG.

But LO is equal to AE since it is also equal to AX, so the product AE times ZH is equal to the product BE times EG. But the product BE times EG is equal to the product AE times ED,[54] so ED is equal to the assumed (line) ZH. That is what we wanted to prove.

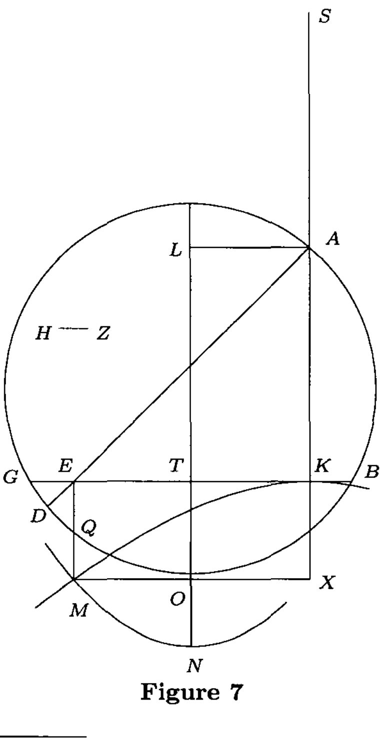

Figure 7

[50] $AK^2 + SX \cdot XK = AX^2$ by Euclid, *Elements* II:5, see [4, vol. 1, p. 382].

[51] Apollonius, *Conics* I:11 [22, p. 21]

[52] The following sentences of this paragraph explain or repeat what has already been said.

[53] Euclid, *Elements* II:5 [4, vol. 1, p. 382].

[54] Euclid, *Elements* III:35 [4, vol. 2, p. 71].

< 2 > (Figure 8) Lines GA, AB are assumed and contain an angle BAG. We want to draw from point G a line, say GZ, such that the product GZ times AB plus the square of AZ is equal to the square of AB.[55]

Let us extend AB towards D and we make AD equal to AB. We join DG and we construct about triangle BDG circle $BEDG$. Then circle $BEDG$ is known in position, point G on it is known, and segment BED of this circle is assumed. Then we join from point G a straight line towards arc BED such that the straight line which falls between this arc and its chord BD is equal to AB. Thus let it be drawn, and let that (line) be GZE. I say that GZ times AB plus the square of AZ is equal to the square of AB.

Proof of this: GZ times ZE is equal to the product BZ times ZD.[56] We add the square of ZA, then the product GZ times ZE plus the square of ZA is equal to the product BZ times ZD plus the square of ZA, that is to say, equal to the square of AB. Thus the product GZ times ZE plus the square of ZA is equal to the square of AB. But ZE is equal to AB. Thus the product GZ times AB plus the square of AZ is equal to the square of AB, and that is what we wanted to prove.

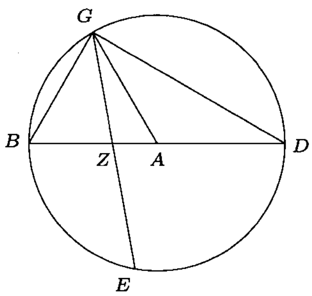

Figure 8

< 3 > (Figure 9) Triangle ABG is right-angled and its angle A is the right angle, and it[57] is known in magnitude and in shape. A line AD has been drawn in it and the ratio of BG to AD is a known ratio. How do we draw from point G a line GZE such that the ratio of GE to AZ is equal to the ratio of BG to AD?[58]

This problem requires that lemma which we have mentioned, because it requires that we draw from point D a line such that the part of it which falls between lines AB and AG is equal to line BG. By means of the lemma which we have mentioned, this and other things can be drawn, in a right-angled triangle as well as in a triangle

[55] Al-Jurjānī assumes $GA = AB$.

[56] Euclid, *Elements* III:35 [4, vol. 2, p. 71].

