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The Toledan *Regule* (*Liber Alchorismi*, part II): A Twelfth-century Arithmetical Miscellany

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The mid-twelfth century was a critical period for the introduction of Arabic mathematics to the West. The forms of the Hindu-Arabic numerals had become known in the late tenth century when they were introduced onto the counters of a peculiar kind of abacus called the 'Gerbertian abacus'(Ifrah 1998: 579-85). The details of calculation using these numerals with place value were not known to Latin scholars until an Arabic version of al-Khwarizmi's account of Indian arithmetic (possibly the *Kitāb fi'l-jam' wa-l-tafrīq*, 'Book on addition and subtraction') was translated (Folkerts/Kunitzsch 1997: 169). While the Arabic text itself is lost, at least four Latin versions of it are known: a literal, but interpolated, translation ('Dixit Algorizmi' = DA), a text incorporated into an introduction to the four mathematical arts of the quadrivium ('Liber Ysagogarum Alchorismi' = LY), and the *Liber Alchorismi* (LA) and the *Liber pulveris* (LP) which share a common source. All these Latin versions have been critically edited by André Allard (Allard 1992). The *Liber Alchorismi*, however, is followed in all the manuscripts (except Dresden C 80, which is fragmentary) by a miscellany of arithmetical texts which have been called 'the second book' of the *Liber Alchorismi*. This has long been known to scholars, ever since its transcription, from Paris, BNF, Bibliothèque nationale de France, lat. 7359, by Baldassare Boncompagni (Boncompagni 1857). No critical edition, translation or study of this text has been made up to now.[1] We offer here a working edition based on what appears to be the best manuscript, Paris, Bibliothèque nationale de France, lat. 15461, together with an English translation and a mathematical commentary. The work is important not so much because of any originality or ingenuity in its mathematical content, as because it documents the kinds of problem that were being tackled in arithmetic, number theory and algebra, just at the time when Hindu-Arabic numerals, calculation with numerals of place value, and algebra were being adapted to a Latin context. As will be argued, this context would appear to be Toledo in the third quarter of the twelfth century, when the new arithmetic of the Arabs was being brought into the Latin curriculum. A humanistic Latin context is evident from the literary quality of the language and the presence of a philosophical

[1] André Allard has promised a critical edition of the text.

essay on the rightness of there being nine digits and three orders of numbers.[2]

Although the work always follows the *Liber Alchorismi de pratica arismetice* of 'magister Iohannes', and can, in some way, be regarded as a continuation of the work, it is never called 'the second book' in the manuscripts. Instead we find the incipit: 'Hic incipiunt regule et primum (prius) de aggregatione' in manuscripts P and M, and 'Incipiunt regule' in E; NU have no title. No manuscripts contain a title at the end, which may confirm the status of this work as a series of appendixes attached to the *Liber Alchorismi*; there is no proper ending to the text. Since 'regule' on its own would be an insufficiently distinctive title for the work, we have decided to call it the 'Toledan *regule*' out of consideration for its provenance. The work occurs in the following manuscripts:[3]

Erfurt, Ea, Q 355, s.14, fols 105r-115r (= **A**)
Florence, BN, Conv. soppr. J.V.18, s.13/14, fols 65v-70r (= **c**)
Oxford, BL, Selden supra 26, s.13, fols 96r-100r (= **E**)
Paris, Mazarine, 3642, s.13, fols 113v-117v (= **M**)
Paris, BNF, lat. 7359, ca.1300, fols 101v-111r (= **N**)
Paris, BNF, lat. 15461, s.13, fols 9v-14v (= **P**)
Paris, BNF, lat. 16202, s.13, fols 73v-81v (= **U**)
Salamanca, BU, 2338, s.14, fols 30r-49v (= **S**)
Vatican, Pal. lat. 1393, s.13, fols 51r-60v (= **L**)

Characteristics of the individual manuscripts

Of the manuscripts that have been fully collated, **P** (Paris, BNF, lat. 15461) has been carefully written and corrected. Its contents and context will be described below.

E (Oxford, Bodleian Library, Selden supra 26) may contain the earliest copy of the text. The manuscript is composite. The relevant codex consists of the folios numbered in a recent hand from 96 to 129. The original order of the quires of which this codex consists appears to have been quires II and III (fols 106-29) and

[2]This essay – part F below – has been described in detail in Lampe 2005. A text with the incipit of <35.3> in part F ('Cum enim in omni lingua ⋯') is listed amongst the works lent to Franciscan and Dominican friars by the archbishop of Santiago de Compostela in 1225: see Luis García-Ballester, 'Nature and Science in Thirteenth-Century Castile. The Origins of a Tradition: the Franciscan and Dominican Studia at Santiago de Compostela (1222-1230)', in *Medicine in a Multicultural Society: Christian, Jewish and Muslim Practitioners in the Spanish Kingdoms, 1222-1610*, Aldershot, 2001, article II.

[3]We are very grateful to Menso Folkerts for the loan of microfilms and the relevant pages of a history of mathematics in the Middle Ages being prepared by H. L. L. Busard and himself.

I (fols 96-100). In this rearranged order the codex contains an incomplete copy of the first part of the *Liber Alchorismi* (106-121v),[4] an incomplete set of instructions for the use of the Pisan tables (122-129v) and the Toledan *regule* (96-100).[5] The codex is characterised by the use of the eastern forms of Hindu-Arabic numerals, which bring it into the ambit of mathematicians associated with Abraham Ibn Ezra in Pisa and Lucca in the mid twelfth century (Burnett 2002a). The text appears to be derivative from that represented by PN. E.g., in most cases 'morabotini', a coinage distinctive of the Iberian peninsula, has been replaced by 'aurei', a term of more general use. Only parts A and C (see next section) are fully represented, but the broken off fragment of part B suggests that more of the text was available in the exemplar. **E** shows some evidence of intelligent engagement with the subject matter. Paraphrases are added to some passages, and the word 'res' for the (square of) the unknown, in part C, is replaced by 'quadratus'. The order in **E** is as follows: **A** 1.1-2, 3.1-3, 2.1-3, 3.4-6, alternate version of 3.5-6, 4.1-5.2, 5.3-4, 6.1-7.2, 8.2-3, 8.1, 8.4, 9.1-12.8. 12.10-14. 7.1-2 (=E^2), **C** 18.1-7, **B** 13.1-4 (breaks off incomplete).

N (Paris, BNF, lat. 7359) provides a text which is very close to that of P, and reproduces some of its mistakes (such as 'numerum eorum' for 'numerorum' in 17.11; 'multiplitio' for 'multiplicatio' in 23.4; the spelling 'duodenerii' in 27.5) including errors in reading the Hindu-Arabic numerals ('3' for '2' in 17.10) and in calculation (in 20.2, 32.3). Occasionally N gives the correct reading where P is wrong (24.2 ductu; 30.1 ut; 30.2 qui: 34.1 aggregata; 34.3 qui; 44.1 etiam constituta) which suggests that it cannot be a direct descendant of P. Moreover, it provides an extra paragraph at the end of the text. On the whole, however, its readings are less reliable, and scribal error is more frequent. Among the variants and characteristic traits of this text (most of which are not mentioned in the apparatus criticus) are: Roman numerals where P has Hindu-Arabic (4.2: ii, 4, 6, 8, 10), or *vice versa* (11.6 '30' for 'xxx'; in 7.2 'iii' has been expunged and replaced by '3'!), or numerals written in full (9.6: 'septem' for '7') or *vice versa* (12.3: '6' for 'sex'), confusion of Hindu-Arabic numerals with each other (especially '2' and '3'; 27.5 '100' for '900') or with the tironian 'et' (read as '2' in 7.2, as '3' in 21.3 and as '7' in 26.3), ambiguity in the use of Hindu-Arabic numerals (e.g. 6.7: '3. 9. 2. 7. 8. 1. 2. 4. 3' for '3, 9, 27, 81, 243'); misreading of abbreviations in the original (17.12 'modo' for 'ergo'; 37.1 '-ant' for '-atur'; 43.4 'qui' for 'quia'; N could be correct here), misreading of minims (4.1. 'inde' for 'vide'), reading 'quod' for 'quot' (6.1 and 6.3 quodlibet), 'id est' for 'scilicet' (16.3), -ationem' for '-antem' (19.1 'denominationem' for 'denominantem'), and so on. N favours spellings with single consonants ('quatuor', 'milia', 'galina'), but writes 'hiis' for 'his'. Frequently phrases that the scribe had at first omitted are

[4]One folio has been cut out between fols 121 and 122.

[5]Fol. 100v is blank, while fols 101-105v contain an acephalous text written in a later hand ('...quanta fuerit differentia partis et cursus mediate...et ad evidentiam directionem quandam subtraximus').

added by him in the margin, or a phrase is written twice (23.5: et fiunt 148 (va-) quos duos productos simul aggrega (-cat) centum 48 —the two halves of 'vacat' added as superscript indicate the redundant text).

Nature of the Text

The repetition and the abrupt changes in subject matter suggest that we are dealing with at least seven distinct elements (indicated by the letters A to G); the multiplication tables between A and B and the numerical magic square at the end of the text, all of which have no accompanying text, may be regarded as further elements. The distinguishable elements may be classified as follows:

A 1.1-12.15. Arithmetical rules, especially on progressions
Multiplication tables for sexagesimal and decimal calculation respectively
B 13.1-17.18. Further arithmetical rules, with some overlap with A but with the addition of fractions, and examples from real life.
C 18.1-7. Algebra
D 19.1-25.2. Further arithmetical rules, on division and multiplication
E 26.1-34.6. Further arithmetical rules, on establishing the 'places'('columns') in multiplication. and on finding a 'hidden' number.
F 35.1-44.7. A philosophical introduction to the principles of Hindu-Arabic numeration
G 45.1-46.2. Brief arithmetical rules
A numerical magic square.

