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The Sector Theorem Attributed to Menelaus

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The Sector Theorem is now generally known as the Menelaus Theorem. At first glance, it appears to be one of the few pieces of mathematics that we find both in Ptolemy and his sources. Unfortunately, the textual transmission of the theorem turns out to be quite involved and it has now become clear that we do not possess any version of it that can simply be taken as that which Menelaus wrote. Nevertheless, it is possible, by examining the textual dependencies of the theorem as it was transmitted by the Arabic mathematicians, to discern basic types of the theorem and to decide which one of these should be attributed to Menelaus.

This exercise allows us to discuss the relationship between Menelaus' and Ptolemy's treatment of the theorem, and to situate their differing approaches to this particular proposition within the broader context of their mathematical and scientific aims. The tendency of Greek mathematical authors to differentiate their texts by subject areas allows us to see that the theorem was intended for quite different purposes by Menelaus and Ptolemy.

The opinion put forward by Neugebauer [1975, 301] favors Menelaus as the author of the eponymous theorem.[1] Although some historians are not convinced by this position, no one since has cogently argued against it. An examination of Menelaus' version of the sector theorem and its function in his *Spherics*, however, shows that it is unlikely that he intended the sector theorem to be read as his original contri-

[1]Before Neugebauer published *A History of Ancient Mathematical Astronomy*, it was common to hear doubts expressed as to Menelaus' authorship of the theorem. For example, Bulmer-Thomas [1974, 299] argues that it was known before Menelaus in the *Dictionary of Scientific Biography*. After 1975, however, it became more common for scholars to simply accept that Menelaus had written the theorem. Toomer [1984, 69, n. 84], for example, states this as a fact in his translation of the *Almagest*. Another example of the weight of Neugebauer's authority is shown in two studies carried out by Nadal and Brunet. Starting from the hypothesis that Hipparchus used a star globe to solve the problems encountered in writing his *Commentary on the Phenomena of Aratus*, they use statistical analysis to describe certain physical characteristics of the hypothetical globe [Nadal and Brunet 1984, 1989]. They do not, however, take into consideration the fact that there are numerous possibilities for Hipparchus' actual practice: various methods of calculation, multiple star globes, other modes of analog calculation and so forth.

bution. An examination of the astronomical evidence in the writings of Hipparchus and others gives support to the conclusion that the sector theorem was known to, and used by, Hipparchus.[2] Ptolemy's presentation of his spherical astronomy, for which the sector theorem is fundamental, gives evidence for an older tradition that relied exclusively on the sector theorem to solve just those problems that we know Hipparchus also solved using precise methods of computation.

We will see that the most likely story is that Menelaus found the sector theorem as a well known tool of predictive spherical astronomy and applied it to his own needs in the production of a new theory of advanced spherical trigonometry. These findings support other, independent reasons for believing that the trigonometric methods based on the sector theorem were available to Hipparchus [Sidoli 2004]. Once we understand Hipparchus' role in the history of the sector theorem, it becomes easier to understand Ptolemy's approach to spherical astronomy in books I, II and VIII of his *Almagest* [Toomer 1984, 64–130, 410–417].

I The sector theorem

The sector theorem was the fundamental theorem of ancient spherical trigonometry. It asserts a compound proportion that holds for combinations of the chords of six arcs of great circles forming a concave quadrilateral on the surface of a sphere. The most important form of the theorem was known to the ancient and medieval mathematical astronomers as *Disjunction*.[3] It is always the first, and often the only, form of the theorem that is fully demonstrated on the basis of the underlying chords.

Consider Figure 1. Where the arcs of the figure are less than semicircles and $Crd(\alpha)$ is the chord subtending arc α, *Disjunction* asserts that

$$\frac{Crd(2\,\overset{\frown}{GE})}{Crd(2\,\overset{\frown}{EA})} = \frac{Crd(2\,\overset{\frown}{GZ})}{Crd(2\,\overset{\frown}{ZD})} \times \frac{Crd(2\,\overset{\frown}{BD})}{Crd(2\,\overset{\frown}{BA})}.$$

By disjunction, we mean that the two components of one of the outer arcs of the sector figure, $\overset{\frown}{GE}$ and $\overset{\frown}{EA}$, are taken separately; that is disjointly. In *Disjunction*, a

[2] Björnbo [1902, 72 ff.] in his seminal study of Menelaus' *Spherics*, had already come to this opinion, in which he was followed by Heath [1921, vol. 2, 270]. Rome [1933, 42] agreed that this was a real possibility. Schmidt [1943, 66–68] also had no problem considering the sector theorem as one of the first tools of quantitative spherical astronomy.

[3] Neugebauer [1975, 28] called this combination of the arcs *Menelaus Theorem II* (M.T.II). My terminology follows Lorch [2001, 2–4]. *Sector theorem* itself is a common Arabic name for the theorem (الشكل القطاع). *Conjunctus* and *disjunctus* are the Latin technical terms for ratio types that Greek authors generally denoted with the terms σύνθεσις and διαίρεσις. This way of naming the two canonical forms of the theorem is found already in Theon's commentaries to the *Almagest* [Rome 1931–1943, 558, 562].

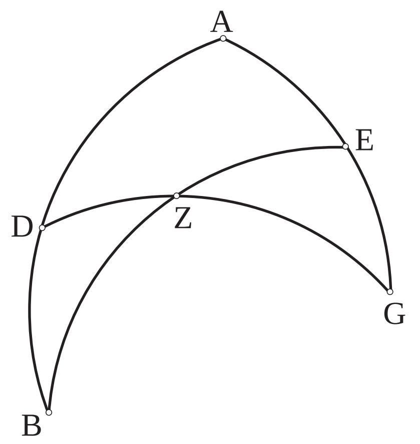

Figure 1: The Sector Theorem

ratio of the two parts of one of the outer arcs is compared to a ratio of the same two parts of the facing inner arc and corrected by a ratio of components of the other outer arc. Hence, it involves two outer arcs and one inner arc; mathematical astronomers would use *Disjunction* where they were given more outer arcs than inner arcs.

A second combination of the arcs, called *Conjunction*, was asserted by Ptolemy and became canonical in the traditions that came under the influence of the *Almagest*.[4] Where the arcs of the figure are less than semicircles, *Conjunction* asserts that

$$\frac{Crd(2\,\overset{\frown}{GA})}{Crd(2\,\overset{\frown}{EA})} = \frac{Crd(2\,\overset{\frown}{GD})}{Crd(2\,\overset{\frown}{DZ})} \times \frac{Crd(2\,\overset{\frown}{ZB})}{Crd(2\,\overset{\frown}{BE})}.$$

By conjunction, we mean that the first term of the proportion takes the two components of one of the outer arcs together; that is conjointly, as $\overset{\frown}{GA}$. In *Conjunction*, the ratio of the whole of an outer arc to one of its parts is compared to a ratio of the same parts of the facing inner arc and corrected by a ratio of components of the other inner arc. Hence, it involves two inner arcs and an outer arc; mathematical astronomers would use *Conjunction* where they were given a preponderance of inner arcs.

For either combination, it is immaterial which outer arc we begin with, so long as the relative position of the arcs is maintained. We will refer to these two basic divisions of the theorem as *combinations*.

Each of the two combinations occurs in a number of different geometric cases. For *Disjunction*, there are three cases. Considering Figure 1, chord AD either meets a certain radius of the sphere (1) in the direction of B, (2) in the direction of G,

[4]Neugebauer [1975, 28] calls this combination of the arcs *Menelaus Theorem I* (M.T.I).

or (3) the two lines are parallel. For *Conjunction*, an unnamed colleague of Thābit Ibn Qurra showed that there are sixteen valid cases [Lorch 2001, 47–49, 165]. For our purposes, the only relevant cases are the three belonging to *Disjunction.* This was largely so during the ancient and medieval periods as well, since, from the time of Theon, *Conjunction* was not demonstrated case by case, but on the basis of the proof of *Disjunction.* We will refer to the divisions of the theorem based on the underlying geometry of the sphere as *cases.*

The operations of ancient ratio theory allow the mathematician to reorder the terms of one of the two basic combinations to form other compound ratios obtaining between almost all other pairs in the original proportion. Thābit demonstrated that, given any compound proportion in six terms, there are eighteen such permutations, from which another eighteen can be trivially obtained by inversion [Lorch 2001, 74–95]. Furthermore, he proved that these eighteen permutations exhaust the valid possibilities [Lorch 2001, 96–111]. We will call such modifications to a basic form *permutations.*

These terminological niceties, although based on real mathematical distinctions, are not exhibited or maintained in the linguistic practices of the ancient and medieval authors. Their vocabulary for handling the mathematical classifications of the sector theorem are neither systematic nor consistent.

We will use the term *version* to designate a particular textual tradition of the sector theorem. There is some unfortunate, but unavoidable, ambiguity here because a version may be something as precise as a particular edition or as broad as a textual tradition of which there are various different exemplars.

II A preliminary history

Because of its role in spherical astronomy, and the intense interest in this field on the part of the Arabic mathematical astronomers, the transmission history of the sector theorem is quite involved. Fortunately, it is not necessary to give an exhaustive account of the issues here. An overview of the known stages of the early medieval transmission, and an examination of the most important texts, will allow us to form an idea of the different versions of the theorem. From this we may determine which versions of the theorem are closest to that found in Menelaus' *Spherics.* The material in this section is largely based on the research of Krause [1936] and Lorch [2001].[5]

In surveying all the material related to the sector theorem in the ancient and medieval periods we find three basic traditions: (1) astronomical, (2) geometrical, and (3) didactic. The astronomical tradition is exemplified in Ptolemy's *Almagest* but probably had its origins earlier, in the Hellenistic period. The didactic tradition

[5]Other useful overviews of the transmission are given by Hogendijk [1996, 18–21] and Taha and Pinel [1997, 151–155].

is first represented by the *Commentary* of Theon of Alexandria and came to its apex in the work of Thābit. The geometric tradition was begun by Menelaus' *Spherics* and brought to fruition by the Arabic mathematicians. The principal difficulty in studying this material is that practitioners worked in more than one tradition, so that the three did not remain distinct.

The Greek text of Menelaus' *Spherics*, composed around 100 CE, has been lost, with the exception of a few fragments of the first book preserved by Theon in his commentary to the *Almagest* [Björnbo 1902, 22–25]. Nevertheless, the text is found in a number of Arabic versions and Latin and Hebrew translations made from one of these, where the sector theorem generally appears as *Spher.* III 1.[6] It is used as a fundamental theorem for the development of a new theory of spherical trigonometry based on the properties of the spherical triangle. The sector theorem itself, however, is conspicuous in being the only theorem in the text that neither concerns nor relies on spherical triangles [Björnbo 1902; Nadal, Taha and Pinel 2004].

The earliest Greek version of the sector theorem that we possess is that of *Almagest* I 13, written in middle of the 2nd century [Toomer 1984, 64–69]. Ptolemy's treatment of the theorem is straightforward and concise. The theorem itself is preceded by a series of lemmas, of which there are three types: (1) the plane sector theorems, (2) application lemmas, and (3) computation lemmas. The plane sector theorems demonstrate *Conjunction* and *Disjunction* for the plane configuration of the subtended chords, which also form a concave quadrilateral (*Alm.* I 13.1 & 13.2). These are followed by two lemmas showing that ratios which hold for certain line segments in a circle can be applied to the chords of double arcs that are related to those lines (*Alm.* I 13.3 & 13.4). There are also two brief corollaries to these, which would be useful in applying the sector theorem to calculations (*Alm.* I 13.3c & 13.4c).[7] Ptolemy demonstrates the spherical configuration of *Disjunction* for the first geometric case and simply asserts that *Conjunction* can be shown on the basis of the plane sector theorem (*Alm.* I 13.5 & 13.6).