[57] According to the manuscript, the angle is known in magnitude and in shape. I have emended the Arabic *hiya* to *huwa*. The emended text means that the triangle is known in magnitude and in shape.

[58] The scribe wrote $\frac{BG}{14}$ and $\frac{AD}{6}$ in the margin of the manuscript. In his figure, $BG : AD$ is approximately $14 : 6$.

which is not right-angled. The reason is that the circumscribed circle of triangle ABG is a known circle.

Thus let us circumscribe about this triangle circle $BTGA$. Let us extend line AD towards point T, then point T on the circle is known, and segment BAG is a known segment. So let us draw from point T a straight line towards arc BAG in such a way that the part of it which falls between arc BAG and chord BG is equal to AD. If such a line can be drawn other than TD, the problem is solvable, but if it cannot be drawn, it (the problem) is impossible, and this is easy (to verify) for someone who thinks (about it) and who uses in it the lemma which we have mentioned.[59]

Thus let us draw this line TLK, and let LK be equal to AD. We join BK, GK, then it is clear that angle BKT is equal to angle BAD because they stand on the same segment.[60] Thus we construct on line AD a triangle similar to triangle BKL; it is clear that one of its sides is part of line BA. Let it be triangle AMD. Let us extend line MD to meet line AG at point S. Then I say that we have drawn through point D a straight line MS such that the part of it which falls between lines AB, AG is equal to line BG.

Proof: Triangle DAM is similar to triangle BKL and line KL is equal to line AD, so line MD is equal to BL. In the same way we also prove that DS is equal to GL. Thus MS is equal to BG, and we have drawn from point D a line MDS equal to BG.

So let us now draw from point G line GZE parallel to MS. I say that GZE is what was desired. Proof: We draw from point Z line FZQ parallel to line BG. Then the ratio of BG to AD is equal to the ratio of FQ to AZ, and the ratio of MS to AD is equal to the ratio of GE to AZ, but MS is equal to BG, so the ratio of BG to AD is equal to the ratio of GE to AZ, and that is what we wanted to prove.

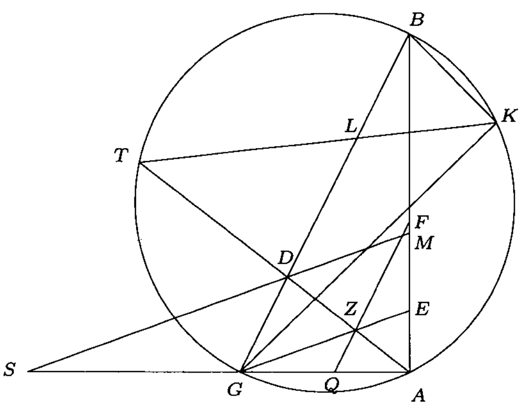

Figure 9

[59]Following a suggestion of Professor O. Naises, I have emended *ta'ammala* in the manuscript to *yata'ammalu*. Without emendation, the text means: "... who thought (about it). In this (construction) the lemma which we have mentioned is being used."

[60]Euclid, *Elements* III:27 [4, vol. 2, p. 58].

< 4 > (Figure 10) If in triangle ABG its altitude AD is drawn and if BA is extended in a straight line towards E such that AE is equal to AG: the excess of twice the rectangle EB times BA over twice the rectangle GB times BD is equal to the excess of the square of BE over the square of BG.

Proof of this: Twice the rectangle BE times BA is equal to twice the square of BA and twice the rectangle BA times AE. Twice the rectangle BG times BD is twice the square of BD and twice the rectangle BD times DG.[61] But the excess of twice the square of AB over twice the square of BD is twice the square of AD. Thus the excess of twice the square of AB plus twice the rectangle BA times AE over twice the square of BD plus twice the rectangle BD times DG is twice the square of AD plus the excess of twice the rectangle BA times AE over twice the rectangle BD times DG. But the square of BE is equal to the (sum of the) squares BA, AE plus twice the rectangle BA times AE, and the square of BG is equal to the (sum of the) squares BD, DG plus twice the rectangle BD times DG. The excess of the square of AB over the square of BD is equal to the square of AD, and the excess of the square of AE, which is equal to AG, over the square of GD is equal to the square of AD. Thus the excess of the square of BE over the square of BG is equal to twice the square of AD and twice the excess of the rectangle BA times AE over the rectangle BD times DG, and this is equal to the excess which we have obtained before. That is what we wanted to prove. End of the treatise.