Within each of these elements a certain amount of ordering and planning may be discerned. There are slight changes of terminology between sections. D and E employ the phrase 'multiply a numeral (*figura*) by a numeral (*figura*)', while A and B use 'number' (*numerus*) in the same context. On one occasion (41.4) the author of F seems to be using 'nodus' in place of 'articulus', though he generally uses 'articulus'. But on the whole the terminology and the style are the same from section to section. There are, however, internal disruptions and hesitations in regard to order: at the end of 9.2 P adds a reference mark, while N and E each indicate that 'another method' is about to be presented; phrases have been copied in the wrong order in 11.1; there is break between 15 and 16, where P has placed a reference sign whose referent is not clear. The difference in order in E may also indicate an earlier different arrangement in the texts and within the texts. Perhaps what we are dealing with are 'working copies' of arithmetical texts, dating from a time when Hindu-Arabic numerals were still a novelty, and different ways of teaching and learning their use were being experimented with. Only one source of this material is mentioned: namely, the 'liber qui dicitur gebla mucabala', or 'book on algebra'

(18.1), from which the excerpts in **C** are taken.[6]

There is little that is unique to the *Regule*. Progressions are found widely. The rule for multiplying digits by each other (9.1-9.6) can also be found in LY II (Allard 1992: 27, 22-23) and *Helcep Sarracenicum* 43-45 (Burnett 1996: 272-3). Notable is that the examples in the short section on algebra do not all follow al-Khwarizmi's algebra but include one example which is also in Ibn Turk (18.5).

The context of the text

To what extent can we say that the context is Toledan? The majority of the manuscripts indicate that the diffusion of these texts is within and from North Italy. In E the material is associated with the tables drawn up for Pisa. N includes some mathematical works translated in Italy. P belongs to a group of manuscripts copied in Northeast Italy — probably in the vicinity of Padua. This latter group, however, consists of texts which are distinctive products of Toledo. Two of the group (Paris, BNF, lat. 9335 and Vatican City, BAV, Ross. lat. 579) contain only translations of mathematical texts made by Gerard of Cremona (1114-87). P itself does not include any text by Gerard, but rather a calendar, whose saint days mark it out as belonging to Toledo Cathedral after 1156 (the date of the transfer of the relics of St Eugenius, Toledo's patron saint), and a computus referring to the 'present years' of 1143 and 1159. The other text in the manuscript is the *Liber mahameleth*, which shares passages both with the *De divisione philosophiae* of the archdeacon resident at Toledo cathedral, Dominicus Gundissalinus (d. after 1180) and with the Toledan *regule*.[7] Both the *Liber mahameleth* and the Toledan *regule* refer, in their examples, to 'morabotini/morabatini', the coinage introduced into Arabic Spain by the Almoravids (after which it is named), and used by both the Islamic and the Christian kingdoms within the Iberian peninsula, but not elsewhere.

Characteristics of the text:

1) The term used for x^2 is *res* ('thing'), where other Latin texts use *substantia* or *census* (both suggesting 'wealth'): see 18.1-6. The 'thing' in Arabic (*shai'*) is, rather, the root, of which the square is the *mal* ('wealth'). Nevertheless, in both cases it is the 'unknown' that is the 'thing' and, in that the 'unknown' is what is ultimately sought, it is more properly the square than the root. On the other hand, 'res' can also mean 'property' (i.e., wealth) in legal language. To indicate this special use of *res* 'thing' we have put the 'thing' in italics in the translation.
2) 'cifre' together with the symbol '0' is used to indicate an 'empty place' which is to be filled by the result of the calculation; i.e., it serves as the unknown (19.1).

[6] Høyrup 1998: 16-17 places this excerpt on algebra in context.

[7] See Burnett, 2002b: 66-70.

3) 'Limes' is used as a category of number, rather than with the meaning '(decimal) place' which is found, e.g., in *Helcep Sarracenicum* (cf. Burnett 1996: 262-3: 5 Ordines igitur numerorum sive limites a primis numeris, qui digiti vocantur et sunt novem...6 Sunt autem in unoquoque limite numerorum novem termini...). For the latter the text uses 'differentia'. The four categories of number are *digitus*, *articulus*, *limes*, and *compositus* (26.1). While the other categories are those found generally in works on the algorism, *limes* is an extra one, which is used in the sense of the *first* number in each decimal place: i.e., 10, 100, 1,000, 10,000 etc. (this is implied in 14.3, 23.2). In other words it is, strictly speaking, a subcategory of 'articulus' which is *any* number that is divisible by 10. On the other hand, it could be interpreted as any number which is followed by more than one zero (when 'articulus' is a number followed by a single zero). In section F it is specifically used to define the genus to which the ten, the hundred and the thousand belong (35.5), and acquires the meaning of 'place' as in other arithmetical texts (43.5-6).
4) 'Articulus' sometimes appears to be used for any number divisible by 10, at other times, only those belonging to the tens (10, 20, 30, 40, 50, 60, 70, 80 and 90). As such it belongs to the series *digitus*, *articulus*, *centenus*, *mille* (27.2-29.2). The hesitation between accepting 'articulus' as only referring to tens and as referring to any number divisible by 10 is indicated by the fact that on one occasion (31.1) 'vel centeni' has been added after 'articulus', to restrict it to the tens.
5) The 'multiplicatus' is the smaller number, while the 'multiplicans' is the larger number (30.1-3, 30.5, 32.1-2).
6) 'Sequens' ('following') denotes progression towards the left (the hundreds 'follow' the tens): 22.1, 26.2-3, 26.5, 30.1-2.
7) 'Figura' is used for any numeral ('0' is not regarded as a numeral), clearly indicating written numbers rather than the counters on an abacus.
8) 'Vel', in most cases, does not introduce a different procedure, but expresses the same procedure in different words. It is equivalent to 'i.e.'
9) Numerals always take a plural verb. Hence the apparent incongruity of 15.16: 'cuius sexta pars sunt binarius'.
10) 'Unus' is usually used as the adjective (17.16, 19.4, 34.1), 'unum' as the noun (2.2, 7.2, 13.3, 18.7, 42.9; exceptionally 'unus' is a noun in 30.2 and 36.6).
11) Some confusion remains (perhaps originating in the exemplar) concerning the use of zero. It may be omitted where it is required ('3' for '30' in 19.2) or added where it is not required (32.2, 32.3).
12) Illustrative figures which originally must have been separate from the text (probably 'boxed off' from the text) have erroneously been incorporated, wholly or partially, into the text. E.g., the figure accompanying 17.12 must be reconstructed from two fragments in the margin (20–12 and 34, 2, 7 under each other) and a '7' in the text of P; in N 20, 12 and 7 have all been incorporated into the text and 34, 2, 7 are missing. In 23.2 the figure demonstrating the four numbers in proportion appears

as four number added to the end of the last sentence.
13) The text is written in good Latin. The sentences are joined by conjunctions or relatives ('ergo' is used more than 'igitur', 'vero' and 'autem' are of about the same frequency). There is much subordination, and conditional, final clauses (etc.) are generally correctly constructed.

Editorial Principles

The text is based on P, all readings of which have been reported. Specimens of the readings of M and U have been given for the opening paragraphs. N and E have been fully collated against P, but their readings have been reported only if (1) they have been adopted when P is clearly corrupt, (2) they offer a plausible alternative to the readings of P. Scribal errors in EN and variants which are purely orthographic, have not been reported. Specimens of the readings of M and U have been given for the opening paragraphs.

The orthography of P has been followed. This entails the use of 'e' for Classical Latin 'ae' and 'oe'. However, where P has been inconsistent, the most common spelling has been adopted (e.g. 'millia', 'millies' and 'millium' instead of 'milia', 'milies' and 'milium'). We have, however, not followed P in writing 'vicessima' instead of 'vicesima', though we have kept the variation between 'octuagesimus' and 'octogesimus' that P shares with N.

In most cases in the manuscripts numbers are set off from the text by being surrounded by *puncta*: this applies as much to Hindu-Arabic numerals ('.2.') as to Roman numerals ('.ii.'), and is even found where the numerals are written out in code (34.1: '.dxp.'). The *puncta* have been omitted in this edition. Angle brackets indicate editorial additions; square brackets, deletions. [8]

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[8]This research was made possible by a British Academy Visiting Fellowship which enabled Ji-Wei Zhao to spend four months working with Charles Burnett at the Warburg Institute, London. Burnett is primarily responsible for the introduction, and the edition of the Latin text, parts A-E and G, Zhao is responsible for the mathematical translation and notes, and Lampe for the edition and translation of part F. All three scholars have collaborated on the article as a whole.

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A

Hic[1] incipiunt regule et primum[2] de aggregatione.
<**1.1**> Omnis numerus naturali dispositione circum se positorum numerorum sibi ipsis aggregatorum usque dum occurrens unitas terminum ponat, medietas est. <**1.2**> Ut quinarius quaternarii et senarii inter se coniunctorum medietas est, similiter ternarii et septenarii coniunctorum, similiter[3] binarii et octonarii, similiter unitatis et novenarii et similiter[4] de omnibus.

<**2.1**>[5] Item omnis numerus in se multiplicatus tantam summam reddit quanta ex circumpositorum inter se et differentiarum inter se, quas habent circumpositi ad medium, multiplicatione provenerit.[6] <**2.2**> Tantum enim fit[7] ex quinquies 5 quantum ex quater sex adiuncta multiplicatione differentiarum quas habent ad 5, id est unum et unum, que multiplicata in se non faciunt nisi unum. Et quantum provenerit[8] ex ter septem, vel ex bis octo, vel ex uno et novem, set adiecta semper multiplicatione differentiarum quas habent extremitates ad medium. <**2.3**> Item[9] omnis numerus usque ad 10 in se ductus, tantum reddit quantum primi[10] duo circumpositi in se ducti, set unitate dempta de multiplicatione medii.