In the late 4th century, Theon, by way of commentary to Ptolemy's work, gives a treatment of the sector theorem that is more detailed than Ptolemy's. His treatment is sufficient for a reader of the *Almagest* but is not mathematically complete. He gives full enunciations for all the theorems; the effect of this is that, in the case of the sector theorem itself, he becomes so verbose as to be almost unintelligible [Rome 1931–1943, 558, 562–563]. He proves the plane cases for a number of different permutations of the chords in the sector figure, on the grounds that some of these are actually employed by Ptolemy; however, he does not exhaust the possible permutations [Rome 1931–1943, 539–545]. Theon also proves the second of the

[6] In one important text tradition the sector theorem is *Spher.* II 8. For convenience I will, nevertheless, use *Spher.* III 1 to denote an occurrence of the sector theorem found in the *Spherics*.

[7] These computation lemmas are further discussed below; see page 60

three geometric cases of *Disjunction*; however, he seems to believe that the theorem does not hold in the third, parallel case [Rome 1931–1943, 560-562 & 552-554].[8] He gives two proofs of *Conjunction*, again in two cases; the first filling out Ptolemy's sketch, the second based on *Disjunction* and a simple lemma about the chords of supplementary arcs [Rome 1931–1943, 562–570].

Although both Ptolemy and Theon discuss Menelaus in other contexts, neither of them makes any mention of him in regard to the sector theorem. Indeed, Theon's comments to *Alm.* I 13 are disappointing from a historiographic perspective. There is no indication that Theon consulted any works other than the *Almagest* when he wrote this material.

Three Arabic translations of Menelaus' *Spherics* are attested, although they have all been lost. The first, **Ü**$_1$, was apparently made in the 8$^{\text{th}}$ century by an unknown scholar and is generally believed to have come through a Syriac intermediary [Krause 1936, 85].[9] A marginal note in al-Ṭūsī's edition mentions another early translation, **Di**, by Abū ʿUthman al-Dimashqī [Taha and Pinel 1997, 153, n. 10]. In the 9$^{\text{th}}$ century, a third, more literal translation, **bḤ**, was made by Isḥāq ibn Ḥunayn, the son of the great medical author and translator.

In Baghdad under the ʿAbbāsid Califate, mathematical, and hence spherical, astronomy became subjects of great interest and activity. In the late 9$^{\text{th}}$ century, Thābit composed *On the Sector Theorem*, **Th***, an important original treatise that would come to play a vital role in the transmission of the theorem in the Arabic tradition [Björnbo 1924; Lorch 2001; Bellosta 2004]. In his introductory remarks, he states that the sector theorem was studied more thoroughly than the other geometric theorems used in astronomy [Lorch 2001, 43]. Moreover, his remarks on other, contemporary interest in the theorem make it clear that it was the focus of considerably attention in his lifetime. *On the Sector Theorem* provides a mathematical completion to Ptolemy's approach to the theorem. Thābit assumes as demonstrated the material already found in the *Almagest*. He then shows the other two geometric cases of *Disjunction*, determined by the relationship of a chord of the sector figure and a certain radius of the sphere. *Conjunction* is demonstrated on the basis of *Disjunction*, in the same way as Theon. Thābit shows that there are sixteen possible combinations of the arcs and gives proofs for all of these. Although it is not certain that Thābit made direct use of Theon's *Commentary*, it seems likely that the elements of his text that are also found in Theon came to him, by one route or

[8] Although Theon does not say so explicitly, his discussion of the application lemma *Alm.* I 13.3 indicates that he held that the sector theorem could not be applied to the parallel case [Rome 1933, 45–49]. He points out that the relevant application lemma fails in the parallel case. In fact, the theorem does hold in this case; however, the proof must avoid the lemma that Theon warns against.

[9] Sezgin [1996, 159] is not convinced of this view, but Hogendijk [1996, 26] offers an argument for a Syriac provenance based on the transliteration of diagram letters.

another, from the Arabic translations of the *Commentary* [Lorch 2001, 343–347].[10]

Around this time, al-Māhānī compiled **Ma**, an incomplete revision of the oldest Arabic translation of the *Spherics*, $\ddot{\mathbf{U}}_1$. This version was previously thought to be lost, but Lorch [2001, 333–334] has recently shown that, **K**, an extant manuscript, bears a direct relationship to **Ma**, or to Māhānī's source.

About a century later, al-Harawī made **H**, a revision of **Ma**. According to Harawī, **Ma** ended in confusion at *Spher.* III 5.[11] He set about to rectify this situation, providing a full and sound text of the *Spherics*.[12] When it came to the treatment of the sector theorem, Harawī interjected a number of interesting historical comments that underscore the differences between Menelaus' approach and that of Ptolemy [Lorch 2001, 330–331]. The theorem itself, however, is virtually identical in **H** and **K**. Hence, we may presume that this version of the theorem was essentially that found in Māhānī's edition, **Ma**. For the sector theorem, we assume **H** = **K** = **Ma**. We will call this version **Ma-H**. An edition of the sector theorem, as preserved in **H** has recently been made by [Lorch 2001, 340-342]. We will take **Ma-H** *Spher.* III 1 as a first witness of the version of the theorem that Menelaus wrote.[13]

The only critical edition of the *Spherics* was made by Krause on the basis of **N**, Abū Naṣr's version of the text [Krause 1936]. Abū Naṣr largely followed Ibn Ḥunayn, **bḤ**, inserting his own astronomical comments following theorems that have a definite, or important, astronomical interpretation. In the matter of the sector theorem, however, Abū Naṣr appears to have turned away from **bḤ** and consulted an Arabic tradition of the theorem as found in the work of Ibn Sīnā and presumably originating with Thābit's treatise [Lorch 2001, 329–330, 353–355]. Ibn Sīnā, the great philosopher and mystic, included a lengthy version of the sector theorem in the astronomical section of his *Remedy*, **bS***, a voluminous compendium of all knowledge [Madwar et Ahmad 1980, vol. 4, 48–76; Weidmann 1926–1927]. **N** is similar to **bS*** in a number of ways, including a very trivial lemma that first makes its appearance in the sector theorem material in **bS***.[14] Hence, the sector theorem in **N** is quite different from what we find in **Ma-H**. In the whole of *Spher.*

[10] Especially, compare Rome [1931–1943, 567–570] with Lorch [2001, 56–63].

[11] In fact, Harawī says that the text broke off after *Spher.* II 10 [Krause 1936, 26]. The numbering of the different versions, however, is such that II 10 in Harawī is III 5 in almost all other editions [Krause 1936, 8–9].

[12] According to Taha and Pinel [1997, 153, n. 10] a marginal note in al-Ṭūsī's edition claims that Harawī also made use of **bY**, an improved edition of **Di**, made by Ibn Yusif.

[13] The proposition numbering of the Māhānī-Harawī tradition gives the sector theorem as II 10. Here, as elsewhere, I follow the proposition numbering of Björnbo [1902]. Krause [1936, 6–9] gives a correspondence of the different numbering schemes.

[14] This is a proof of the fact that if $A : B = C : D$ and $E : F = 1$, then $A : B = (C : D) \times (E : F)$ [Madwar et Ahmad 1980, 68].

III, Abū Naṣr, following Ibn Sīnā, uses *Sines* to bring the mathematical expression more into agreement with contemporary practices in spherical trigonometry, a field to which he was himself an important contributor.[15]

In the 12th century, Gerard of Cremona produced a Latin translation, **G**, from an Arabic text labeled **D** by Krause, which, although lost, is also attested through other sources. **D** appears to have been a compilation in two parts; **Da** on the basis of **Ma**, and **Db** on the basis of **bḤ**. Until recently, it was held that **D** *Spher.* III 1 belonged to **Da** (= **Ma-H**, for the sector theorem), but Lorch [2001, 332–334] has shown that the theorem is different in the two versions and dispelled any reasons for believing that it does not belong to **Db**. Because of the literalism of Gerard's translations, we may take his version of the theorem as close to that of Ibn Ḥunayn [Kunitzsch 1992; Weber 2002]. On the other hand, in comparing **G** *Spher.* III 1 and **Ma-H** *Spher.* III 1 with the other versions of the theorem, we see structural similarities that differentiate these two from the rest. We will take **G** *Spher.* III 1 as a second witness to the theorem that Menelaus wrote.

In the 13th century, Naṣīr al-Dīn al-Ṭūsī, produced **Ṭ**, an edition of the *Spherics* with commentaries, as part of a larger project to provide a new recension of the canonical works of Greek astronomy. Al-Ṭūsī made his text on the basis of **H** and **N** and was strongly influenced by Abū Naṣr's approach, although he largely preserved the text of **H**. In his proof of *Spher.* III 1, al-Ṭūsī returned to the ancient practice of using the chords of double arcs. Al-Ṭūsī's text has been printed, rather carelessly, in the Hyderabad editions of Arabic texts [al-Ṭūsī 1940].

In this same century, Jacob ben Māḥir made **J**, a Hebrew translation, again on the basis of **D**. This has never been edited, but it served as the basis for a Latin translation of the *Spherics* by Edmund Halley [1758].

Although there are a number of other versions of the *Spherics*, they all depend on these, in one way or another. The early transmission of the work is summarized in Figure 2.[16] This schema is suitable for most of the text, however, in the case of the sector theorem, the didactic tradition founded by Thābit entered into the transmission of the *Spherics* itself. In particular, Abū Naṣr consulted Ibn Sīnā, and al-Ṭūsī made use of Abū Naṣr's work. This effect is so pronounced that it is advisable to draw up a separate schema for the sector theorem itself. The transmission of *Spher.* III 1 is summarized in Figure 3.

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[15] He expresses the relevant ratios in terms of the *Sine of the arc* (جيب قوس), where Thābit uses the *chord of the double arc* (وتر ضعف قوس) and Ibn Ḥunayn the *correspondent to the arc* (نظير قوس). The latter two are different expressions for the same line, $Crd(2\alpha)$, while the first, $Sin(\alpha)$, is the expression for half of this line, which can be used to express the same proportions because $Crd(2\alpha) = 2Sin(\alpha)$.

[16] A more complete schema for the transmission of the *Spherics* is given by [Taha and Pinel 1997, 198].

This figure shows how the astronomical and didactic traditions affected the transmission of *Spher.* III 1. We see that the astronomical tradition, as exemplified by Ptolemy, came to exert a strong influence on the sector theorem as it was handled by the Arabic mathematical astronomers. This resulted in both the astronomical and didactic traditions entering into the geometric tradition of the *Spherics*. For example, the texts of the *Spherics* produced by Abū Naṣr and al-Ṭūsī, which were the only Arabic versions published until quite recently, included a *Spher.* III 1 that was taken over from the didactic tradition, through **bS*** and **Th***. This means that most studies of the sector theorem have been misled by the assumption that the versions of Abū Naṣr and al-Ṭūsī can be essentially attributed to Menelaus.

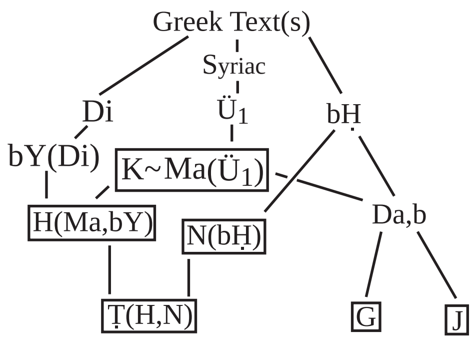

Figure 2: Stema for the early history of Menelaus' *Spherics*. Texts in boxes are still extant. Parentheses indicate that a version is an edition based on the enclosed text(s). A few possible, but for our purposes irrelevant, early versions are omitted. The majority of the late editions are ignored [Taha and Pinel 1997, 198].