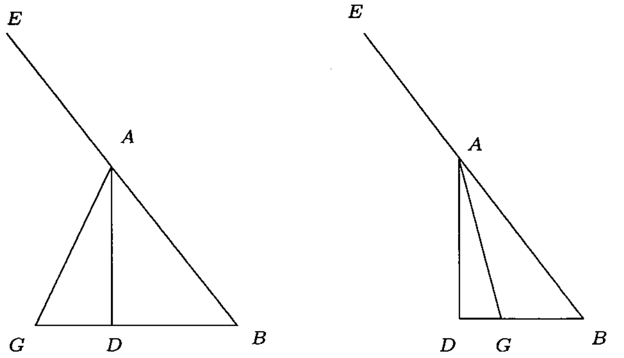

Figure 10

On Monday the 29th of Jumādā I of the year 1153.[62]

[61] The manuscript has two figures, one for the case where D is between B and G (as in the text), and one for the case where G is between B and D. In that case $2BG \cdot BD = 2BD^2 - 2BD \cdot DG$ and $BG^2 = BD^2 + DG^2 - 2BD \cdot DG$, and also $BE^2 - BG^2 = 2BE \cdot BA - 2GB \cdot BD$. I leave the details to the reader.

[62] The date of copying the manuscript corresponds to August 22, 1740 CE.

VIII The *Extraction of the Meridian Line*: Translation

In the name of God, the merciful, the compassionate.

The book of the extraction of the meridian line from the book Anālīmā, and the proof for it by Abū Saʿīd al-Ḍarīr, may God have mercy upon him. It comes out together with a simplification of the procedure.[63]

< 1 > (Figure 11) Let three successive shadows of the gnomon be observed, let the longest of these be AG, the intermediate one AD, and the shortest one AE, and let the gnomon be AB. We join BG, BD, BE and we describe with centre B and radius BE arc EHT.[64] We drop perpendicular TK onto AG and perpendicular HL onto AD.[65] We join KL and GD and we extend them to meet at point S.[66] Then we join SE, it is parallel to the East-West line, and the perpendicular drawn to it from point A is part of the meridian line.

Proof of this: It is clear that the intersection of the plane of circle EHT and the plane of the line which is described by the shadow in the horizon plane is parallel to the equator,[67] and (it is clear) that line HT is the intersection of three planes: the first is the plane of circle EHT, the second the plane HTL and the third the

[63]The simplified procedure is part 4 of the text.

[64]Points H and T are supposed to be on BD and BG, so the circle is the base of a cone with vertex B. The cone is described by the rays through the centre of the sun and the tip of the gnomon in the course of one day. The plane of circle EHT is parallel to the celestial equator.

[65]Here I have followed Schoy's emendation. Muṣṭafā Ṣidqī wrote: "We drop perpendicular TK onto BG and perpendicular HL onto BD." In his figure line BG is perpendicular to TK, line BD is perpendicular to HL, and points K, L are not on lines AG, AD respectively. Schoy's emendation is consistent with the proof in (2) that KL intersects GD. Muṣṭafā Ṣidqī made so many scribal errors in this proof (compare the critical apparatus) that Schoy was unable to make sense of the passage, see [18, p. 266, fn. 1].

[66]Note that line KL is only used for the definition of point S and in the proof that point S exists. Line KL does not play a role in the practical construction of S in section 3 of the text.