<**3.1**> Si vis scire ex aggregatione numerorum ab uno naturaliter se sequentium quanta summa reddatur, ipsum in quo desieris, si par fuerit, multiplica per medietatem sui et adde ipsam medietatem et hec erit summa que ex ipsis efficitur.[11] <**3.2**> Verbi gratia, dispone in ordinem[12] i, ii, iii, iv, 5, 6, 7, 8. Multiplica ipsum in quem[13] desiisti, scilicet octo, per medietatem sui, scilicet per[14] 4, et efficies 32 et adde insuper 4 et hec erit summa que ex aggregatione illorum efficiebatur, scilicet 36. <**3.3**> Vel sequentem imparem per medietatem eiusdem paris multiplica et habebis summam. <**3.4**> Si vero in impari desieris, verbi gratia 1, 2, 3, 4, 5, 6, 7, ipsum,

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[1]E omits

[2]prius M

[3]N adds 'et'

[4]sic E

[5]In E 2.1-3 follows 3.3

[6]proveniunt MU

[7]MU omit

[8]provenit MU

[9]MU omit

[10]M omits

[11]efficietur M

[12]ordine E

[13]quo E

[14]E omits

scilicet[15] septem, multiplica per maiorem partem sui, scilicet 4, 7 enim constant ex tribus et 4, et quater septem efficiunt 28 et hec est summa predictorum. <**3.5**> Vel per medietatem sequentis paris eundem imparem multiplica. <**3.6**> Si autem non ab unitate incipiens continue numeraveris, acsi ab unitate incepisses operandum est et omnes qui sub eo sunt a quo incepisti, aggreges; illud aggregatum a producto subtrahendum, residuum erit summa.[16]

<**4.1**> Si ex solis paribus a binario se per ordinem naturaliter sequentibus aggregatis scire volueris quanta summa reddatur, ipsum in quem desiisti vide quotus est a primo pari et per ipsum a quo denominatur multiplica sequentem se et efficies summam. <**4.2**> Verbi gratia, 2 , 4, 6, 8, 10. Ultimus est 10 et est quintus a primo pari, id est binario. V[17] sequitur senarius. Per 5 ergo a quo denominatur 10 multiplicetur senarius et quinquies sex vel econverso fient 30 et hec erit summa predictorum. <**4.3**> Vel medietatem paris numeri in quo desinis per medietatem proximi sequentis paris multiplica et habebis summam. <**4.4**> Si vero non a binario incipiens continue[18] numeraveris pretereundo impares, acsi a binario incepisses agendum[19] et omnes pares qui sub eo sunt a quo incepisti coacerves; illud[20] coacervatum a producto subtrahendum et residuum erit summa.

<**5.1**>[21] Si autem ex aggregatis imparibus ab uno naturaliter se sequentibus scire volueris quanta summa reddatur, ipsum a quo denominatur ultimus in quem desinis, multiplica in se ipsum et efficies summam. <**5.2**> Verbi gratia, 1, 3, 5, 7, 9. Ultimus est novem et est quintus in ordine. 5 vero[22] a quo denominatur multiplicetur in /9rb/ se et efficies 25. Quinquies enim quinque 25 fiunt[23] et hec erit[24] summa predictorum. <**5.3**> Vel a proximo pari qui est post imparem in quo desinis, medietatem subtrahe, ablatam per se multiplica et habebis summam. <**5.4**> Si vero non ab unitate inicipiens <continue numeraveris>, acsi ab unitate incepisses agendum est et omnes impares qui sub eo sunt a quo incepisti coacerves; illud coacervatum a producto

[15] M omits

[16] E continues with an alternative version of the previous sentences: vel a proximo pare qui est post imparem in quo desinis medietatem subtrahe ab altero per se multiplica et habebis summam. Si vero non ab unitate incipies, acsi ab unitate incepisses agendum est et omnes impares qui sunt sub eo a quo incepisti illum coacervatum a producto residuum erit summa.

[17] E adds 'autem'

[18] continue E, continuo PN

[19] E adds 'est' superscript

[20] illum E

[21] E adds heading 'De imparibus'

[22] ergo E

[23] efficiunt E

[24] erat P

subtrahendum; residuum erit summa.[25]

<**6.1**> Si autem ex aggregatis quotlibet duplicibus a primo per ordinem[26] se naturaliter sequentibus scire volueris quanta summa reddatur, ultimum in quem desinere volueris, multiplica per primum parem, scilicet binarium, et que summa ex eorum multiplicatione provenerit,[27] subtracto primo pari, illa eadem ex predictorum aggregatione colligitur. <**6.2**> Verbi gratia, 2, 4, 8, 16, 32. Ultimum qui est 32 multiplica per binarium et efficies 64 et subtrahe ipsum[28] a quo inceperis et quod fiet hec est[29] summa predictorum, scilicet 62. <**6.3**> Vel aliter: quotlibet duplicium a primo naturaliter se sequentium ultimus duplatus est summa omnium aggregatorum, subtracto primo pari. <**6.4**> Vel aliter generalius: ex quorumlibet duplorum aggregatione si volueris scire quanta summa proveniat, duplica extremum in quem desieris et subtrahe primum a quo inceperis et quod fit ex duplatione ultimi cum subtractione primi, hec est summa quam requiris.[30] <**6.5**> Vel si ab uno incipere volueris, duplicationi ultimi ipsum addere debebis et precedentium summam habebis.

<**6.6**> Si ex quibuslibet aggregatis triplis vis scire quanta summa reddatur, ultimum in quem desieris divide et minorem eius partem triplica et quod ex eius triplicatione provenerit, precedentium summa erit. <**6.7**> Verbi gratia, 3, 9, 27, 81, 243. Ultimi medietas minor 121, hec triplicata reddit 363, quam summam aggregati precedentes tripli reddunt.

<**7.1**> Si volueris[31] scire ex aggregatione omnium quadratorum quorumlibet numerorum ab uno naturaliter se sequentium, quanta summa proveniat,[32] vide prius ex aggregatione ipsorum numerorum quanta summa provenit. Deinde duas tertias numeri, quot ipsi sunt naturaliter se sequentes,[33] addita tertia parte unius, multiplica in predictam summam et quod inde provenerit, summa aggregationis quadratorum ipsorum[34] numerorum erit. <**7.2**> Verbi gratia, sint numeri naturaliter se sequentes isti quattuor, scilicet i, ii, iii, 4, quorum aggregationis summa est 10. Quadrati vero ipsorum sunt 4, scilicet unum et 4 et 9 et 16. Si ergo vis scire ex aggregatione istorum quadratorum quanta summa proveniat, duas tertias partes quaternarii, quot scilicet

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[25] E omits 5.3-4

[26] E omits 'per ordinem'

[27] efficitur E

[28] illud E

[29] erit E

[30] queris E

[31] vis E^2

[32] *provenit* E^2

[33] numeri quot...sequentes] ipsorum numerorum naturaliter se sequentium E^2

[34] E^2 omits

sunt,[35] que sunt 3 minus tertia, multiplica in predictam summam numerorum que erant[36] 10 et cum additione tertie partis unius fiunt[37] 30. Et hec summa provenit si predictos quadratos aggreges, scilicet 1, 4, 9, 16.

<**8.1**>[38] Omnes numeri equa differentia se vincentes,[39] si fuerint in impari numero, tantum reddit medius in se duplatus quantum extremi sibi aggregati et extremi extremorum, ut 2, 4, 6, 8, 10, 12, 14.[40] <**8.2**> Omnes numeri equa proportione a se distantes, si fuerint impares numeri, tantum reddit medius in se multiplicatus quantum duo circumpositi et circumpositorum circumpositi in se ducti usque ad unitatem, ut 2, 4, 8, 16, 32. <**8.3**> Si vero pares fuerint, tantum duo medii in se ducti quantum duo extremi et extremi extremorum in se ducti, ut 2, 4, 8, 16, 32, 64, 128, 256. <**8.4**> Si vero pares[41] fuerint, tantum efficiunt duo medii sibi aggregati quantum extremi et extremi extremorum usque ad unitatem, ut 2, 4, 6, 8.

De multiplicatione digitorum in se[42]

<**9.1**> Omnis[43] numerus infra denarium multiplicatus in se ipsum reddit summam sue[44] denominationis decuplate, subtracta inde multiplicatione differentie ipsius ad denarium facta in se ipsum. <**9.2**> Verbi gratia, sexies sex dicantur fieri 60, que est denominatio a sex decuplata. Differentia autem senarii ad denarium est quaternarius, qui multiplicatus in sex facit 24. /9va/ His ergo 24 de sexaginta subtractis remanent 36 quam summam reddunt sexies sex. Et in omnibus sic ab uno usque ad decem.[45] <**9.3**> Si autem maiorem per minorem vel econverso multiplicare volueris, differentiam maioris ad denarium multiplica in minorem et ipsam multiplicationem subtrahe a denominatione facta a minore et quod remanserit est summa que provenit ex multiplicatione diversorum numerorum. <**9.4**> Verbi gratia, cum multiplicaveris quinquies 7, dic fieri quinquaginta, que est denominatio a quinque qui erat ibi minor numerus. Set differentia maioris numeri, scilicet 7, ad denarium sunt tres. Qui tres multiplicati in minorem numerum, scilicet 5, fiunt 15. His ergo subtractis de 50 remanent 35 et hec est summa quam faciunt quinquies 7 vel econverso. <**9.5**> Vel

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[35] E^2 omits

[36] erat P

[37] fient E^2

[38] E adds heading: 'De equa proportione' and puts 8.1 after 8.2-3

[39] iungentes E

[40] 12, 14] 20, 40 E

[41] impares *MSS.*

[42] De multiplicatione digitorum in se] Item E

[43] N adds 'namque'

[44] due P, sue M

[45] P adds a triangle of dots. N adds 'vel aliter'. E adds 'Item alio modo'

aliter: denominatio fiat a maiore et differentia minoris ad denarium multiplicetur in maiorem et subtrahatur de summa prime denominationis. <**9.6**> Verbi gratia, quinquies 7 dicatur fieri 70 qui denominatur a 7 et est maior in illa multiplicatione. Set[46] differentia minoris, scilicet quinarii, ad denarium, est 5 qui multiplicatus in maiorem, scilicet 7, facit 35. His subtractis a 70 que erat decuplata denominatio a maiore, scilicet septenario, remanent .35. et hanc summam reddunt multiplicati alter per alterum, quinquies 7 vel econverso septies 5.

<**10.1**> Omnium ergo trium numerorum eiusdem proportionis, si multiplicaveris[47] primum in tertium, tantum provenerit[48] ex multiplicatione eorum quantum ex ductu solius medii in se. Quorum si prepositis[49] primo et medio, tertius tantum fuerit incognitus, multiplica medium in se et quod inde provenerit divide per primum et quod exierit de divisione erit tertius. <**10.2**> Aut si primus tantum fuerit incognitus, multiplica medium in se et divide per tertium et exibit primus. <**10.3**> Aut si medius tantum fuerit incognitus, multiplica primum in tertium et radix eius quod inde provenerit est medius, quoniam medius in se ductus tantum reddit quantum duo extremi, quod in prepositis numeris facile notabis.