The schema in Figure 3, however, shows that **Ma-H** and **G** are the versions of the theorem least removed from Menelaus. Hence as stated above, we may take **Ma-H** *Spher.* III 1 and **G** *Spher.* III 1 as the two most important witnesses to the version of the sector theorem that Menelaus included in his *Spherics*.

When we collate the sector theorem in **Ma-H** with that in **G**, the Latin translation descending from **bḤ**, it is clear that these two Arabic translations were, in fact, somewhat different. Indeed, we should admit the possibility that they were made from different Greek recensions. When we make this comparison, however, we also notice that they share a number of characteristics that differentiate them from all other versions of the theorem.[17] In fact, the similarities are greater than the differences. Moreover, it is clear that the proof given in **Ma-H** is simply a sketch of some more complete version. In this regard, **G** will help us see some of what has

[17]Some of these characteristics are already mentioned by al-Harawī who was struck by the difference between Menelaus' and Ptolemy's versions of the theorem [Lorch 2001, 330-331].

been left out. Ultimately, however, we should take the two versions of the theorem in **Ma-H** and **G** as independent witnesses of the theorem Menelaus wrote.

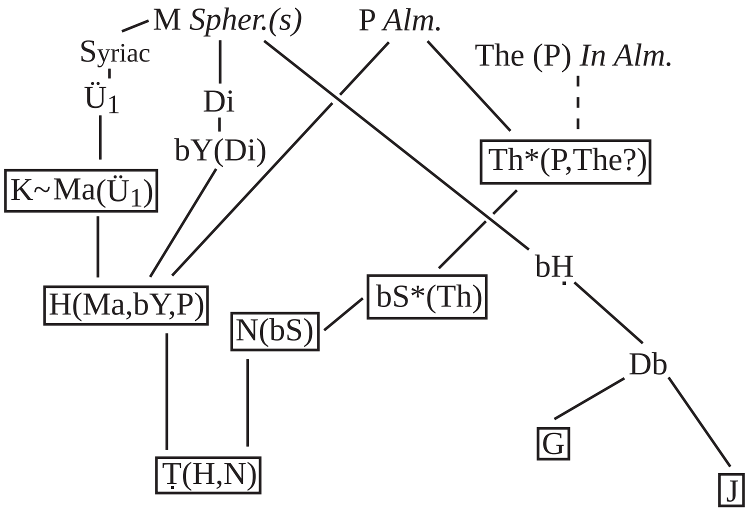

Figure 3: Schema for the early history of *Spher.* III 1. Texts in boxes are still extant. Parentheses indicate that a version makes use of the enclosed text(s). Asterisks indicate original treatises, not translations or editions of the *Spherics*.

III Two versions of *Spher.* III 1

The version of the theorem found in the Māhānī-Harawī tradition is simply an assertion of the proposition followed by a proof sketch. There is almost no argumentation. In only one or two places is an argument even suggested. In most places, however, the arguments missing in **Ma-H** can be supplied by a reading of **G**. Hence, we will not address the details of the mathematics until we have considered both versions.

Only *Disjunction* is mentioned, or demonstrated. The geometrical cases are differentiated on the basis of the relationship between a chord of one of the arcs of the sector figure and a certain radius of the sphere. This feature distinguishes the **Ma-H** version of the theorem from that in Gerard. *Disjunction* is only treated in two cases, although the possibility of the third case is mentioned. The argument given for the parallel case is mere handwaving.

No lemmas are given, but because of the structure of the argument they are hardly needed. This becomes clear when we read Gerard's translation. Where we might expect the use of a lemma, we find instead an explicit reference to the diagram. This appeal to the figure also shows up a number of times in Gerard.

I) **Ma-H** *Spher.* III 1 (*The Māhānī-Harawī edition*):[18]

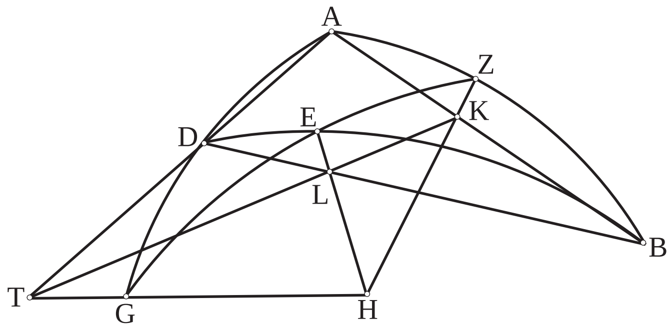

Figure 4: Diagram for the first case of the sector theorem in the Māhānī-Harawī tradition of the Arabic text.

[1] Then, if arc BED intersects arc GEZ between the two arcs BZA, GDA, and each of the four arcs is less than a semicircle, I say that the ratio of the chord of twice AZ to the chord of twice BZ is compounded of the ratio of the chord of twice AG to the chord of twice GD and the ratio of the chord of twice DE to the chord of twice EB.

[2] The proof of it is that we make point H the center of the sphere and we join HZ, HE, HG and we join BD, BA, AD. [3] And let AD, HG meet in one of two directions (في إحدى الجهتين) at T. [4] BD will intersect HE at L; AB will intersect ZH at K. [5] Then points K, L, T are in the planes of both circle ZEG and triangle ABD; hence, line KLT is straight. [6] Then, according to what is in the diagram, the ratio AK to KB is compounded of the ratio AT to TD and the ratio DL to LB; and these are the ratios of the chords of twice the arcs attached to them. [6] Hence, the ratio of the chord of twice AZ to the chord of twice BZ is compounded of the ratio of the chord of twice AG to the chord of twice GD and the ratio of the chord of twice ED to the chord of twice EB.

[7] And if AD is parallel to HG, then the chord of twice AG will equal the chord of twice GD, because the sum of AG, GD is a semicircle. [8] Hence, the ratio AK to KB is the ratio DL to LB. [9] Hence, the ratio of the chord of twice AZ to the chord of twice ZB is the same as the ratio of the chord of twice ED to chord of twice EB. [10] And that is what we wanted to show.

The cases of the theorem are differentiated on the basis of the relationship between the chord AD and radius HG. These lines either meet "in one of two directions" or they are parallel (**Ma-H**[3] & **Ma-H**[7]). In this one respect, this tradition is similar to all other occurrences of the sector theorem. In **G**, on the contrary, the cases are distinguished one the basis of the relationship between chord AD and line

[18]See page 71 for the text.

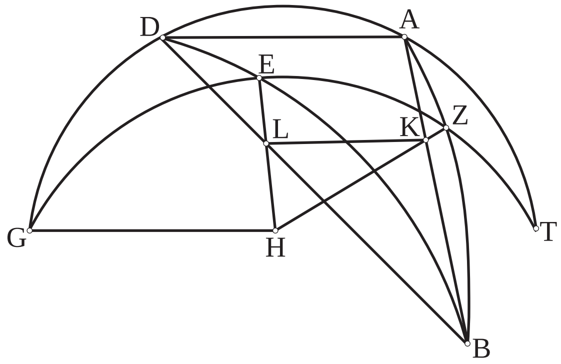

Figure 5: Diagram for the parallel case. The layout of the parallel lines has been modified from the medieval figures; cf. page 72.

KL.[19] As **J** shows, this was found in **Db**, and hence goes back to Ibn Ḥunayn's translation.

In almost all other respects, **Ma-H** *Spher.* III 1 is simply an abbreviated version of **G** *Spher.* III 1. The one exception to this is the explicit reference to "one of the two directions," where the presentation in **Ma-H** is more complete than that in **G**.

On the basis of differences such as these, Lorch [2001, 327–335] considered the version in the Māhānī-Harawī tradition to be more pristine. These considerations will be taken up after we have read Gerard's translation.

The abbreviation encountered in this version of the theorem is so pronounced that Harawī felt the need to give his readers some warning prior to setting out the theorem itself [Lorch 2001, 331]. He remarks on the use of ellipsis in the expression for the chords of the double arcs and claims that statements about the drawing, such as that in **Ma-H**[6], should be read as referring to lemmas for the plane sector figure. Harawī himself included these lemmas, however, he made it clear that they were not originally in the text.

Since there is so little argumentation in **Ma-H**, if we wish to see how the mathematician would have proceeded, we must be familiar with a more complete version of the proposition, such as that in Gerard. Despite the differences between **Ma-H** and **G**, these two versions have a great deal in common. Since **G** is more complete, a reading of this text will reveal the arguments missing in **Ma-H**. A mathematical summary of the text is provided in Appendix B. The summary can also be used to flesh out the argument for the sector theorem in **Ma-H**, making the necessary changes for the differences in construction. Moreover, it includes mathematical interpretations of the difficult passages toward the end.

It should be noted that Gerard uses the strange expression *nadir of an arc* to

[19]See **G**[8] & **G**[19] below, where AD is called NM and KL called SD.

mean the *chord of a double arc*; that is $na(\alpha) = Crd(2\alpha)$. This convention is explained in **G**[4], below.

II) **G** *Spher.* III 1 (*The Gerard translation*):[20]

[1] Let there be two arcs of two great circles on the surface (*superficie*) of a sphere, upon which are NE, LN. [2] And, between them, I produce the two arcs ETA, LTM and they intersect each other at T. [3] Therefore, I say that the ratio (*proportio*) of the *nadir* of arc AN to the *nadir* of arc AL is compounded of the ratio NE to EM and the ratio of the *nadir* of arc MT to the *nadir* of arc TL.[21] [4] And indeed, when I say the *nadir* of an arc, I mean nothing more than the line, which is subtended by the double of that arc; according to which, let that arc be less than a semicircle.

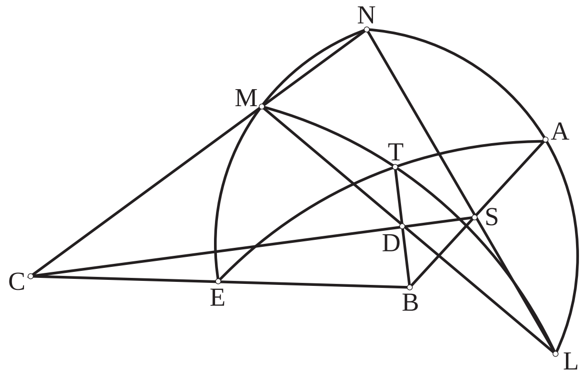

Figure 6: Diagram for the first case of the sector theorem in Gerard's Latin translation.

[5] The demonstration of which is this. [6] I will make point B the center of the sphere. [7] And I will produce lines NL, NM, LM, TB, EB, ASB, SD. [8] And initially (*in primis*), the two lines NM, SD, when extended, will meet at point C, according to what is in the diagram. [9] And I produce line EC. [10] Therefore point C will be in each of the two planes (*superficierum*) of the two arcs ATE, NME, while each of the two points E, B will also be in those two planes. [11] Therefore, CEB is a straight line. [12] And because this diagram is so, the ratio NS to SL is as the ratio compounded of the ratio NC to CM and the ratio MD to DL. [13] In fact, the ratio NC to CM is as the ratio of the perpendicular falling from point N to CEB to the perpendicular falling from point M to line CEB. [14] And likewise, the perpendicular falling from point N to line CEB is half the chord of twice arc NE. [15] And the perpendicular falling from point M to that line is half the chord of twice arc EM. [16] Therefore, the ratio NC to CM is as the ratio of the *nadir* of arc NE to the *nadir* of arc ME. [17] Furthermore,

[20]See page 72 for the text.