[67]The text is confused. It would be correct to say that the plane of circle EHT is parallel to the plane of the celestial equator, so its intersection with the horizontal plane is parallel to the East-West line (Arabic: *khaṭṭ al-iʿtidāl*). Compare the corresponding confused passage in al-Bīrūnī's summary of Diodorus' *Analemma* [1, no. 2, pp. 118:13–119:3]:

ومعلوم ان الفصول المشتركة بين سطح كل دائرة قائمة على سهم مخروط الظل وبين سطح القطع الذي يحدثه رأس الظل في الافق تكون موازية لسطح معدل النهار لان الدائرة موازية له وسهم القطع هو خط نصف النهار

"It is known that the intersections of the plane of every circle perpendicular to the axis of the shadow cone with the plane of the (conic) section which the tip of the shadow produces in the horizon are parallel to the plane of the equator since the circle is parallel to it, and (that ?) the axis plane $HTGD$. Line KL is in plane $HTKL$, but line GD is in plane $HTGD$, and the meeting (point) of the two (lines) is at point S. Thus point S is at the intersection of the three planes we mentioned. So it is clear that line THS is a straight line, since it is the intersection of the above-mentioned planes.

Again, point S is at the intersection of the two planes $HTKL$, $HTGD$, and this intersection is also the intersection of the two above-mentioned planes and the plane of circle EHT.[68] Thus point S is in the plane of this circle and in the plane of the horizon, so it is at the intersection of the plane of the horizon and the plane of this circle. But point E is also in the plane of the horizon and in the plane of circle EHT, and we have proved that point S is also in these two planes. So line SE is the intersection of the plane of the horizon and the (plane of the) circle we have mentioned, and it is parallel to the East-West line.

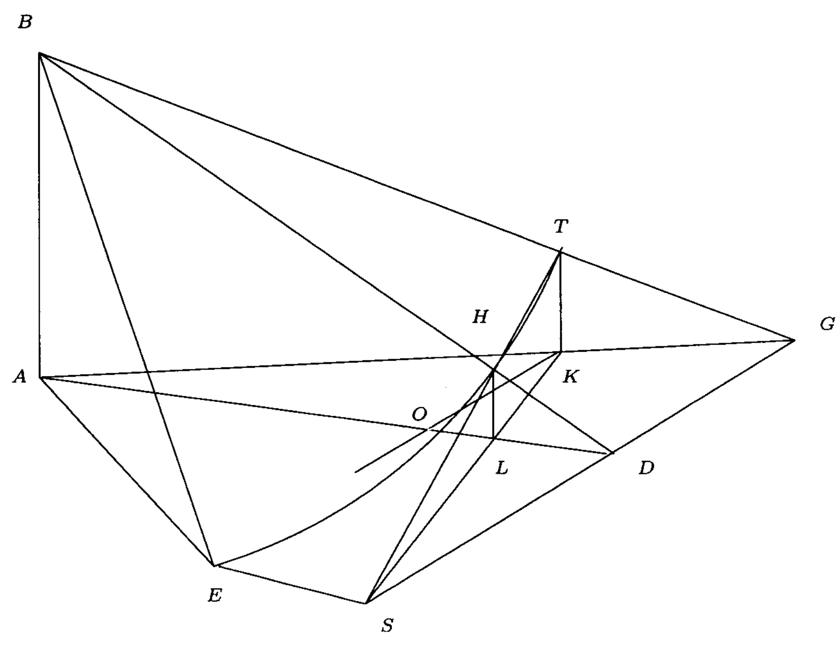

Figure 11

< 2 > Proof that lines GD, LK meet on the side of points L, D: Since line BG is longer than line BD and line BT is equal to BH, TG is longer than DH. So

of the (conic) section is the meridian line" (cf. the translation in [10, vol. 1, p. 164]). It seems to me that al-Jurjānī and al-Bīrūnī tried to make sense of a complicated and perhaps confused passage in the *Analemma* of Diodorus.