<**11.1**> Si ergo aliqui quattuor numeri fuerint proportionales, scilicet ut quomodo se habet primus ad secundum, sic se habet[50] tertius ad quartum, tunc tantum proveniet ex multiplicatione primi in quartum quantum ex multiplicatione secundi in tertium. In his autem 4 terminis socii sunt primus et quartus, secundus et tertius. Unde generaliter: ex omnibus qualiscumque ignoretur unumquemlibet reliquorum duorum per socium ignorati divide et quod exierit per socium dividendi[51] multiplica et proveniet ignoratus terminus. Item si[52] aliquis eorum ignoratur productus ex aliis duobus per ignorati socium dividatur et proveniet incognitus.[53]

<**11.2**> Unde si prepositis[54] tribus quartus fuerit tantum incognitus, multiplica secundum in tertium et quod inde provenerit divide per primum et quod exierit erit quartus. <**11.3**> Aut si primus tantum fuerit incognitus, multiplica secundum in tertium et divide per quartum et exibit primus. <**11.4**> Aut si secundus tantum

[46]set ME, scilicet PN

[47]multiplicabis E

[48]provenit E

[49]propositis E

[50]habeat E

[51]dividendi *scripsi*, dividentis PNE

[52]Item si] Item P, Item \si/ N, Vel cum E

[53]In P these two sentences are reversed, but an inserted (a) and (b) indicate that the order should changed. (b) is missing in N. E has the correct order, followed by our edition.

[54]propositis E

fuerit incognitus, multiplica primum in quartum et divide per tertium et exibit secundus. <**11.5**> Aut si tertius fuerit incognitus, multiplica[55] primum in quartum et divide per secundum et exibit tertius.

<**11.6**> Verbi gratia, ut si 10 modii vendantur pro xxx aureis, tunc pro duobus modiis sex aurei debentur. Hic quattuor numeri sunt proportionales, scilicet 10 modii, 30 aurei, duo modii, sex aurei.[56] Que enim proportio est decem modiorum ad 30 aureos, quod est pretium eorum, eadem proportio est duorum modiorum ad sex aureos quod est pretium eorum. Cum ergo multiplicaveris primum numerum qui est 10 modii in quartum numerum qui est sex morab<otini> proveniunt inde 60. Tantum similiter proveniet ex multiplicatione secundi numeri qui est 30 morab<otini> in tertium numerum qui est duo modii. /9vb/

<**11.7**> Cum ergo aliquis occultans tibi quartum numerum qui est sex aurei, dicat: 'Cum decem modii vendantur pro 30 aureis, quantum debeatur pro duobus?' Multiplica tunc 30 morabo<tinos>,[57] qui est secundus numerus, in duos modios qui est tertius numerus et quod provenerit divide per decem modios qui est primus numerus et exibit quartus,[58] scilicet[59] sex aurei, qui debentur pro duobus. <**11.8**> Similiter si occultans primum[60] qui est decem modii, dicat: 'Duo modii venduntur pro sex aureis, quot modii habebuntur[61] pro 30?' Multiplica tunc secundum numerum qui est 30 aurei in duos modios qui est tertius numerus et quod inde provenerit divide per sex qui est quartus numerus et exibit primus, qui est scilicet 10[62] quot dantur pro 30. <**11.9**> Similiter si occultans secundum numerum qui est 30 morab<otini>[63] dicat: 'Postquam pro duobus modiis habui sex aureos, quot morab<otinos> habeo pro decem modiis?' Multiplica tunc primum numerum qui est decem modii, in quartum numerum qui est sex aurei et quod inde provenerit divide per duos modios qui est tertius numerus et exibit secundus numerus, scilicet 30 morab<otini>,[64] qui debentur pro decem modiis. <**11.10**> Similiter si occultans tertium qui est duo modii, dicat: 'Postquam decem modii dantur pro 30 aureis, pro sex aureis quot modios

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[55]multiplicam P

[56]The four values are numbered 1 to 4 superscript in PN, though N puts the '4' on the line of writing.

[57]aureos E

[58]E adds 'numerus'

[59]scilicet] qui est E

[60]E adds 'numerum'

[61]habentur E

[62]E adds 'modii'

[63]aurei E

[64]scilicet 30 morabotini] qui est 30 aurei E

dabunt[65] ?' Multiplica tunc primum numerum qui est decem modii in quartum numerum qui est sex aurei et divide per secundum qui est 30 morab<otini> et exibit tertius numerus qui est duo modii. <**11.11**> In his autem interrogationibus summopere notandum est quid primum dicatur et quid secundum, scilicet si res prius[66] nominatur aut pretium. Quicquid enim prius dicitur, illud tertio loco repetitur et quod secundo nominatur illud est quartum, sive sit dictum sive occultatum.

<**12.1**> Si autem tertius et quartus ignorantur[67] set eorum aggregatio tibi sola proponitur, cum eos invenire volueris, aggrega primum et secundum et eorum aggregatio respectu prime proposite sit tibi secunda. Deinde multiplica primum numerum in aggregationem primam et quod inde provenit[68] divide per aggregationem secundam et quod de divisione exierit[69] erit numerus tertius. <**12.2**> Similiter ad inveniendum quartum, multiplica numerum secundum in aggregationem primam et quod inde provenerit divide per secundam et quod exierit erit quartus.[70] <**12.3**> Similiter etiam econverso: si ignoraveris primum et secundum, [set] eorum aggregatione cognita, invenies eos per tertium et quartum secundum predictam regulam. <**12.4**> Verbi gratia, sint quattuor numeri, primus duo, secundus quattuor,[71] tertius tres, quartus 6. Si ergo tertius et quartus ignorantur, scilicet tres et sex, set eorum aggregatio tibi ostenditur que est novem, cum eos invenire volueris, aggrega primum et secundum, scilicet duo et quattuor et efficies sex. Deinde multiplica primum qui est duo in aggregationem primam que fuit novem et fient decem et octo, quos divide per aggregationem secundam que est sex et exibit tertius qui est tres. Similiter[72] de aliis, ut subiecta figura declarat:

6 3	4 2
9	6

<**12.5**> Si vero tertius et quartus ignorantur, set quod ex diminutione minoris eorum ex maiore remanet tibi proponitur, si eos invenire[73] volueris, diminue primum de secundo vel econverso, semper[74] minorem de maiore, et quod remanserit respectu prime proposite voca diminutionem secundam. Postea multiplica primum numerum in diminutionem primam et quod inde provenerit divide per diminutionem secun-

[65] dabunt E, dabuntur PN

[66] primum E

[67] P first wrote 'occultantur' and then underscored it

[68] provenerit E

[69] exibit E

[70] 'Similiter...quartus' added in the margin by the same hand in P, with a triangle of dots to connect it to the right position in the text.

[71] quarto PN

[72] E adds 'etiam'

[73] aggregare PNE

[74] scilicet PNE

dam et quod exierit erit numerus tertius. <**12.6**> Similiter ad quartum inveniendum, multiplica numerum secundum in diminutionem primam et quod provenerit divide per denominationem secundam et quod exierit erit numerus quartus. <**12.7**> Similiter etiam econverso: si ignoraveris primum et secundum, ostensa tibi eorum diminutione invenies eos per tertium et quartum secundum predictam regulam, ut in predictis numeris patet. <**12.8**> Si enim ignoraveris tertium et quartum, scilicet tres et sex, set quod remanet ex diminutione alterius ex altero tibi ostenditur, quod est tres, tu minue primum de secundo vel econverso, semper minorem de maiore, et remanent duo que est diminutio secunda. Deinde multiplica primum in diminutionem primam et quod inde provenerit divide per diminutionem secundam et quod inde exierit erit numerus tertius. Similiter etiam in aliis ut subiecta figura declarat.

6 3	4 2
3	2

<**12.9**> Similiter etiam hoc idem invenies, si re/10ra/bus et earum pretiis hoc ipsum adaptes.[75]

<**12.10**> Si vero tertius et quartus tantum fuerint tibi incogniti, set eorum multiplicatio sola tibi proponitur, tu multiplica primum[76] in ipsam multiplicationem et quod inde provenerit divide per secundum[77] et radix eius quod de divisione exierit erit tertius. <**12.11**> Similiter ad inveniendum quartum, multiplica secundum in ipsam multiplicationem et quod inde provenerit divide per primum et radix eius quod de divisione exierit, erit quartus. <**12.12**> Verbi gratia, si predictorum numerorum tertium et quartum, scilicet tres et sex, ignoraveris, set eorum multiplicatio tibi sola proponitur, que est decem et octo, tunc multiplica primum qui est duo in ipsam multiplicationem et provenient 36.[78] Quos 36 divide per secundum qui est quattuor et exibunt de divisione novem, cuius radix, scilicet ternarius, est tertius. <**12.13**> Similiter ad inveniendum quartum, multiplica secundum qui est quattuor in ipsam multiplicationem et provenient 72.[79] Quos divide per primum et exibunt 36 cuius radix est senarius et hic est quartus, ut subiecta figura declarat.

6 3	4 2
18	8

<**12.14**> Si vero duo medii fuerint incogniti, set eorum aggregatio tibi sola proponitur, tu multiplica primum in quartum et quod inde provenerit pone per se. Deinde divide ipsam aggregationem in tales duas partes quarum altera multiplicata in alteram reddat multiplicationem primi in quartum et ipse partes erunt medii incogniti.

[75] E omits 12.9

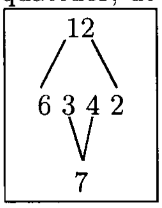
[76] E adds 'numerum'

[77] secundam PNE

[78] 36 is placed in a box in PN

[79] 72 is placed in a box in PN.