[21]The text should read $\frac{na(AN)}{na(AL)} = \frac{na(NE)}{na(EM)} \times \frac{na(MT)}{na(TL)}$; see **G**[18], below.

in the same way, it is shown that ratio NS to SL is as the ratio of the *nadir* of arc NA to the *nadir* of arc AL and that the ratio MD to DL is as the ratio of the *nadir* of arc MT to the *nadir* of arc TL.[22] [18] Therefore, the ratio of the *nadir* of arc NA to the *nadir* of arc AL is as the ratio compounded of the ratio of the *nadir* of arc EN to the *nadir* of arc ME and of the ratio of the *nadir* of arc MT to the *nadir* of arc TL.

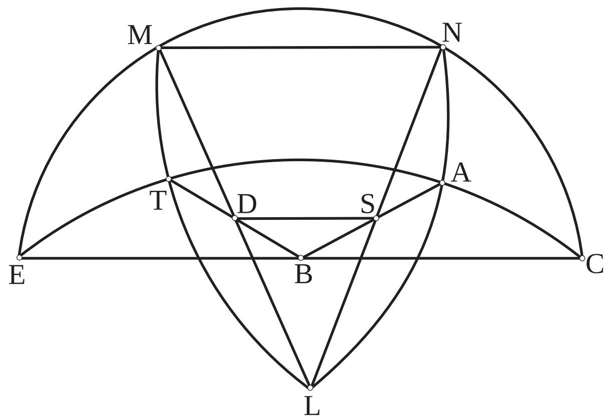

Figure 7: Diagram for the parallel case. The layout of the parallel lines has been modified from the medieval figure; cf. page 74.

[19] And likewise, we make line SD equidistant from line NM and we will complete the two halves of the two circles ETC and ENC, according to what is in the second diagram. [20] And since, in the two planes ENC, ETC, there are two equidistant lines, which are SD, MN, the common section of those two planes, which is line EC, will be equidistant from the two lines SD, MN. [21] And since the perpendicular falling from point N to line CBE is half the chord of twice arc EN and is likewise half the chord of twice arc EM, the *nadir* of arc EN will equal the *nadir* of arc EM. [22] And since line MN is equidistant from line DS, ratio NS to SL, which is as the ratio of the *nadir* of arc NA to the *nadir* of arc AL, will be as the ratio MD to DL, which is as the ratio of the *nadir* of arc MT to the *nadir* of arc TL. [23] Therefore, the ratio of the *nadir* of arc NA to the *nadir* of arc AL is as the ratio compounded of the ratio of the *nadir* of arc MT to the *nadir* of arc TL and the ratio of the *nadir* of arc NE to the *nadir* of arc EM, since it is equal to it.

[24] And by way of this approach, likewise is shown the rest of the things that result from this case of the proportion (*proportionis*) in respect to the *nadirs* of these arcs. [25] And we know that from the arrangement of the lines which intersect each other in the plane that we discussed.

[26] And the remaining cases (*species*) of this configuration are shown just as we

[22] For the details of the compact logic of this step, see Appendix B.

show in this diagram. [27] Since the ratio of the *nadir* of arc AL, also, to the *nadir* of arc AN is as the ratio compounded of the ratio of the *nadir* of arc LT to the *nadir* of arc TM and the ratio of the *nadir* of arc ME to the *nadir* of arc EN. [28] And that is because we just now showed that the ratio of the *nadir* of arc NA to the *nadir* of arc AL is as the ratio compounded of the ratio of the *nadir* of arc MT to the *nadir* of arc TL and the ratio of the *nadir* of arc NE to the *nadir* of arc ME. [29] Therefore, by inverse ratio (*convertendo proportione*), the ratio of the *nadir* of arc AL to the *nadir* of arc AN will be as the ratio compounded of the ratio of arc LT to the *nadir* of arc TM and the ratio of the *nadir* of arc ME to the *nadir* of arc EN.

One of the interesting features of Ibn Ḥunayn's translation is his attempt to reproduce the advantage of the technical expression *the under the double of AB* (ἡ ὑπὸ τὴν διπλῆν τῆς AB) used by the Greek mathematicians in place of the more unwieldy *the line under the double of arc AB* (ἡ ὑπὸ τὴν διπλῆν τῆς AB περιφερείας εὐθεῖα).[23] The former expression has the advantage of keeping the reader's attention on the lettered objects, unencumbered by unnecessary words. It relies, however, on features of the Greek language that cannot be reproduced in standard Arabic syntax. Ibn Ḥunayn apparently attempted to produce a similar effect through the expression *the correspondent to arc AB* (نظير قوس اب).[24] He then gave an explanation of his terminology. Gerard simply transliterated the first word in *naẓīr qaws* to produce *nadir arcus*. Perhaps he did not fully understand how the Arabic was functioning and, since a clear explanation of the terminology was given in the text, saw no need to translate what was obviously a technical expression.

G proves the theorem in two cases without relying on lemmas. Only *Disjunction* is shown, although there is a reference to the possibility of other combinations in the final passages. In **G**[24], the text refers to "the rest of the things that occur in the matter of the proportion of the sphere." This is an obscure phrase, but it likely refers to *Conjunction*, the other combination of the arcs of the spherical sector figure. **G**[25] then asserts that this combination can also be shown on the basis of the underlying plane figure, in a manner similar to *Disjunction*. Finally, **G**[26]–[29] is a muddled attempt to show how, for any given arrangement of the plane figure, different permutations of the arcs can be derived directly by manipulation of the compound proportion. In fact, the derived permutation is simply the inversion of the original proportion, so that the intent of the passage is almost lost.

Ma-H introduces the first case by noting, in **Ma-H**[3], that the two relevant lines meet "in one of two directions" (في إحدى الجهتين). In **G**, this reference has almost disappeared. In fact, however, the phrase that has been translated with "initially"

[23]The concise idiom is more common. Also quite common is the partially elided *the under the double of arc AB* (ἡ ὑπὸ τὴν διπλῆν τῆς AB περιφερείας) [Heiberg 1898–1903, eg. 71].

[24]The Arabic expression is preserved in al-Mu'taman's *Conclusion* [Hogendijk 1996, 40].

(*in primis*), at **G**[8], is probably a translation of the first part of a similar Arabic phrase. Whatever the case, the absence of any mention of two possible directions is not, on its own, sufficient reason for considering **H** more pristine than **G** [Lorch 2001, 334]. This is, in fact, the only place where **Ma-H** *Spher.* III 1 contains more information than **G** *Spher.* III 1.

As Lorch [2001, 328] notes, **G** is unusual in differentiating the cases on the basis of the relationship of lines NM and SD. In all other versions, the cases are based on the relationship between NM and BE. As Lorch himself acknowledges, however, all other versions, except **Ma-H** and **G**, have come under the influence of Ptolemy and his commentators. Hence, the similarity of **Ma-H** to these, in this regard, has no bearing on the authority of **Ma-H** as a witness to Menelaus' text. We simply have two independent sources attesting two different ways of setting out the theorem. There is no historical or mathematical reason to prefer one to the other.

It should be noted, however, that every other version of the parallel case that is structurally similar to **Ma-H** and gives a complete argument, proceeds by an indirect proof. Since Menelaus himself, in the opening remarks of the *Spherics*, mentions that he avoided indirect proof, all these arguments can be taken as spurious. For the parallel case of **Ma-H** *Spher.* III 1, however, there is no argument at all. Moreover, it is also possible to supply a direct argument for this arrangement. Hence, these considerations do not help us decide between the authority of the two versions. Nevertheless, we should bear in mind that any indirect arguments in the *Spherics* should be considered suspect.

With the exception of these two differences, **Ma-H** and **G** are quite similar. Neither of them makes any use of lemmas. Both of them make direct reference to the diagrams as part of the argument. Both of them prove the theorem in only two cases, although **Ma-H** makes mention of the fact that another case is possible. Both of them only prove *Disjunction*, although **G** mentions the other combination and claims that it will be demonstrated along similar lines. We should consider these basic similarities to be well attested for the theorem that Menelaus actually wrote.

In the *Almagest*, Ptolemy provided lemmas for the equivalents of **G**[12]–[16] & [17]. In the didactic tradition we find a lemma of the step asserted in **G**[21]. The argumentation in **G**, however, makes it clear why Menelaus considered these lemmas unnecessary.

The lemmas asserting the plane configuration of the sector theorem were probably part of the toolbox of advanced geometric work. Already at the beginning of the Hellenistic period, Euclid assumed his readers were familiar with propositions dealing with similar material in his *Porisms*.[25] The lemmas that allow the application

[25] Heath [1921, vol. 2, 270] claimed that one of Pappus' lemmas to the *Porisms*, *Coll.* VII 137, was actually the plane sector theorem. In fact, they are not quite the same theorem, although, as Schmidt [1943, 68] points out, one can be shown from the other as a corollary. For Pappus, see

of ratios obtaining on the chords to the subtending arcs were handled separately by Ptolemy and most authors following him. Menelaus, however, appears to have incorporated them nicely into the text itself. Menelaus simply imagines perpendiculars dropped from the endpoints of the arcs onto a diameter of the sphere (**G**[14], [15] & [17]). In both cases, this construction produces similar triangles. Menelaus leaves it to the reader to supply the full argument based on these triangles. A similar construction, in **G**[21], makes it obvious that supplementary arcs have equal corresponding chords. Since the lemmas were all either part of the toolbox of advanced mathematical work, or apparent from the construction itself, Menelaus saw no need to establish them separately.

This discussion has allowed us to develop a view of the structural form of the theorem Menelaus used as the basis of his theory of spherical trigonometry. This was the beginning of the geometric tradition of the sector theorem. By comparing this version of the theorem with the well known versions from the astronomical tradition, we may get a sense of how the theorem was meant to function in the two different traditions.

IV The early Greek traditions of the theorem

The earliest extant example of the astronomical version of the sector theorem is that in Ptolemy's *Almagest*. For Ptolemy, the sector theorem is a computational device that solves three primary problems of spherical astronomy: (1) computation of the rising times of arcs of the ecliptic, (2) instantaneous determination of the position of the ecliptic relative to the local horizon, and (3) calculation of simultaneous rising times for fixed stars and the related phenomena of stellar visibility.

There is little doubt that the sector theorem was used in a similar way to solve some or all of these problems by mathematical astronomers prior to Ptolemy. We are told by both Porphyry and Paul of Alexandria that the astronomer Apollonarius, who was probably about contemporary with Menelaus, used similar methods as Ptolemy to derive rising time values for the signs of the zodiac [Boer and Weinstock 1940, 212; Boer 1958, 1–2]. These two authors are grouped together by Porphyry as "moderns" who worked to bring precision to the problem of rising times by using precise geometric methods.[26] Moreover, we are told by Pappus that Menelaus wrote a treatise on rising times [Hultsch 1876–1878, 600–601]. No one has ever questioned the assumption that this text made use of the sector theorem to produce

Jones [1986, 270].

[26] According to Porphyry, they worked "by means of the methods of lines" (διὰ τῶν γραμμικῶν ἐφόδων) [Boer and Weinstock 1940, 212]. This is probably a technical expression for a construction that can be solved using ancient trigonometric methods. See note 29, below, for a discussion of the related phrase "by means of lines" (διὰ τῶν γραμμῶν).

computational solutions.