[68] Al-Bīrūnī's proof ([10, p. 166], [1, no. 2, pp. 117–119]) corresponds to this proof by al-Jurjānī *minus* the sentence "Again ... circle EHT." This sentence is mathematically redundant and probably did not belong to Diodorus' *Analemma*.

the ratio of GT to TB is greater than the ratio of DH to HB. Componendo,[69] the ratio of GB to BT is greater than the ratio of DB to BH. But the ratio of GB to BT is as the ratio of GA to AK, and the ratio of DB to BH is as the ratio of DA to AL. Thus the ratio of GA to AK is greater than the ratio of DA to AL. So if we draw from point K line KO parallel to line GD, point O falls between points A and L. Thus angles OKG, DGK are (together) equal to two right angles, but they are (together) greater than (the sum of) angles DGK, GKL, so (the sum of) angles DGK, GKL is less than two right angles. Therefore lines GD and LK will meet[70] in the direction we have indicated, and that is what we wanted to prove.

$< 3 >$ (Figure 12) Again, we redraw the three shadows in the same way, and we describe in the plane of the horizon with centre G and radius GB arc ZM, and in the same way, with centre D and radius DB arc ZO.[71] Then they meet at Z. We join GZ, DZ and we cut off from lines ZG, ZD (two segments) ZT, ZH equal to line BE. We join TH, GD and we extend them to meet at S. I say that this point corresponds exactly to point S in the first figure.

Proof of this:[72] Line GZ is equal to GB in the first (figure), and DZ in this (figure) is equal to DB in that (figure), and each of the lines ZH, ZT here is equal to each of the lines BH, BT in that (figure). Thus the sides of triangle GZD here are equal to the corresponding sides of triangle GDB there.[73] So angle GZD is equal to angle GBD there, lines TZ, ZH are equal to lines TB, BH respectively, so HT here is equal to HT there. But TG here is equal to TG there, and similarly HD is the same in both (figures), and angles ZGD, ZDG here are equal to angles BGD, BDG there. Therefore HT here meets GD at S and line GS here is equal to line GS there, since GT in both (figures) is equal, and angle ZGD here is equal to angle BGD there, and angle GTH here is equal to angle GTH there, so line TH meets GD at a point whose distance to G is equal to the distance to G of the corresponding point in the first figure, and that is point S. That is what we wanted to prove.

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[69] See Euclid, *Elements* V:18 [4, vol. 2, p. 169].

[70] Lines GD and KL meet by Euclid's parallel postulate, see [4, vol. 1, pp. 155, 202–220].

[71] Presumably, the shadows AG, AD, AE are measured in the horizontal plane, and BE, BD, BG are then constructed from the length of the gnomon AB by the theorem of Pythagoras. For sake of clarity, I have drawn as dashed lines all lines in the figure which are not mentioned in the text.

[72] The proof could have been shortened. It follows from the construction of the figures that triangles BGD and ZGD are congruent, and also triangles BTH and ZTH. Thus GT and angles TGD and TGH are the same in both figures, so the triangles GTS in both figures are congruent. Perhaps the proof was revised by a scribe.

[73] "Here" and "there" are used in a similar way by al-Bīrūnī in his paraphrase of Diodorus' construction, cf. [1, no. 1, pp. 119:9–12].