<**12.15**> Verbi gratia, ut si quattuor et tres ignores, set eorum aggregatio tibi proponitur, scilicet septem, multiplica primum qui est duo in quartum qui est sex et efficies 12. Deinde propositam aggregationem que est septem divide in tales partes que in se multiplicate efficiant[80] 12 et ipse erunt medii numeri incogniti, tres igitur et quattuor, ut subiecta figura declarat.[81]



gradus	minuta	secunda	tertia	quarta	quinta	sexta	septima	octava	nona
minuta	secunda	tertia	quarta	quinta	sexta	septima	octava	nona	dec
secunda	tertia	quarta	quinta	sexta	septima	octava	nona	dec	undec
tertia	quarta	quinta	sexta	septima	octava	nona	dec	undec	duodec
quarta	quinta	sexta	septima	octava	nona	dec	undec	duodec	tertiad
quinta	sexta	septima	octava	nona	dec	undec	duodec	tertiad	quartad
sexta	septima	octava	nona	dec	undec	duodec	tertiad	quartad	quintad
septima	octava	nona	dec	undec	duodec	tertiad	quartad	quintad	sextad
octava	nona	dec	undec	duodec	tertiad	quartad	quintad	sextad	septimad
nona	dec	undec	duodec	tertiad	quartad	quintad	sextad	septimad	octavad.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

B

<**13.1**> /10rb/ Omnis numerus usque ad decem in se ductus tantum reddit quantum primi duo circumpositi in se ducti, si dempta unitate de multiplicatione medii. <**13.2**> Vel aliter generalius: omnis numerus in se ductus tantum reddit quantum duo extremi et extremi extremorum usque ad unitatem, set adiectis differentiis in

[80]efficiunt E

[81]E continues with a second copy of 7.1-2.

se ductis, quas habet medius ad extremos.[82] <**13.3**> Tantum enim reddit 5 in se ductus quantum quater sex cum differentiis in se ductis que sunt due unitates, et quantum ter septem cum differentiis in se ductis que sunt duo binarii, et quantum bis 8 cum ductis in se differentiis que sunt duo ternarii, et sic usque ad unum. <**13.4**> Omnis numerus tantum reddit in se ductus quantum duo in se[83] ducti equa proportione ab illa distantes.[84]

<**14.1**> Omnis numerus in se ductus tantum efficit quantum due partes eius, si utraque in se ducatur et altera in alteram bis. <**14.2**> Omnis numerus ductus in alium tantum reddit quantum ductus in omnes partes eius. <**14.3**> Cum aliquis numerus multiplicat alium, tantum provenit quantum si idem multiplicet limitem, subtracto eo de summa quod differentia multiplicati ad limitem, ducta per multiplicantem efficit.

<**15.1**> Omnis numerus per alium dividendus aut est ei equalis aut maior aut minor. Si est equalis, tunc unicuique dividentium singule unitates dividendi proveniunt. <**15.2**> Si vero maior, tunc quotiens dividens in dividendo fiunt (sic!), tot integri unicuique dividentium proveniunt. <**15.3**> Si vero aliud superfuerit, per fractiones dividendum erit. <**15.4**> Ita ut quota vel quote partes minor numerus est maioris, tota pars vel partes unicuique dividentium proveniunt. <**15.5**> Cum fractiones fractionibus aggregas, si idem est numerus fractionum et denominationis earum, tunc ex aggregatione integer surgit. <**15.6**> Ut ex tribus tertiis vel quattuor quartis unum integrum redditur. <**15.7**> Si vero minor est numerus fractionum numero denominationis, tunc qua proportione se habet numerus fractionum ad numerum denominationis earum, eadem proportione habent se fractiones ille ad integrum, <**15.8**> ut sex duodecime sic se habent ad integrum ut senarius ad duodenarium. Sunt ergo eius medietas. <**15.9**> Si vero maior fuerit, tunc quotiens maior fuerit tot integra fractiones aggregate constituent. <**15.10**> Ut sex tertie duo integra restituunt, quoniam numerus fractionis bis continet numerum denominationis. <**15.11**> Si vero continet eum aliquotiens et insuper aliquam[85] vel aliquas eius partes, tunc quotiens eum continet, tot integra constituunt fractiones aggregate et insuper totam vel totas partes unius integri quota vel quote partes est numerus ille qui superest numeri denumerantis fractioness. <**15.12**> Ut est octo tertie, octonarius bis continet ternarium et eius duas tertias.

<**15.13**> Si scire volueris quomodo pars partis cuiuslibet se habet ad integrum, numeros a quibus denominantur fractiones in se multiplica et qualiter unitas se

[82]extrema E

[83]E breaks off here, leaving the rest of the page blank.

[84]distante s P

[85]aliqua P

habuerit ad summam illam, sic pars partis habebit se ad integrum. <**15.14**> Ut tertia pars unius quarte, duodecima est unius integri. Nam ter quattuor duodecim fiunt. <**15.15**> Si scire volueris partes partis cuiuslibet quomodo se habent ad integrum, numeros a quibus denominantur fractiones in se multiplica et qualiter numerus coacervans[86] habet se ad numerum iam productum, sic fractiones ille aggregate ad integrum. <**15.16**> Ut due tertie partes unius quarte, sexta pars sunt unius integri. Nam ter 4 12 sunt, cuius sexta pars sunt (*sic*) binarius.

<**15.17**> Si vero a duobus diversis numeris denominantur fractiones, tunc qua proportione maior numerus se habet ad minorem, taliter fractio denominata a minori habet se ad fractionem denominatam a maiore. <**15.18**> Ut tertia pars alicuius continet duas sextas eius. Nam senarius continet duos ternarios. <**15.19**> Si diversorum numerorum vel diversarum quantitatum fractiones denominantur ab eodem numero, sicut integra habent se ad invicem, sic et fractiones et econverso. <**15.20**> Nam sicut duodenarius habet se ad novenarium, sic tertia pars duodenarii ad tertiam novenarii et econverso. <**15.21**> Sin autem fractiones quot/l0va/cumque a diversis nominibus denominatas coacervare volueris, numeros a quibus denominantur fractiones coacerva et per summam inde provenientem coacerva fractionem denominatam a numero qui surgit ex multiplicatione numerorum fractiones denominantium. <**15.22**> Nam si scire volueris tertia pars et quarta aggregate quid efficiant, numeros a quibus denominantur fractiones, scilicet ternarium et quaternarium, aggrega et fiunt 7. Quo septenario coacerva[87] fractiones denominatas a numero qui surgit ex multiplicatione numerorum fractiones denominantium, scilicet duodecim. Nam ter 4 12 fiunt. Sunt igitur tertia et quarta pars alicuius septem duodecime partes, que coacervate quid constituant superius ostensum est.[88]

<**16.1**> ...Vel ipsum per se multiplica et multiplicationi inde provenienti ipsum adde et hoc totum in duo equa divide et illa medietas est tota summa illius et omnium infra se ipsum contentorum. <**16.2**> Si ex numeris equa differentia se vincentibus sibi aggregatis vis scire quanta summa reddatur, si in impari numero coacervandi fuerint, vide quot sint et toto numero medium aggregandorum multiplica et tota est summa. <**16.3**> Verbi gratia, sint tres, 4, 5 aggregandi. Ternario igitur quot ipsi sunt medium eorum, scilicet[89] 4, multiplica et tanta erit summa quam reddunt prepositi numeri aggregati. <**16.4**> Vel sint tres, 5, 7, 9, 11 aggregandi. Quinario igitur, tot enim sunt aggregandi, medium eorum, scilicet septem, multiplica et tanta erit summa prepositorum aggregatorum. <**16.5**> Vel sint 2, 5, 8 aggregandi. Ternario igitur medium eorum, scilicet 5, multiplica et tanta erit summa illorum. <**16.6**>

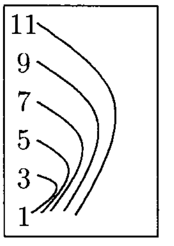
[86] coacervans N, 'coacreveras' corrected from 'coacerveris' P

[87] coacreva P

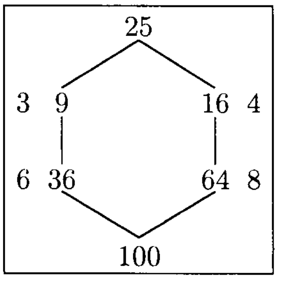
[88] This is followed in P by a siglum (a fancy equals sign), whose significance is not clear.

[89] id est N

Sin autem pari numero sunt aggregandi, sub eadem proportionalitate unum maiorem prioribus adiunge. Deinde vide quot sint et toto numero medium eorum multiplica et ablato illo quem illis aggregandis adiunxeras summa aggregatorum efficitur.



<16.7> Ut sint 2, 4, 6, 8 aggregandi. Proportionalitatis illius proximum, id est x, adiunge. Quinario itaque, quia tot sunt aggregandi, medium eorum, scilicet senarium, multiplica eruntque 30. Huic itaque summe denarium quem adiunxeras aufer et 20 qui remanent summa sunt aggregatorum. **<16.8>** Si impar quilibet cum omnibus imparibus sibi subpositis et unitate aggregatur summa que excrescit numerus quadratus erit.



<17.1> Si duorum numerorum quadrata pariter accepta fuerint numerus quadratus, necesse est quorumlibet numerorum duorum eadem proportionalitate ad se relativorum quadrata pariter accepta, numerum quadratum esse. **<17.2>** Si ad quantitatem aliquam quantitates quotlibet proportionentur diversis set notis proportionibus, que pariter accepte summam notam faciunt, eadem prima quantitas quanta sit invenire. **<17.3>** Verbi gratia, sit a quantitas sitque etiam ut b et c et d proportionibus notis ad ipsum a proportionentur. Sit quoque ut b et c et d[90] pariter accepte quantitatem componant g, que g quanta sit notum sit. Proponitur itaque ut etiam quanta a fuerit inveniatur. **<17.4>** Ut si proponatur Socrates bis tot nummos habere quot Plato et insuper eorum quos Plato habet duas tertias partes, totumque quod Socrates habet simul acceptum quindecim nummos esse. Proponitur itaque ut quantum Plato habeat inveniatur. Sumo itaque numerum ad quem duo numeri predictis proportionibus proportionantur. Hic[91] autem ternarius est. Senarius enim bis ipso maior est et binarius eiusdem due tertie partes. Hos itaque duos numeros, senarium et binarium, coniungo et fiunt 8. Considerato igitur qua proportione 15 ad 8 se habeat, eadem proportione Platonem ad tres nummos se habere pronuntio. Quindecim enim continet totum octo et eius septem octavas. Similiter quinque nummi et obolus et quarta pars oboli tres nummos et[92] eorum septem octavas continent. Dico itaque Plato 5 nummos et obolum[93] et quartam partem oboli habere. Bis enim tantum et eiusdem due tertie partes 15 nummi sunt. Si enim quinque nummos et obolum[94] et quartam partem oboli bis sumpseris xi nummi et quarta pars unius nummi erunt. Eorumdem etiam quinque nummorum et