The form of the sector theorem in the *Almagest* reflects the needs of this computational tradition. For the purposes of calculation, both *Disjunction* and *Conjunction* are used depending on which arcs are known and which sought. Hence, Ptolemy explicitly states each combination. The theorem is proceeded by a number of lemmas useful to a readership that wishes to follow the argument but may not be familiar with the advanced mathematical literature. Two of these lemmas are actually not necessary for the logic of the proof (*Alm.* I 13.3c & 13.4c) [Toomer 1984, 66-68]. In fact, they are useful in applying the sector theorem to calculations. Using the compound proportion of the sector theorem, it is generally necessary to know five of its terms to solve for the sixth. These two lemmas show, however, that all six terms can also be determined when we know four terms and the sum or difference of the other two. Every time Ptolemy himself uses the sector theorem, five given terms are used to solve for the sixth. Hence, the computation lemmas must have been written by some other mathematical astronomer who actually used them in his spherical astronomy. As I have shown elsewhere, a type of computation that Hipparchus tells us he made in his work on spherical astronomy can best be recomputed using the sector theorem and these lemmas [Sidoli 2004].

Ptolemy's presentation of the sector theorem is written for an astronomical audience. It assumes the reader has no mathematical knowledge beyond Euclid's *Elements*. The entire presentation is geared toward the use of the sector theorem as a computational device. Menelaus' version of the sector theorem, on the other hand, is aimed at a mathematical audience. He assumes his readers have a more advanced knowledge base than that provided by the *Elements*. He skims over a number of trivial details. He takes it as obvious that the proof provided can be applied to the other combination of the arcs of the sector figure. Menelaus' proof is at once more elegant and more general.

The sector theorem plays an interesting role in the context of Menelaus' spherical trigonometry. It is the first theorem in *Spher.* III and hence the foundation of the entire theory. In general, *Spher.* III develops a metrical theory of spherical triangles. The sector theorem, however, is the only proposition in the entire work that neither uses nor discusses spherical triangles. Moreover, it does not depend on any previous theorems in the *Spherics*. Hence, its role in *Spher.* III is essentially that of a lemma. Both its subject and its conclusion are only logically useful to the theory of *Spher.* III; it is not itself a result of this theory.

Only one of the possible combinations of arcs is actually demonstrated or stated in Menelaus' version of the sector theorem. Nevertheless, he meant his proof of *Disjunction* to serve for all combinations. A statement to this effect is still preserved in Gerard's version (**G**[24]–[25]), although its import has become obscured through translation. Menelaus' actual use of the sector theorem in the proofs of the remainder of *Spher.* III, however, dispels any doubts we might have that this was his intention.

The sector theorem is used in the proofs of four later propositions: *Spher.* III 2, 3, 10 & 15.[27] In the first two of these, the combination used is *Disjunction* as demonstrated in *Spher.* III 1. In *Spher.* III 3, however, after the compound proportion is stated, it is immediately manipulated into another permutation of the same arcs.[28] In *Spher.* III 10, the sector theorem is used three times. All three of these are occurrences of a form of *Conjunction.* In the last of these, the compound proportion is inverted but it is the same basic permutation. In *Spher.* III 15, the sector theorem is used twice in permutations that are only established in the didactic tradition. This shows that Menelaus meant his proof to serve for both *Disjunction* and *Conjunction*, as well as all possible permutations of these. It is this wide applicability that is being discussed in the final passages of the Gerard version of the sector theorem.

For the geometrical purposes of *Spher.* III, the permutations of the sector theorem are just as, if not more, important than the two combinations. Menelaus, however, is introducing the theorem as preparatory to the more important work of spherical trigonometry dependent on it. He makes his proof as short as possible, despite the fact that it will serve for both combinations and all the permutations. He assumes his readers are familiar with the plane versions of the theorem and can readily see how his proof is applied to all of the spherical cases he will need. In all likelihood, those features of Menelaus' treatment that allow it to be at once both more concise and more general than the other versions of the theorem are due to the fact that he assumed his readers would already be familiar with the theorem.

V An historical conjecture

Based on Menelaus' treatment of the sector theorem and its function in his spherical trigonometry, it is improbable that he was the original author of the theorem. Menelaus probably took a well known theorem of quantitative spherical astronomy and adapted it to his purposes. On this foundation, he built his new theory of spherical trigonometry. The sector theorem itself, however, was sufficiently simple and useful that, when Ptolemy came to produce his own spherical astronomy, he based everything on this theorem. Indeed, few traces of the other ancient traditions of spherical geometry are present in the *Almagest.* For example, there is no use of the techniques of projective geometry found in Ptolemy's *Planisphere.* Only one partial analemma is used, in *Alm.* II 5, to compute the ratio of the length of a gnomon to

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[27]For *Spher.* III 2 & 3, the numbering in **G** and **N** is the same, however, however, **G** III 10 = **N** III 13 and **G** III 15 = **N** III 22–25. In the **Ma-H** tradition, there are only two books. See note 13.

[28]The second time this occurs, Abū Naṣr informs us that this manipulation is demonstrated in material preliminary to the sector theorem [Krause 1936, 66]. He attributes this to "the author of the book."

the length of the principal shadows, given the latitude. In fact, this computation is not used in the *Almagest* and it is probably a gesture toward the tradition of gnomonics, the theory of sundial construction that relied on the analemma. On the other hand, Ptolemy generally simplifies his approach to spherical astronomy by making use of some of the elementary theorems of the pure spherical geometric tradition represented by the *Spherics* of Theodosius and Menelaus.

We are now in a position to make a strong case that the tradition of applying the sector theorem to problems of spherical astronomy goes back at least as far as Hipparchus. Already at the beginning of the 20$^{\text{th}}$ century, Björnbo [1902, 65–88] made a case for this position and a number of the arguments in this section have their origins in his work. There are two basic sources for these claims: (1) remarks about Hipparchus' mathematical work on spherical astronomy, and (2) Ptolemy's treatment of the subject in *Alm.* I, II & VIII.

In his *Commentary on the Phenomena of Aratus*, Hipparchus refers the reader to previous work that he produced on simultaneous risings. There are four references to this material [Manitius 1894, 128, 148, 150 & 184]. On the basis of his statements, we know that the work contained methods for determining the rising, setting and culminating points of the ecliptic when a fixed star, given in equatorial coordinates, rises or sets. The presentation was evidently mathematical. In three cases, Hipparchus tells us that this work used proofs; and in the last of these cases, he tells us that he proceeded by exact, trigonometric techniques [Manitius 1894, 128, 148 & 150].[29]

Hipparchus' treatment of a related problem is mentioned in the Oxyrhynchus papyri. *P. Oxy.* 4276 tells us that a "compilation (συντάγματος) of Hipparchus" was used to find the rising point of the ecliptic, the ascendant, "accurately calculated" (ψηφισθεὶς ἀκρειβῶς) for a given time at the latitude of Egypt [Jones 1999, 418]. The calculated value was $28;35°$; the precision was to either $0;05°$, or $0;01°$ and rounded. Since Hipparchus himself lived at the latitude of Rhodes, and all of the work in his *Commentary* was carried out for this latitude, this compilation must have provided the mathematical tools for finding the rising point of the ecliptic as a function of both date and latitude and was probably one of his works on simultaneous risings. I have shown elsewhere that some the mathematical problems handled in these treatises can be solved by means of the sector theorem, but not by the other ancient methods of performing calculations on the sphere [Sidoli 2004].

[29] The phrase that he uses literally means "by means of lines" (διὰ τῶν γραμμῶν) and was a technical expression denoting a geometric construction susceptible to chord table computations. In every instance where it appears in the Greek technical literature, this phrase denotes a solution that uses either the plane trigonometric methods of the analemma or the spherical trigonometric methods of the sector theorem [Björnbo 1902, 83–83; Luckey 1927; Neugebauer 1975, 301–302; Sidoli 2004].

Two of the most important topics addressed by the ancients using spherical astronomy were the rising times of arcs of the ecliptic and the simultaneous risings of points of the celestial sphere. We know that Hipparchus produced works that gave computational solutions to the problems arising in both of these areas. In what was apparently his usual style, he presented this material in separate works devoted to each topic, perhaps more than one.

At this point, some background remarks on numerical studies of rising times in Greek authors will be helpful.[30] Two primary methods for determining a numerical value for the rising time of an arc of the ecliptic are preserved in technical Greek astronomical texts. They are founded on different mathematical assumptions and produce contradictory results. Hypsicles, in his *Ascensions*, gave the derivation of a numerical scheme which, although adjusted for the latitude of Alexandria, had its origins in the work of the Babylonian mathematical astronomers [de Falco, Krause and Neugebauer 1966]. Ptolemy, on the other hand, derives his table of rising times from the geometrical considerations of the two-sphere model of the Greek cosmos [Toomer 1984, 90–103].

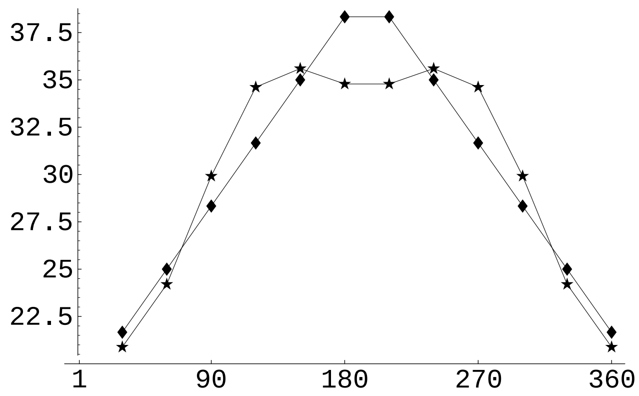

Figure 8: Plot of Hypsicles' rising times compared with Ptolemy's. ◆ = Hypsicles; ⋆ = Ptolemy.

In his *Ascensions*, Hypsicles gives a value for the rising time of each zodiacal sign at the latitude of Alexandria. This is simply a linear zigzag function that increases from the vernal equinox to the autumnal equinox and then symmetrically decreases to complete the cycle; see Figure 8 [de Falco, Krause and Neugebauer 1966, 38]. As can be seen from the comparison of these times with Ptolemy's, for the most part the numbers are fairly good. Although such numbers were probably adequate for most astronomical and astrological purposes, the overall pattern they exhibit is, as Ptolemy says in his *Tetrabiblos*, "not even close to the truth" [Robbins 1994, 94]. We may see the structural problems by comparing the values in the *Almagest* with those in the *Ascensions*. We compare Hypsicles' numbers with those for the rising times

[30]A useful study of the subject of rising times is provided by Brunet and Nadal [1981].

of the zodiacal signs at Alexandria with Ptolemy's; see Figure 8 [Toomer 1984, 101]. This shows that, far from being a maximum, the autumnal equinox is actually a local minimum. The effect becomes more pronounced as the observer moves toward the equator.

By perusing Ptolemy's table of rising times, *Alm.* II 8, we can discern the overall behavior of the rising times of arcs of the ecliptic. Figure 9 plots, for every other latitude, the column of the rising times of 10° intervals of the ecliptic, which Ptolemy included in the table in order to exhibit the functional pattern that we prefer to find in a graph. At the equator, the rising times are symmetrical about the cardinal points of the ecliptic; the solstices are equal maxima, the equinoxes equal minima. As the observer moves north, however, we notice two distinct patterns. (1) The minimum of the spring equinox decreases while the minimum of the autumnal equinox increases. (2) The maxima increase together and move from the solstices toward the spring equinox. This means that for all latitudes, other than the equator, the rising times are only symmetrical with respect to the equinoxes. Moreover, the movement of the maxima toward the autumnal equinox means that, for more northerly latitudes, the rising times about the autumnal equinox are greater than those about the solstices. Similar patterns also apply to the southern hemisphere; however, such considerations would have been irrelevant to the ancients.