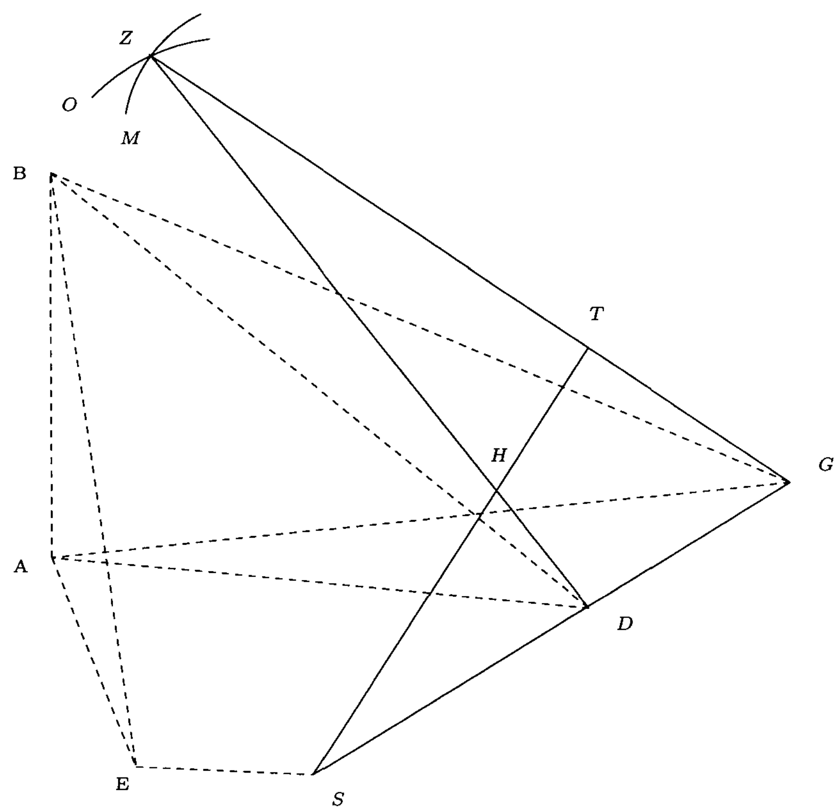

Figure 12

< 4 > (Figure 13) For the extraction of the meridian line, let circle $ABGD$ be on a plane tablet, (the circle being) divided into quadrants by means of diameters AEG, BED, and its circumference (being) divided into three hundred and sixty degrees. We take arc AZ in the amount of the latitude of the locality, we join EZ, and we drop perpendicular AH onto EZ. We assume AT equal to the declination of the sun on that day, and we drop perpendicular TK onto AE. We cut off from AH (segment) AL equal to KT, and we draw through L perpendicular LM towards AH, to meet AE at point M. We construct on line AE semicircle ANE. We draw AN equal to AM and we join NE. Then we cut off from line AN line AS with the magnitude of the gnomon, and we draw perpendicular SO to meet line AE at O. Then we describe in the plane of the horizon a circle at the place we want, with magnitude (defined by) the radius SO. We place at its centre as a gnomon a perpendicular to the plane of the horizon, with magnitude AS. Then we observe its shadow before noon or after noon, until it falls on the circumference of this circle. We join its location with the foot of the gnomon by a straight line: it is the East-West line. We draw from the centre a perpendicular to that line; then it is the

meridian, and that is what we wanted to construct. God Most High knows best. The end.

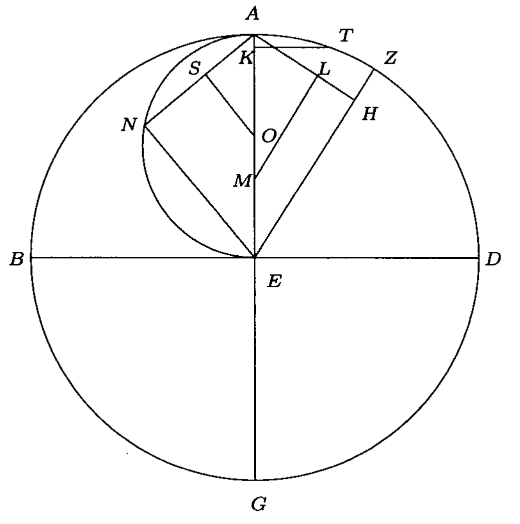

Figure 13

On Saturday the 21st of the month Rabīʿ II of the year 1153.[74]

[74]The date of copying the manuscript corresponds to July 16, 1740 CE.

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