[90] *a* PN

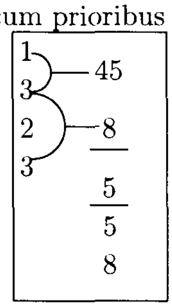
[91] his P, hiis N

[92] et *bis* P

[93] obulum P

[94] obulum P

oboli et quarte partis oboli, si duas tertias sumpseris, tres nummos et tres quartas unius facient. **<17.5>** Quod sic accipe: novem obolorum duas tertias[95] partes accipe, id est vi obolos, et sunt tres nummi. Reliquus vero nummus et quarta pars /10vb/ oboli novem octave partes unius sunt nummi. Quoniam due tertie partes, id est sex octave nummi, tres quarte partes unius nummi sunt, quas si cum prioribus undecim nummis et quarta parte nummi et tribus nummis iunxeris, 15 nummi sunt. **<17.6>** Vel aliter facilius: multiplica numerum denominationis partium, scilicet tres, in 15 et proveniunt 45. Deinde eundem ternarium multiplica in <duo et> duas tertias et proveniunt octo. Nam tres in duo fiunt 6 et additis duabus tertiis fiunt octo. Postea divide primum productum, scilicet 45, per ultimum productum, scilicet 8, et exeunt 5 et quinque octave, hoc modo:



<17.7> Cum queritur quot minora sint in aliquot maioribus, per numerum minorum qui sunt in uno maiorum maiora multiplices et numerus qui excrescet quot minora sint in tot maioribus ostendet. **<17.8>** Si vero quot maiora in aliquot minoribus queritur, per numerum minorum qui sunt in uno maiorum minora divides et numerus qui de divisione exibit quot maiora sint in tot minoribus ostendet.[96] **<17.9>** Verbi gratia, solidus minus est quam libra. Ergo si queritur quot solidi sint in c libris, quere quot minora sint in maiori aliquo. Per numerum itaque minorum, id est solidorum, qui sunt in uno maiorum, id est libra una, id est per xx, viginti enim solidi faciunt libram unam, numerum maiorum, id est librarum, videlicet centenarium, multiplices et sunt 2.000. Scito igitur quia tot solidi, videlicet 2.000, sunt in c libris. **<17.10>** Rursus sit questio quot libre sint in 24[97] milibus nummorum. Queritur ergo quot maiora sint in aliquot minoribus, eo quod nummi minus sunt quam libre. Per numerum itaque minorum, id est nummorum, qui sunt in uno maiorum, id est in libra una, scilicet per ccxl, tot enim denarii sunt in una libra, numerum minor<um>, id est denariorum, videlicet 24.000,[98] divides et exibunt inde 100. Scito ergo quia tot libre, id est c, sunt in 24.000[99] nummorum.

<17.11> Cognita summa quotarumlibet partium alicuius totius, totum ipsum invenire. Primo numeros partes propositas denominantes aggrega. Deinde alterum per alterum multiplica et sic habebis quattuor proposita; summam scilicet partium propositarum, summam numerorum partes denominantium, et aggregatum ex eis-

[95] tertias *bis* P

[96] 'Si vero quot maiora in aliquot minoribus queritur, per numerum minorum qui sunt in uno maiorum minora divides et numerus qui de divisione exibit quot maiora sint in tot minoribus ostendet' repeated in PN.

[97] 34 PN

[98] 34,000 PN

[99] 34,000 PN

dem, quartum est totum quod ignoratur. Nam que proportio est totius ad summam partium propositarum, eadem est producti ex numeris partes denominantibus ad totum aggregatum ex eisdem. Multiplicetur ergo summa partium propositarum per productum ex numeris eas denominantibus et productus inde [7][100] dividatur per aggregatum ex easdem denominantibus et exibit totum quod ignoratur, per precedentem regulam quattuor numerorum[101] proportionalium: si primus fuerit[102] incognitus, multiplica secundum in tertium et divide per quartum; exibit primus.

<**17.12**> Verbi gratia, sint tertia et quarta peccunie mee 20 nummi. Proponitur ergo invenire quota sit summa peccunie. Aggregatis ergo denominantibus, scilicet ternario et quaternario, fiunt septem. Eorumdem alterum per alterum multiplica et productus erit 12. Habes ergo quattuor proposita. Multiplica ergo secundum per tertium primo, id est 20 per 12, et producentur 240. Hoc ergo divide per 7 et exeunt 34 et due septime, qui numerus in eius generis rebus constituendus est, nummis scilicet vel solidis vel libris, in quo partes proposite fuerunt et hec est summa quesita.[103]

20 12

240

7

34

$\frac{2}{7}$

<**17.13**> Ex tribus numeris proportionalibus si primus ducatur in tertium exibit quadratus medii.

<**17.14**> Pluribus hominibus diversas summas peccunie ad lucrandum simul conferentibus, si ex lucro quod ex tota collectione provenerit vis scire quanta sors unumquemque eorum iure contingit, portiones quas apposuerunt aggrega et portionem cuius volueris in summam lucri multiplica. Deinde quod ex multiplicatio/11ra/ne fit, per aggregatum divide et quod ex divisione exierit, hec illius cuius sortem multiplicasti pars erit. <**17.15**> Vel econverso: divide sortem per aggregatum et quod exierit in summam lucri multiplica et quod inde provenerit, hec ipsius sors erit. Similiter de ceteris singillatim facies. <**17.16**> Verbi gratia, tres mercatores peccuniam ad lucrandum contulerunt, unus sex solidos, alius 8, alius 12 qui omnes fiunt 26. Ex his lucrati sunt lx. Si vis ergo scire quantum de lucro contingat quemque secundum quantitatem collate peccunie, partes omnium simul aggrega et fiunt 26. Deinde multiplica per se unamquamque partem quam quisque contulit in summam lucri. Deinde divide id quod ex multiplicatione provenit in summam collati capitalis, scilicet 26, et quod de divisione exierit, hoc est quod debetur illi cuius sortem multiplicasti. Sic facies de unoquoque per se. Sunt ergo hi quattuor

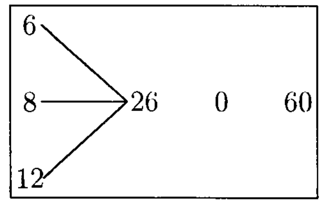
[100] inde 7] in.30127. N

[101] numerorum] numerum eorum PN

[102] fuit P

[103] NP add (P *in marg.*): Hic numerus continet totum 20 et insuper eius 5 septimas sicut duodecim septem.

numeri, scilicet sors cuiuslibet et 26, tertius est incognitus, quartus 60, qui sunt proportionales per suprapositam regulam. Multiplica igitur primum in quartum, id est partem cuiuslibet in 60 et productum divide per secundum, scilicet 26, et exibit tertius, scilicet sors que contingit eum cuius partem posuisti primum terminum.



<17.17> Si vis scire de aliqua certa summa multis hominibus debita quantum proveniat aliquibus illorum, numerum ipsorum aliquorum de quibus vis scire in ipsam summam multiplica et quod ex multiplicatione provenerit in totum numerum multorum divide et quod provenerit, hoc est quod debetur illis. **<17.18>** Verbi gratia, vigintiquattuor nummi debentur octo hominibus et vis scire quantum proveniat tribus illorum. Multiplica ergo ipsos tres in 24 et quod ex multiplicatione provenerit, divide per 8 et videbis quid proveniet illis. Et secundum hanc regulam similiter probabis quantum proveniat aliis quinque, ipsos scilicet quinque multiplicando per 24 et quod ex eorum multiplicatione provenerit dividendo per octo. Sunt igitur hi tres termini 8, 24, 3, quartus ignoratur, scilicet quantum debetur tribus. Per precedentem ergo regulam multiplica secundum in tertium et productum inde divide per primum, scilicet 8, et exibit quartus incognitus, scilicet pretium trium. Sicut enim octo habet se ad tres, sic 24 ad pretium trium. Continet ergo illud bis et insuper eius duas tertias partes.

8	24	3	0

C

<18.1> Exceptiones de libro qui dicitur gebla mucabala. Fit hic quedam trimembris[104] divisio per opposita. Quia queritur[105] aut que res[106] cum totiens[107] radice sua efficiat numerum, aut que res[108] cum tali[109] numero efficiat totiens[110] radicem, aut que totiens radix cum tali numero efficiat rem[111] .

<18.2> Queritur ergo que res cum x radicibus[112] suis, id est decies accepta[113] radice

[104] trimenbris P

[105] E adds 'ut'

[106] que res] queres N, quadratus E

[107] E omits

[108] que res] queres N, quadratus E

[109] E omits

[110] E omits

[111] aut...rem] aut radix cum numero efficiat quadratum E

[112] que res...radicibus] quis quadratus est qui cum radicibus decem E

[113] multiplicata E

sua, efficiat 39.[114] Ad hoc inveniendum medietatem radicum prenominatarum multiplica in se et quod inde provenerit adde priori numero et eius quod inde excreverit[115] accipe radicem et de ipsa radice minue[116] medietatem radicum prenominatarum et quod inde remanserit est radix rei,[117] quam si in se multiplicaveris, res provenit quam[118] queris. <**18.3**> Verbi gratia, quoniam superius decem radices fuerant[119] proposite, medietatem earum que est 5 si in se multiplicaveris, 25 efficis.[120] Quos adde predicto numero qui est[121] 39 et efficies[122] 64, cuius radix est octo. De qua radice, scilicet 8, si minueris[123] medietatem radicum[124] que est quinque, remanent tres, qui sunt radix rei,[125] scilicet novenarii, qui cum x radicibus suis, id est x ternariis, efficit[126] 39. Ergo novenarius est res que[127] queritur.