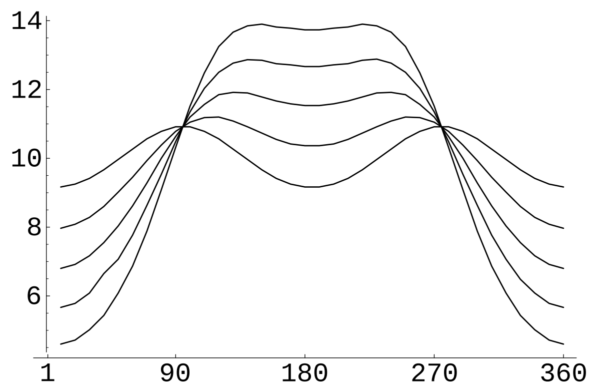

Figure 9: Plot of five geographic latitudes from Ptolemy's table of rising times. The lowest minimum at the autumnal equinox, $\lambda = 180°$, is at the equator ($\varphi = 0°$). The other latitudes, increasing with this local minimum are $16;27°$, $30;22°$, $40;56°$, and $48;32°$.

Hipparchus' work on rising times is discussed by Pappus in his commentary to Euclid's *Phaenomena* 12 [Hultsch 1876–1878, 598–602]. Pappus explains why Euclid only provides a proof for particular rising-time phenomena for the semicircle of the ecliptic following Capricorn ($270° \rightarrow 90°$) and the related setting-time phenomena for the opposite semicircle of the ecliptic, that following Cancer ($90° \rightarrow 270°$). Specifically, Euclid shows that, in $270° \rightarrow 90°$, equal arcs "at the points of contact of the tropics" (πρὸς ταῖς συναφαῖς τῶν τροπικῶν) always set in more time than those "at the equator" (πρὸς τῷ ἰσημερινῷ). Pappus intends to explain why Euclid does not discuss the rising times of this semicircle in a similar manner, and generally to discuss work on the conditions of solvability (διορισμός) for problems involving rising times.

Part of the explanation lies in work carried out by Hipparchus. Pappus tells us that,

> [1] Hipparchus, in *On the Ascensions of the 12 Signs of the Zodiac*, demonstrates, through calculation (δι᾽ ἀριθμῶν), that equal arcs of the semicircle following Cancer that set maintaining some time relation (σύγκρισιν) to one another do not rise in the same way; [2] for there are some latitudes (τινας οἰκήσεις), in which the equal arcs of the semicircle following Cancer that are nearer the ecliptic always rise in more time than those at the points of contact of the tropics.

The fist sentence, [1], claims that Hipparchus used numbers, or calculation, to demonstrate the difference between patterns of rising and setting times in the interval $270° \rightarrow 90°$. He showed that a pair of equal arcs, $\alpha_1 = \alpha_2$, whose *setting* times have a certain relation at all latitudes, $\sigma(\alpha_1) \lesseqgtr \sigma(\alpha_2)$, will not necessarily have the same relation between *rising* times at all latitudes.[31] This claim may be easily seen in Figure 9. Since every arc of the ecliptic sets in the same time as the diametrically opposite arc of the ecliptic rises, the pattern for the setting times on the interval $270° \rightarrow 90°$ can be seen by looking at the rising times for $90° \rightarrow 270°$. Hipparchus was pointing out that while the relation, $\lesseqgtr$, between any two setting times for equal arcs in $90° \rightarrow 270°$ is always the same relation at every latitude, this is not the case for rising times in this interval. In the next sentence, [2], Pappus explains how this relation changes for rising times at different latitudes.

The Greek that has been translated as "through calculation" literally reads "by numbers," *di arithmōn*. Neugebauer read this is as "through arithmetical methods" and took these to be in the tradition of Babylonian mathematical astronomy, as opposed to geometrical methods [Neugebauer 1975, 301].[32] There are two problems

[31] Björnbo [1902, 75] interpreted the Greek differently. He took the qualification "maintaining to one another some time comparison" (ἔχουσαί τινα πρὸς ἀλλήλας χρόνου σύγκρισιν) as specifying overall conditions for the arcs, under which the statement would hold. He took this to mean *"insofern diese* (die Tierzeichen) *eine gegenseitige Zeitkomparation haben,"* and argued that Hipparchus was excluding from the discussion polar latitudes, where certain arcs of the ecliptic will neither rise nor set.

[32] There are at least two other Babylonian schemes for rising times [Rochberg 2004]. One of them has essentially the same structure as that in Hypsicles, but is not strictly linear [Neugebauer 1975, 368]. This scheme is attested in a number of non-technical Greek authors [Neugebauer 1975, 712–

with this reading.

The first is that the pattern Hipparchus is discussing is not evident in the linear schemes. In the Babylonian methods, although the relation will be the opposite for rising and setting times it will be the same relation for all latitudes. That Hipparchus was concerned with the variation of this relation between latitudes is made clear by the next sentence of the text, [2].

The second issue is linguistic. I am aware of no instance in the Greek technical literature of *di arithmōn* being used to denote Babylonian style arithmetical methods, whereas, in a number of cases, it refers to calculation. The phrase *di arithmōn* can also be used in a general sense to refer to a statement that may be exemplified, or justified, through numbers, and Heron uses the phrase in both senses.[33] Eutocius uses the phrase to indicate Apollonius' method of finding the ratio of the circumference of a circle to its diameter with great precision [Heiberg 1973, vol. 3, 258]. He is certainly talking about some sort of calculation. The reference to calculation is supported by comparison with related phrases. The phrase *dia tōn arithmōn* is used twice in the *Almagest* to refer to a trigonometric calculation carried out in order to confirm a statement that has already been established through pure geometry [Toomer 1984, 157 & 211]. It is used in the same sense by both Pappus and Theon in their commentaries, and by Pappus elsewhere in the *Collection* [Rome 1931–1943, 17, 29, 57, 58, 61, 111, 123, 776, 282, 293, 890, 891 & 1084; Hultsch 1876–1878, 42]. The adjective *arithmētikos* is also used by Ptolemy to refer to calculation [Toomer 1984, 604]. Hence, there is good reason to believe that Pappus here intended us to understand that Hipparchus proceeded by means of calculations carried out on the basis of geometric models.

According to sentence [2], there are "some latitudes" for which an arc of the semicircle $90° \to 270°$ beginning at the summer solstice, $90°$, rises in more time than an equal arc nearer the autumnal equinox, $180°$. The fact that the statement is qualified to *some* latitudes, implies that there are other latitudes for which this is not the case. The context of the discussion as an explanation for *Phean.* 12 makes it clear that the arcs *nearer* the autumnal equinox are the same arcs Euclid treats as having an endpoint at the equinox.

The statement is again best explained by referring to Figure 9. In the southerly latitudes, we see that the rising times in the vicinity of autumnal equinox, $180°$, are less than those around the solstices, $90°$ and $270°$. At a latitude equal to the obliquity of the ecliptic, the rising times at $180°$ are equal to those at $90°$ and $270°$; however, because the maxima are closer to the solstices, the rising times of arcs beginning at

724]. An even cruder scheme is preserved in the Cuneiform sources but does not seem to have been transmitted to the Greco-Roman world [Rochberg 2004, 89].

[33] The two uses in *Definitions* simply mean "through numbers," [Heiberg 1907, 140]. There is a use in *On Measurements*, however, that clearly refers to calculation, [Schöne 1903, 160].

90° will still be greater than equal arcs with an endpoint at 180°. Since the minimum at 180° increases for more northerly latitudes, while the maxima move toward this point, at some northerly latitude the rising time of every arc with an endpoint at 180° will be greater than an equal arc starting at 90°. Pappus' remark indicates that Hipparchus articulated these features of rising time phenomena on the basis of numbers.

This reading of Pappus' text is at variance with two earlier interpretations. Björnbo [1902, 74–75] took [2] to mean that, for some latitudes, for any two equal arcs on the interval 90° $\rightarrow$ 180° the one closer to 180° rises in more time than that toward 90°. Neugebauer [1975, 301] construed the passage to mean that the rising times monotonically increase on the interval 90° $\rightarrow$ 180°, and claimed that Hipparchus put forward a Babylonian numerical scheme.[34]

Both of these readings involve taking πρὸς as indicating direction *toward* and ἔγγιον as *nearer*. Although direction toward is the primary meaning of πρὸς when it governs the accusative case, where it is used with the dative, as here, it generally denotes true proximity. Furthermore, as stated above, while Pappus speaks loosely using "nearer," the broader context reveals he is talking about the same arcs that Euclid described as "at the equator" (πρὸς τῷ ἰσημερινῷ).

Neugebauer's reading makes nonsense of the restriction to *some* latitudes, since, if Hipparchus had proposed a Babylonian scheme along the lines of Hypsicles', [2] would be true everywhere. Moreover it would make no sense for Pappus to refer to the use of Babylonian style rising time schemes in order to explain Euclid's *Phaenomena*; they simply will not serve this purpose. Björnbo's view puts him in the awkward position of having to explain where Hipparchus went astray, which he does by invoking an inexact chord table [Björnbo 1902, 75]. The reading proposed above has two advantages. It stays true to the sense of the Greek and it finds Hipparchus making a statement that is both true and evident on the basis any reasonably accurate table of rising times.

This passage shows that Hipparchus' treatment of rising times was based on the geometrical properties of the two-sphere model. His study allowed him to make sophisticated statements about rising time phenomena that would best be exhibited, using ancient methods, in a table of rising times. There is no reason not to believe that Hipparchus derived his table of rising times using the sector theorem. We will see below that Ptolemy's treatment of rising times also supports this conjecture.

The conjecture that Hipparchus treated these problems by means of the sector

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[34] In fact, his summary of [2] describes a reading that agrees with the translation given above. Nevertheless, he immediately interprets this in a way that does not follow. This issue is exacerbated by a second summary of the passage that Neugebauer [1975, 768] gives when discussing Pappus. Here, his reading of the passage clearly agrees with mine, but he again takes no notice of the qualifying *some*.

theorem is again born out by Ptolemy's treatment of this topic in the *Almagest*. Ptolemy develops a complete theory of spherical astronomy in *Alm.* I 14–16, II 1–13 & VIII 5-6 [Toomer 1984, 69–130, 410–417]. Unlike a number of other theories in the *Almagest*, the presentation of this material is exactly modeled on the deductive structure of pure mathematical texts in the Greek tradition. There are relatively few mathematical prerequisites. Along with the *Elements* and chord table, necessary for all the mathematics in the text, this theory requires the sector theorem and a toolbox of elementary theorems from the *Spherics* of both Theodosius and Menelaus [Heiberg 1927; Krause 1936]. The final sections of *Alm.* I develop two tables that are used in almost all the following calculations. These list the declination and right ascension of degrees of the ecliptic. Following this preliminary matter, the theory divides into five topics: (1) the defining characteristics of geographic latitude (*Alm.* II 1–6), (2) rising times (*Alm.* II 7–8), (3) miscellaneous problems (*Alm.* II 9), (4) instantaneous positions of the ecliptic (*Alm.* II 10–13), and (5) simultaneous risings and stellar visibility phenomena (*Alm.* VIII 5–6). Ptolemy's presentation is very efficient and there is little material that is not used to derive further results. All of the excess that does occur can be attributed to historical strata in the text, or gestures to historical topics that Ptolemy does not consider worth developing in full.[35]

The most impressive mathematical results of Ptolemy's spherical astronomy are the table of rising times and the table of instantaneous positions of the ecliptic (*Alm.* II 8 & 13). They tabulate functions of two strong variables. In both cases, the derivations are demonstrated through the example of a single latitude. Instead of his own latitude of $30;22^\circ$ for Alexandria, Ptolemy uses the latitude of Rhodes, 36°. This is the latitude at which Hipparchus carried out his astronomical work.[36]

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[35] Of 41 units (theorems, calculations, metrical analyses and lists) of mathematical text in the sections of the *Almagest* on spherical astronomy, only seven are not used later in the text. Three of these, in *Alm.* I 13 & II 11, are very trivial corollaries, taking up less than ten lines of Heiberg's text (H71,14–H72,10; H73,11–H74,8; H156,3–H156,9). These were probably carried over from one of Ptolemy's predecessors. (The first two are discussed above on page 60.) The only other excess material is the gesture toward gnomonics, *Alm.* II 5, discussed above (page 61), a surplus computation of rising times, in *Alm.* II 7 (H120,23–H124,22), and the treatment of simultaneous risings, *Alm.* VIII 5 & 6, both discussed below (page 69).