<**18.4**> Item que est res que[128] cum novem sibi additis efficit sex radices[129] sui?[130] Ad hoc inveniendum medietatem[131] radicum multiplica in se et ex eo quod inde provenerit numerum predictum diminue[132] et eius quod[133] remanserit radicem de medietate radicum minue et quod remanserit erit radix rei quam[134] queris. <**18.5**> Verbi gratia, sex radices fuerunt proposite, quarum medietatem que est[135] tres in se multiplica et efficies 9, de quibus 9 predictum numerum, scilicet 9, minue et remanet nichil. Et huius quod remanet, scilicet nichil, /11rb/ radicem que similiter

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[114]R<espondetur> E

[115]provenerit E

[116]diminue E

[117]quadrati E

[118]res provenit quam] quadratus provenit quem E

[119]fuerunt E

[120]efficies E

[121]predicto numero qui est] prenominate summe scilicet E

[122]efficis E

[123]minuis E

[124]radicis E

[125]quadrati E

[126]efficit E] efficis PN

[127]quadratus qui E

[128]Item...que] Similiter queritur quis est quadratus qui E

[129]sex radices] sexies radicem E

[130]R<espondetur> E

[131]medietatem *bis* PN

[132]minue E

[133]quod E] que PN

[134]quadrati quem E

[135]sunt E

est nichil de medietate radicum que est[136] tres minue et quia de tribus nichil minuisti, remanent tres qui sunt radix rei quam[137] queris, scilicet novenarii, qui[138] cum novem sibi additis fit 18, qui sunt sex radices novenarii, id est sexies tres, qui ternarius est[139] radix novenarii.[140]

<**18.6**> Item[141] que sunt[142] radices que cum 4 sibi additis efficiunt rem suam?[143] Ad hoc inveniendum multiplica medietatem radicum in se et quod inde provenerit adde predicto numero et radicem eius quod inde excreverit,[144] adde medietati radicum et quod inde excreverit[145] est radix rei quam[146] queris. <**18.7**> Verbi gratia, tres radices fuerunt preposite[147] quarum medietas est unum et dimidium, que multiplicata in se efficiunt duo et quartam, que adde priori numero qui est 4 et efficies sex et quartam. Cuius radix est duo et dimidium, quam radicem[148] adde medietati radicum, que est unum et dimidium et fient 4, qui sunt radix rei quam[149] queris, scilicet sedecim.[150] Quos sedecim[151] efficiunt tres radices sui, id est tres quaternarii vel ter quattuor cum[152] additis sibi 4.[153]

D

<**19.1**> Quamcumque datam quantitatem secundum quascumque datas proportiones si dividere volueris vel numerum indivisum ad modum divisi dividere, primo

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[136]sunt E

[137]quadrati quem E

[138]E adds 'novenarius'

[139]ternarius est] sunt E

[140]E adds 'Similiter in aliis omnibus'

[141]Item] Similiter queritur E

[142]EN adds 'tres'

[143]qudratum suum. R<espondetur> E

[144]radicem...excreverit] eius quod inde provenerit radicem E

[145]provenerit E

[146]quadrati quem E

[147]proposite E

[148]E omits

[149]quadrati quem E

[150]sexdecim P (*passim*)

[151]Quos sedecim] Quem quadratum E

[152]E omits

[153]E continues with: 'Sicut in predictis numeris ita et in omnibus aliis invenire poteris, in rebus et in earum pretiis si diligenter observaveris has predictas regulas. Explicit Liber algvorismorum.' This is followed by the beginning of **B** above, which ends abruptly after a few lines.

proportiones partium propositarum in terminis sunt ordinande, deinde termini aggregandi, aggregatum vero primo ponendum. Numerus vero proportionis cum proportionalis ei quod queritur secundo ponendus. Datam vero quantitatem tertio ponendum. Cifre vero quarto[154] ponendum. Multiplicatur igitur data quantitas per numerum proportionis et productum inde[155] dividatur per aggregatum ex ipsis terminis et exibit quod queritur, per regulam quattuor proportionalium. Si quartus ignoratur, multiplicetur tertius in secundum et productum dividatur per primum et exibit quartus. Quod si per unum inventum cetera habere volueris, multiplicia habebis multiplicando[156] ipsum per numerum denominantem proportiones. Multiplicia autem dicuntur que aliquotiens continent aliquem numerum. Submultiplicia dicuntur que aliquotiens continentur. Si vero aliquis numerus post divisionem remanserit, ille numerus erit fractionum quantitatis in divisione a dividente denominatarum.

<**19.2**> Verbi gratia, proponitur nobis dividere 40 solidos 4 hominibus ita quod secundus habeat quadruplum ei quod habet primus, tertius vero quincuplum secundo, quartus triplum tertio. Multiplicentur ergo quattuor per unum et exeunt 4. Set sit primus terminus unum, secundus vero quattuor. Item multiplicentur[157] 4 per 5 et fiunt 20. Erunt ergo 20 tertius terminus. Item multiplicentur[158] 20 per 3[159] et fiunt 60. Erunt ergo 60 quartus terminus. Aggregatis autem terminis, fiunt 85. Ponatur ergo primo[160] 85. Si vis autem scire quid accidat secundo, pone secundum terminum secundum. Similiter si vis scire quid accidat tertio, pone tertium terminum secundum. Similiter de singulis.

	primus	1			
85	secundus	4	quadruplus	40	0
	tertius	20	quincuplus		
	quartus	60	triplus		

<**19.3**> Pone ergo secundum terminum, scilicet 4, secundum, datam vero quantitatem pone tertio, çifre[161] vero quarto. Multiplicetur data quantitas, scilicet 40, per 4 et fiunt 160. Dividatur autem 160 per 85 et exeunt unum et 75 octuagesime quinte. Qui numerus in eius generis rebus constituendus est, nummis vel solidis vel libris in quo data quantitas proposita fuit. Et hoc est quesitum. <**19.4**> Si vis autem scire per hoc quid accidat primo numerum inventum, id est unum inte-

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[154] quartum PN

[155] in P

[156] dividendo P

[157] multiplicantur N

[158] multiplicantur N

[159] 30 P

[160] primum PN

[161] ziffre N

grum, scilicet 12 denarios qui sunt unus solidus, divide per 4 et exeunt 3 d<enarii.> Item numerum numerantem fractiones, scilicet 75, per eundem 4 divide et exeunt 18 octuagesime quinte et tres quarte unius octogesime quinte et hec est pars primi, scilicet 3 d<enarii.> et 18 octogesime quinte et tres quarte unius octogesime quinte. <**19.5**> Item si per numerum prius inventum cetera habere volueris, scilicet multiplicia eius, ipsum numerum inventum, scilicet 1 et 75 octogesimas quintas, multiplica per numerum denominantem tertiam proportionem, verbi gratia per 5, et fiunt 9[162] solidi et 35 octogesime quinte et hec est portio tertii. Et ita de singulis multiplicibus. <**19.6**> Si vero submultiplicia eius habere volueris, ipsum numerum prius inventum per numerum denominantem proportionem divide. <**19.7**> In probando /11va/ adde integra integris.

<**20.1**> Si quascumque proportiones datas ad quemcumque terminum minorem continuare volueris, primam datarum per terminum minorem multiplica et productum per sequentem multiplica et ita procede multiplicando productum sequentis per sequentem et productum ex penultimo sequenti multiplicatum per ultimum sequentem, erit primum continuandorum. Hoc facto secundum datarum per eundem terminum minorem multiplica et productum per sequentem multiplica et ad modum predictum procede multiplicando et productum ex penultimo sequenti per ultimum sequens multiplicatum erit secundum continuandum. Item tertiam datarum per eundem terminum minorem multiplica et productum per sequentem multiplica et ad modum predictum procede multiplicando et productum ex penultimo sequenti per ultimum sequens multiplicatum erit tertium continuandorum. Et ita de singulis. <**20.2**> Verbi gratia, sint date proportiones 4, 3, 2, 1 et sit minor terminus 6. Multiplicetur 4 per 6 et erit productum 24. Hoc autem multiplica per tres et productum erit 72.[163] Quod multiplicetur per 2 et fient[164] 144.[165] Et si hoc multiplicetur per 1, exibit[166] idem. Hoc ergo, scilicet 144,[167] est primum continuandorum. <**20.3**> Item multiplicetur 3 per 6 et erit productum 18, hoc autem multiplica per 2 et fit 36 secundum continuandum. <**20.4**> Item multiplica duo per 6 et fit 12. Erit ergo 12 tertium continuandorum. <**20.5**> Item multiplica 1 per 6 et fit 6. Erit ergo quartum continuandorum.

<**21.1**> Si radicem cuiuslibet propositi quadrati invenire volueris propositum quadratum per quemlibet[168] alium quadratum multiplica et producti radicem ex-

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[162] 5 PN

[163] 64 PN

[164] fiet P

[165] 124 PN

[166] N adds 'unum'

[167] 24 P, 124 N

[168] quelibet P

trahe et radicem per radicem quadrati per quem multiplicasti quadratum propositum divide. Quod exibit erit radix quesita. <**21.2**> Verbi gratia, si radicem duorum et quarte invenire volueris, multiplica illud in aliud quadratum, qui scilicet sit 4. et proveniunt 9. Duo enim in quattuor fiunt 8 et quarta in quattuor fit unum; unde fiunt 9. Horum autem 9 radix sunt 3. Quos divide per radicem quaternarii que est duo. Exibit unum et dimidium, hoc modo:

1
3
2

<**21.3**> Si radicem propinquioris quadrati invenire volueris, 3[169] numerum propositum per quemlibet quadratum multiplica et predicto modo operare.

<**22.1**> Si articulum in articulum vel digitum in digitum vel compositum in compositum multiplicare volueris, figuram per figuram multiplica. Deinde numeros denominantes differentias aggrega, ab aggregato unum subtrahe et principium sequentis differentie ab eo quod remanet denominate totiens excrescit quot unitates excreverint ex multiplicatione figurarum et quot excreverint denarii, totiens principium sequentis differentie excrescit.

<**23.1**> Cum multiplicaveris unum numerum in alium, vide quilibet eorum quota pars sit alicuius articuli sive limitis et accipe tantam partem de altero eorum, quam multiplica in illum articulum vel limitem et productum inde est id quod ex ductu unius in alterum provenit. <**23.2**> Verbi gratia, si multiplicaveris 32 in 25, quota pars est 25 de centum limite, scilicet quarta, tanta accipe de 32, scilicet quartam que est octo. Quos octo multiplica in centum et productum inde est id quod ex ductu 32orum in 25 provenit. Probatio. Sicut enim habet se 8 ad 32 sic 25 ad centum. Tantum ergo fit ex ductu duorum mediorum in se quantum ex ductu duorum extremorum per predictam regulam 4 numerorum proportionalium et sic in omnibus. <**23.3**> Similiter etiam de articulo. Verbi gratia, 25 medietas est de quinquaginta. Set medietas de 32 sunt 16. Quos multiplica in 50. Idem provenit quod ex ductu 32 in 25. $\boxed{8\ \ 32\ \ 25100}$

<**23.4**> Duorum numerorum compositorum ex diversis vel eisdem digitis set eodem articulo vel limite /11vb/ cum multiplicaveris unum in alium, ut sedecim in 18 et huiusmodi, multiplica digitum in digitum et articulum in articulum et productos inde aggrega. Deinde digitum digito aggrega et aggregatum in articulum vel limitem multiplica et productum inde priori aggregationi adde et illud totum aggregatum est summa que unius compositi in alium multiplic<at>io[170] efficit. <**23.5**> Verbi gratia, proponantur multiplicandi 16cim in 18. Multiplicetur ergo digitus in digitum, id est sex in octo, et fiunt 48. Deinde articulum in articulum, id est decem in decem.