[36] As Neugebauer [1975, 733] points out, 36°, may simply have been a standard convention taken as the reflection in the quadrant of the latitude of the arctic circle, having no important connection to Hipparchus' work. It is also possible, however, that this value gained currency as a standard for both astronomical and geographic studies because Hipparchus made much use of it. It is worth noting that there is a clear connection between the treatment of terrestrial latitudes which Strabo attributes to Hipparhus and that found in *Alm.* II 6 and especially the table of rising times, *Alm.* II 8 [Jones 1917–1932, vol. 1, 506–520; Toomer 1984, 82–90 & 100–103; see also Jones 2002].

The table of rising times, *Alm.* II 8, is the centerpiece of Ptolemy's spherical astronomy and simplifies the computations of a number of subsidiary problems, as exemplified in *Alm.* II 9. A desire to determine the local position of the ecliptic would have been made the more urgent by Ptolemy's decision to use ecliptic coordinates for his star catalog, whereas Hipparchus generally worked with equatorial coordinates [Duke 2002]. The table of the instantaneous positions of the ecliptic, *Alm.* II 13, gives a numeric solution to this problem. A treatment of the characteristics of geographic latitude, *Alm.* II 6, is preliminary to both of these topics. For the purposes of the *Almagest*, however, the mathematical treatment of simultaneous risings, *Alm.* VIII 5, could be dealt with in a cursory manner. This is presumably because the mathematics of this topic is so cumbersome when compared to list based approaches such as those of his own *Phases* or Hipparchus' *Commentary on Aratus*.

In general, Ptolemy's approach to spherical astronomy was to use elementary theorems from Theodosius and Menelaus combined with calculations made with the sector theorem. This allowed him to use the symmetry of parallel circles and the geometry of spherical triangles to simplify his approach. In just those two topics that we know Hipparchus treated mathematically, however, Ptolemy relies only on the sector theorem.

In fact, Ptolemy gives two methods for calculating a table of rising times, whereas for his purposes one is perfectly sufficient. In the middle of *Alm.* II 7, Ptolemy shows how the sector theorem alone can be used to calculate the rising times of individual signs and he points out that this method can also be used to calculate the rising times of smaller arcs [Toomer 1984, 92–94]. In the final part of *Alm.* II 7, however, he sets out another method that he considers to be "easier and more methodical" (εὐχρηστότερον καὶ μεθοδικώτερον) [Heiberg 1898–1903, p. 1, 125]. It is a batch calculation that is simplified by considerations of symmetry based on the mathematics of Theodosius and Menelaus. This is the only place in Ptolemy's spherical astronomy where he gives two methods for calculating the same table. Björnbo [1902, 74] took *Alm.* II 7 as evidence that Hipparchus' table of rising times only included the rising times of whole signs. It is probable, however, that Ptolemy's concession that we could use the sector theorem alone to find the rising times of smaller arcs was a reference to the fact that the older tables were so produced, despite the laboriousness of the method.

In the final sections of *Alm.* VIII, Ptolemy shows how the sector theorem can be used to treat simultaneous risings and stellar visibility phenomena [Toomer 1984, 410–417]. The computations are quite involved, using at least two applications of the sector theorem. Ptolemy has no interest in actually carrying out these laborious calculations. He merely goes through the metrical analyses, showing in each case how one would proceed. These sections are probably a gesture to an important historical topic, which Ptolemy feels should be mentioned even though it was unnecessary for the *Almagest*. Ptolemy's own mathematical treatment of these problems was evidently given in a separate treatise discussed at the beginning of the *Phases* Heiberg [1907, part. 2, 3-4]. I have argued elsewhere that Hipparchus' work on simultaneous risings was mathematically related to this work [Sidoli 2004]. Ptolemy's treatment of this material makes sense when we realize that he is giving a short summary of mathematical methods that his predecessors developed in full detail.

VI Conclusion

By examining the early transmission of Menelaus' *Spherics* and its relation to other work on the sector theorem, we are able to select two versions that are likely to contain a text of the sector theorem close to that which Menelaus actually wrote. In fact, these two versions exhibit characteristic features that distinguish them from all other versions of the theorem. A close reading of these two versions in the context of the *Spherics* as a whole makes it clear that Menelaus did not intend his readers to take the sector theorem as his original work. On the contrary, he relies on the reader's familiarity with the theorem to move through it as quickly as possible.

Menelaus treats the sector theorem as a stock component of mathematical astronomy. Our evidence for the history of spherical astronomy before Ptolemy suggests that the use of the theorem goes back at least as far as Hipparchus. An investigation of the few ancient references to Hipparchus' spherical astronomy shows that he used computational methods at least as powerful as the sector theorem. Ptolemy's treatment of spherical astronomy corroborates this suggestion. He seems to have intended his work to be read in comparison with that of his predecessors. In every case where he solves a problem that we know Hipparchus also solved, he shows how it can be done using the sector theorem alone, even though this is not usually his own method.

A thorough appraisal of the evidence shows that we should admit that the spherical trigonometry based on the sector theorem predated Menelaus and that in all likelihood it went back at least as far as the work of Hipparchus in the late Hellenistic Period.

Acknowledgments

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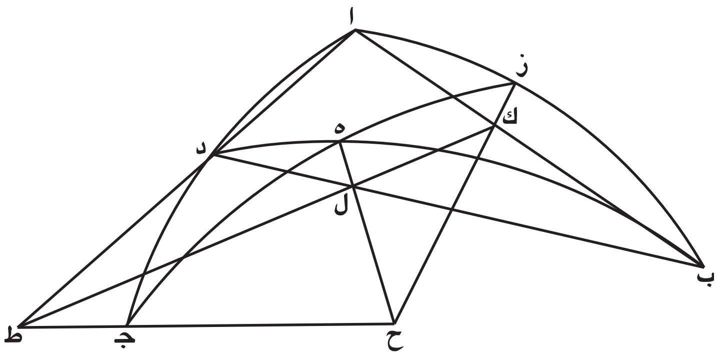
Appendix A: Texts of *Spher.* III 1

The Arabic Version

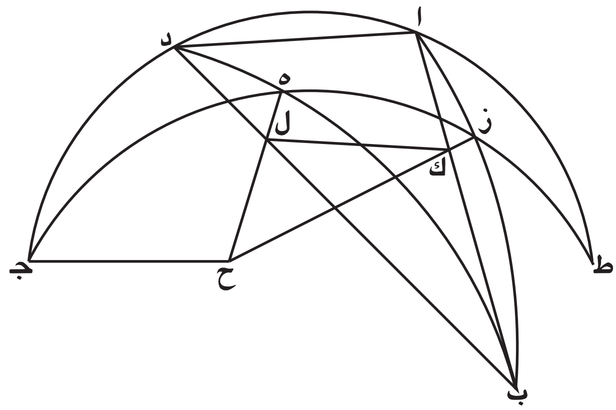
I) **Ma-H** *Spher.* III 1 (*The Māhānī-Harawī edition*):[37]

فلتقطع الآن قوس ب ه د قوس جـ ه ز بين قوسي ب ز ا جـ د ا وكل واحدة من القسي الأربع أصغر من نصف دائرة، أقول إن نسبة وتر ضعف ا ز إلى وتر ضعف ب ز مؤلفة من نسبة وتر ضعف ا جـ إلى وتر ضعف جـ د ومن نسبة وتر ضعف د ه إلى وتر ضعف ه ب، برهانه أن نجعل مركز الكرة نقطة ح ونصل ح ز ح ه ح جـ ونصل ب د ب ا ا د وليلق ا د ح جـ في إحدى الجهتين على ط وب د يقطع ح ه على ل وب ا يقطع ز ح على ك فنقط ك ل ط في سطحي دائرة ز ه جـ ومثلث ا ب د فخط ك ل ط مستقيم فبحسب ما في الرسم نسبة ا ك إلى ك ب مؤلفة من نسبة ا ط إلى ط د ومن نسبة د ل إلى ل ب وهذه هي نسب أوتار أضعاف القسي المركبة عليها فنسبة وتر ضعف ا ز إلى وتر ضعف ب ز مؤلفة من نسبة وتر ضعف ا جـ إلى وتر ضعف جـ د ومن نسبة وتر ضعف ه د إلى وتر ضعف ه ب .



[37]This text was edited by Lorch [2001, 340–342]. In the **Ma-H** tradition, the sector theorem is *Spher.* II 8.

وإن كان ا د يوازي ح ج فوتر ضعف ا ج يساوي وتر ضعف ج د لأن مجموع ا ج ج د نصف دائرة فتكون نسبة ا ك إلى ك ب هي نسبة د ل إلى ل ب فتكون نسبة وتر ضعف ا ز إلى وتر ضعف ز ب هي كنسبة وتر ضعف ه د إلى وتر ضعف ه ب وذلك ما أردنا أن نبين.



The Latin Version

II) **G** *Spher.* III 1 (*The Gerard translation*):[38]

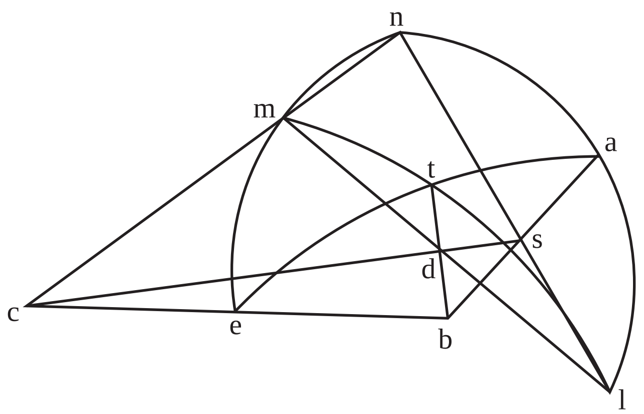
Sint in superficie spere duo arcus duorum circulorum magnorum super quos sint $.\overline{n.e}.\overline{l.n}.$ Et protraham inter eos duos arcus $.\overline{e.t.a}.\overline{l.t.m}.$ et secent se super punctum $.\bar{t}.$ Dico ergo quod proportio nadir arcus $.\overline{a.n}.$ ad nadir arcus $.\overline{a.l}.$ est composita ex proportione $.\overline{n.e}.$ ad $.\overline{e.m}.$ et ex proportione nadir arcus $.\overline{m.t}.$ ad nadir arcus $.\overline{t.l}.$ Et ego quidem non significo cum dico nadir arcus nisi lineam, que subtenditur duplo illius arcus. Secundum quod sit ille arcus minor semicirculo. Cuius hec est demontratio. Ponam centrum spere punctum $.\bar{b}.$ Et protraham lineas $.\overline{n.l}.\overline{n.m}.\overline{l.m}.\overline{t.b}.\overline{e.b}.\overline{a.s.b}.\overline{s.d}.$ Et concurrant in primis due linee $.\overline{n.m}.\overline{s.d}.$ cum protrahuntur super punctum $.\bar{c}.$ secundum quod est in forma. Et protraham lineam $.\overline{e.c}.$ Ergo erit punctum $.\bar{c}.$ in unaquaque duarum superficierum duorum arcuum $.\overline{a.t.e}.\overline{n.m.e}.$ at unumquodque duorum punctorum $.\bar{e}.\bar{b}.$ iterum erunt in istis duabus superficiebus. Ergo est $.\overline{c.e.b}.$ linea una recta. Et cum hac forma sit ita, tunc proportio $.\overline{n.s}.$ ad $.\overline{s.l}.$ est sicut proportio composita ex proportione $.\overline{n.c}.$ ad $.\overline{c.m}.$ et ex proportione $.\overline{m.d}.$ ad $.\overline{d.l}.$ Verum proportio $.\overline{n.c}.$ ad $.\overline{c.m}.$ est sicut proportio perpendicularis cadentis ex puncto $.\bar{n}.$ super $.\overline{c.e.b}.$ ad perpendicularem cadentem ex puncto $.\bar{m}.$ super lineam $.\overline{c.e.b}.$ Iterum

[38] Paris BN Lat. 9335 48v-49r.

3 $.\overline{a.l}.$] $.\overline{e.m}.$ MS. A marginal note states that $.\overline{a.l}.$ is found "in alio."

4 $.\overline{m.t}.$] $.\overline{s.t}.$ MS.

et, perpendicularis cadens ex puncto .$\overline{n}$. super lineam .$\overline{b.e.c}$. est medietas corde dupli arcus .$\overline{n.e}$. Et perpendicularis cadens ex puncto .$\overline{m}$. super illam lineam est medietas corde dupli arcus .$\overline{e.m}$. Ergo, proportio .$\overline{n.c}$. ad .$\overline{c.m}$. est sicut proportio nadir arcus .$\overline{n.e}$. ad nadir arcus .$\overline{m.e}$. Et similiter etiam declaratur quod proportio .$\overline{n.s}$. ad .$\overline{s.l}$. est sicut proportio nadir arcus .$\overline{n.a}$. ad nadir arcus .$\overline{a.l}$. et quod proportio .$\overline{m.d}$. ad .$\overline{d.l}$. est sicut proportio nadir arcus .$\overline{m.t}$. ad nadir arcus .$\overline{t.l}$. Ergo proportio nadir arcus .$\overline{n.a}$. ad nadir arcus .$\overline{a.l}$. est sicut proportio composita ex proportione nadir arcus .$\overline{m.t}$. ad nadir arcus .$\overline{t.l}$. et ex proportione nadir arcus .$\overline{e.n}$. ad nadir arcus .$\overline{m.e}$.

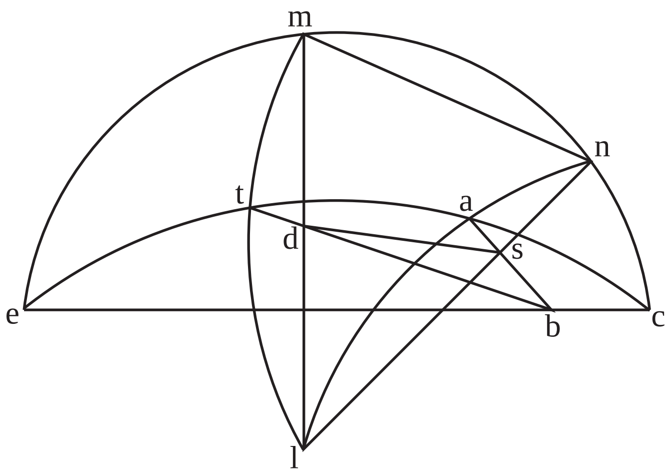


Et iterum nos ponemus lineam .$\overline{s.d}$. equidistiantem linee .$\overline{n.m}$. et complebimus duas mediatates duorum circulorum .$\overline{e.t.c}$. et .$\overline{e.n.c}$. secundum quod est in forma secunda. Et quoniam in duabus superficiebus .$\overline{e.n.c}.\overline{e.t.c}$. sunt due linee equidistantes, que sunt .$\overline{s.d}.\overline{m.n}$. erit sectio communis istis duabus superficiebus, que ist linea .$\overline{e.c}$. equidistantes duabus lineis .$\overline{s.d}.\overline{m.n}$. Et quoniam perpendicularis cadens ex puncto .$\overline{n}$. super linea .$\overline{c.b.e}$. est mediatas corde dupli arcus .$\overline{e.n}$. et est iterum mediatas corde dupli arcus .$\overline{e.m}$. erit nadir arcus .$\overline{e.n}$. equalis nadir arcus .$\overline{e.m}$. Et quoniam linea .$\overline{m.n}$. est equidistans linee .$\overline{d.s}$. erit proportio .$\overline{n.s}$. ad .$\overline{s.l}$. que est sicut proportio nadir arcus .$\overline{n.a}$. ad nadir arcus .$\overline{a.l}$. sicut proportio .$\overline{m.d}$. ad .$\overline{d.l}$. que est sicut proportio nadir arcus $\overline{.m.t.}$ ad nadir arcus .$\overline{t.l}$. Ergo proportio nadir arcus .$\overline{n.a}$. ad nadir arcus .$\overline{a.l}$. est sicut proportio composita ex proportione nadir arcus .$\overline{m.t}$. ad nadir arcus .$\overline{t.l}$. et ex proportione nadir arcus .$\overline{n.e}$. ad nadir arcus .$\overline{e.m}$. cum sit ei equalis. Et per huius modi uiam iterum declarantur reliqua que accidunt de hac specie proportionis in nadir horum arcuum. Et sciemus illud ex dispositione linearum que iam secuerunt se in superficie quem diximus. Et declaruntur relique species huius descriptionis sicut nos declaramus in hac forma. Quoniam proportio nadir arcus .$\overline{a.l}$. iterum ad nadir arcus .$\overline{a.n}$. est sicut proportio composita ex pro-

30 .$\overline{e.n}$.] .$\overline{c.n}$. MS.

31 .$\overline{e.m}$.] .$\overline{e.n}$. MS.

portione nadir arcus $.\overline{l.t}.$ sicut nadir arcus $.\overline{t.m}.$ et ex proporione nadir arcus $.\overline{m.e}.$ ad nadir arcus $.\overline{e.n}.$ Et illud est quoniam iam nuper ostendimus quod proportione nadir arcus $.\overline{n.a}.$ ad nadir arcus $.\overline{a.l}.$ est sicut proportione composita ex proportione nadir arcus $.\overline{m.t}.$ ad nadir arcus $.\overline{t.l}.$ et ex proportione nadir arcus $.\overline{n.e}.$ ad nadir arcus $.\overline{m.e}.$ Cum ergo convertendo proporione erit proportio nadir arcus $.\overline{a.l}.$ ad nadir arcus $.\overline{a.n}.$ sicut proportio composita ex proportione nadir arcus $.\overline{l.t}.$ ad nadir arcus $.\overline{t.m}.$ et ex proportione nadir arcus $.\overline{m.e}.$ ad nadir arcus $.\overline{e.n}.$



Appendix B: Mathematical Summary of G *Spher.* III 1

It will be helpful to give a mathematical overview of the argument in **G** *Spher.* III 1. Because the text does not make every step of the proof explicit, there are places where the summary is mathematically more complete than the text itself. In the final passages, the summary gives a plausible mathematical interpretation of some obscure remarks.

I adopt the following notational conventions. $na(AB)$ is the *nadir* of arc AB, such that $na(AB) = Crd(2AB)$. $\perp (N, AB)$ is the perpendicular dropped from point N to line AB. All sentence numbers refer to the translation of **G** *Spher.* III 1 (see pages 55–57).

Mathematical Summary:

[1]–[4]: Given Figure 6, to show:

$$\frac{na(AN)}{na(AL)} = \frac{na(NE)}{na(EM)} \times \frac{na(MT)}{na(TL)}.$$

42 $.\overline{t.m}.$] $.\underline{\overline{a.n}}\overline{t.m}.$ MS.

42 $.\overline{m.e}.$] $.\overline{m.m.e}.$ MS.

[5]–[8]: We complete Figure 6. [Case A]: let NM and SD meet at C.

[9]–[11]: Since point C is in both planes ATE and NME, while points E and B are likewise in both planes ATE and NME, therefore CEB is the rectilinear intersection of these planes.

[12]: By the plane sector theorem,

$$\frac{NS}{SL} = \frac{NC}{CM} \times \frac{MD}{DL}.$$

[13]–[15]: Because of similar triangles,

$$\frac{NC}{CM} = \frac{\perp (N, CB)}{\perp (M, CB)};$$

but

$$\perp (N, CB) = na(EN)$$

$$\perp (M, CB) = na(EM);$$

therefore

$$\frac{NC}{CM} = \frac{na(EN)}{na(EM)}.$$

[17]: Because of similar triangles,

$$\frac{NS}{SL} = \frac{\perp (N, AB)}{\perp (L, AB)},$$

while

$$\perp (N, AB) = na(NA)$$

$$\perp (L, AB) = na(AL);$$

and, again because of similar triangles,

$$\frac{MD}{DL} = \frac{\perp (M, TB)}{\perp (L, TB)},$$

while

$$\perp (M, TB) = na(MT)$$

$$\perp (L, TB) = na(TL).$$

[18]: Therefore,

$$\frac{na(AN)}{na(AL)} = \frac{na(NE)}{na(EM)} \times \frac{na(MT)}{na(TL)}.$$

[19]: [Case B]: let $NM \parallel SD$. We complete Figure 7.

[20]: Since NM and SD are in planes ENC and ETC,

$$NM \parallel SD \parallel EB.$$

[21]: Since $NM \parallel EB$,

$$\perp (M, EB) = \perp (N, EB).$$

[22]: Again since $NM \parallel SD$, and because of similar triangles,

$$\frac{NS}{SL} = \frac{MD}{DL} = \frac{\perp (N, AB)}{\perp (L, AB)} = \frac{\perp (M, TB)}{\perp (L, TB)};$$

but

$$\perp (M, TB) = na(MT)$$

$$\perp (L, TB) = na(LT),$$

and

$$\perp (N, AB) = na(NA)$$

$$\perp (L, AB) = na(AL).$$

[23]–[24]: Therefore

$$\frac{na(AN)}{na(AL)} = \frac{na(MT)}{na(TL)} \times \frac{na(NE)}{na(EM)},$$

because

$$\frac{na(MT)}{na(TL)} = \frac{na(MT)}{na(TL)} \times \frac{na(NE)}{na(EM)}.$$

[24]–[25]: In the same way, we demonstrate the other proportion that holds for the *nadirs* of the arcs in the spherical sector figure. We make this demonstration on the basis of the plane sector theorem.

[26]–[29]: Moreover, the permutations of either arrangement can be shown on the basis of the plane figure for that arrangement. We simply manipulate the compound proportion according to the operations of ratio theory.

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