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[169] et N

[170] multiplitio PN

et fiunt 100. Quos duos productos simul aggrega et fiunt centum 48. Deinde digitum digito aggrega et fiunt 14. Quos in articulum, scilicet decem, multiplica et fiunt 140. Quos aggrega priori aggregationi que fuit 148 et fiunt 288. Et hec est summa que ex ductu 16 in 18 provenit.

<**24.1**> Cum volueris multiplicare radices aliquorum numerorum, ipsos numeros in se multiplica et producti radix est productus ex ductu unius radicis in aliam. <**24.2**> Verbi gratia, si volueris multiplicare radices denarii et quadraginta, multiplica 10 in 40 et proveniunt 400. Horum autem 400orum radix est 20. Qui 20 sunt numerus productus ex multiplicatione radicis denarii in radicem 40 per regulam trium numerorum proportionaliter se habentium, qui cum sicut se habet primus ad secundum, sic secundus ad tertium, tunc quantum fit ex ductu medii in se, tantum ex ductu[171] extremorum hoc modo: $\boxed{10\ 20\ 40}$

<**25.1**> Si vis scire quantum vixerit qui vivens tantum quantum vixit et iterum tantum et dimidium tanti et dimidium dimidii c annos complet, aggrega que proponuntur et per aggregatum divide summam que completur et quod exierit hoc est quod vixit. <**25.2**> Verbi gratia, cum proponitur tantum quantum vixit et iterum tantum et dimi<dium> tanti et dimi<dium> dimidii, simul[172] aggregata quattuor minus quarta fiunt, que sunt 15 quarte. Per quas 15 quartas si dividis centum conversum prius in quartas, exeunt 26 et due tertie. Que quater <minus quarta> simul accepta, centum complent et hoc est quod vixit.

E

<**26.1**>...Vel aliter: quoniam omnis numerus vel est digitus vel articulus vel limes vel compositus, ideo quotiens multiplicatur numerus in numerum, aut multiplicatur digitus in digitum vel in articulum vel limitem vel compositum vel econverso, aut compositus in compositum vel articulum vel limitem vel digitum et econverso. <**26.2**> Cum autem articulum in articulum multiplicare volueris, figuram per figuram multiplica. Deinde quarum differentiarum sint ipsi articuli considera et numeros a quibus denominantur eorum differentie aggrega. Ab aggregato autem unum subtrahe et in differentia denominata a numero remanenti productum ex multiplicatione figurarum si fuerit tantum digitus pone. Si autem articulus tantum in sequenti eam. Si vero digitus et articulus, digitus in differentia denominata a numero remanenti, articulum vero ponatur in differentia sequenti. Et quod ibi significaverit est summa que ex ductu unius articuli in alium provenit. <**26.3**> Verbi gratia, si multiplicare volueris 20 per 70, figuras quibus representantur, scilicet 2 et 7, in se multiplica et fiunt 14. Set quia secunde differentie est utraque que est decenorum, ideo numeros denominantes differentiam utriusque, scilicet

[171] ducto P

[172] similiter PN

duo et duo, secunda enim a duobus denominatur, aggrega et fiunt quattuor. A quibus quattuor subtracto uno remanent tres. A quibus denominatur tertia differentia, que est centenorum. Et quia ex ductu figurarum in se provenerat[173] 14 qui est digitus et articulus, ideo digitum, scilicet 4, in eadem differentia, scilicet tertia, pone et articulum, scilicet decem, in sequenti que est quarta et habebis 1,400. Et hec est summa que ex ductu unius articuli in alium provenit, scilicet 20 per 70. Similiter etiam faciendum est si digitus in articulum vel limitem vel compositum multiplicetur et econverso. <**26.4**> Cum autem /12ra/ compositum in compositum multiplicare volueris, predictam regulam observabis, hoc adiecto, ut unusquisque superiorum multiplicetur in unumquemque inferiorum, videlicet digitus in digitum et articulus et articulum in digitum et articulum, quotquot fuerint, singuli superiorum in omnes inferiores. <**26.5**> Verbi gratia, 23 in 64 cum multiplicare volueris, digitum superiorem, scilicet tres, in digitum inferiorem, scilicet 4, multiplica et fiunt 12. Per priorem ergo regulam, digitus erit in prima differentia, articulus in secunda. Deinde eundem digitum, scilicet 3, multiplica in figuram inferioris articuli, qui est sex et fiunt 18. Aggregatis autem numeris denominantibus differentias, scilicet uno et duobus, articulus enim est secunde differentie et digitus prime, fiunt tres. De quibus subtracto uno remanent duo, a quibus denominatur secunda differentia. Pone ergo in secunda digitum, scilicet 8, et in sequenti, scilicet tertia, 1. Deinde figuram articuli superioris que est 2 multiplicabis in inferiorem digitum, qui est 4, et fiunt 8. Aggregatis ergo numeris denominantibus differentias, scilicet uno et duobus, fiunt tres. A quibus subtracto uno remanent[174] duo, a quibus denominatur secunda. Ideo digitus, scilicet 8, ponatur in secunda. Postea figuram suprapositi articuli multiplicabis in figuram subpositi, scilicet 6, et fiunt 12. Et quoniam uterque est secunde differentie, ideo aggregatis numeris denominantibus differentias, scilicet duobus et duobus, fiunt 4. A quibus subtracto uno remanent tres, a quibus denominatur tertia differentia. Pone ergo digitum in tertia, scilicet duo, et articulum in sequenti, scilicet quarta, et fiunt hoc modo:

		1	2
	1	8	
		8	
1	2		

<**26.6**> Que sic posita aggrega et fiunt 1472. Et hec est summa que provenit ex multiplicatione 23 in 64. <**26.7**> In hac eadem regula docetur qualiter etiam compositus in articulum vel limitem vel digitum multiplicetur.

<**27.1**> Cum multiplicaveris digitum in digitum aut proveniet tantum digitus aut tantum denarius aut digitus cum denario semel vel aliquotiens aut denarius multotiens. <**27.2**> Cum multiplicaveris digitum aliquem in aliquem articulorum qui sunt usque ad centum, multiplica figuram in figuram et quot unitates fuerint in

[173]provenerit N

[174]remanet P

digito qui provenerit, tot denarii erunt. Quot autem denarii fuerint in articulo qui provenerit, tot centenarii erunt. <**27.3**> Verbi gratia, cum multiplicare volueris septem in 70 multiplica figuram in figuram et proveniunt 49. Novem autem unitates sunt in digito. Tot ergo denarii erunt, qui sunt 90. Quater autem denarius[175] est in articulo, tot ergo centenarii erunt, qui sunt quadringenti. Ergo 490 est summa que ex ductu illorum in se provenit. Similiter fit in omnibus aliis. <**27.4**> Cum multiplicaveris digitum in aliquem centenorum qui sunt usque ad mille, multiplica figuram in figuram et quot unitates fuerint in digito si provenerit, tot centenarii erunt. Quot autem denarii in articulo si provenerit, tot millenarii erunt. <**27.5**> Verbi gratia, si multiplicaveris 3 in 900 multiplica figuram in figuram et fiunt 27. Septem sunt unitates in digito et duo denarii[176] in articulo. Ergo 2,700 est summa que ex ductu illorum in se provenit. Similiter fit in omnibus aliis.

<**28.1**> Cum multiplicaveris unum articulorum in alium de his qui sunt usque ad centum, multiplica figuram in figuram et quot unitates fuerint in digito qui provenerit, tot erunt centenarii. Quot autem denarii in articulo, tot erunt millenarii. <**28.2**> Verbi gratia. si multiplicaveris 30 in 70, multiplica figuram in figuram et fiunt 21. Duo denarii sunt in articulo et unitas semel in digito. Ex ductu igitur priorum proveniunt duo millia et centum.[177] Similiter in omnibus aliis. <**28.3**> Cum multiplicaveris aliquem articulorum qui sunt usque ad centum in aliquem centenorum qui sunt usque ad mille. figuram in figuram multiplica /12rb/ et quot unitates fuerint in digito. tot erunt millenarii. Quot autem denarii in articulo. totiens decem millia. <**28.4**> Verbi gratia. si multiplicaveris 30 in 500, multiplica figuram in figuram et proveniunt 15. Et quia quinque sunt unitates in digito. erunt quinque millia. Denarius autem est semel in articulo. Ex ductu igitur priorum proveniunt quindecim millia hoc modo: 15.000. Similiter fit in omnibus.

<**29.1**> Cum multiplicaveris aliquem centenorum qui sunt usque ad mille in alium ex his multiplica figuram in figuram et quot unitates fuerint in digito qui provenerit, totiens erunt decem millia. Quot autem denarii in articulo, totiens erunt centum millia. <**29.2**> Verbi gratia. cum multiplicaveris 300 in 500. multiplica figuram in figuram et proveniunt 15. Quinque autem unitates sunt in digito et denarius semel in articulo. Ex ductu igitur suprapositorum proveniunt 150,000, que sunt centum quinquaginta millia.

<**30.1**> Cum multiplicaveris aliquem digitorum in aliquem articulum millenorum, ut[178] decies vel vigies millia et huiusmodi, aut iteratorum millium, ut decies millies

[175] denerius P

[176] duodenerii PN

[177] cetum P

[178] vel P

millium et quantum iterare mille volueris, multiplica figuram in figuram et digitum si provenerit pone in differentia multiplicantis et articulum in sequenti. <**30.2**> Verbi gratia, sex si multiplicaveris in triginta millia, multiplica figuram et figuram et proveniunt 18. Digitus ergo, qui est octo, ponatur in eadem differentia multiplicantis qui[179] est tres et articulus qui est unus in sequenti hoc modo: 180.000. et proveniunt centum octoginta millia. <**30.3**> Cum multiplicaveris aliquem articulorum in aliquem sepe repetitorum millium, ut decies vel vigies millies millies millium et quotiens mille iterare volueris, multiplica figuram in figuram et digitum si provenerit pone in differentia secunda a multiplicante, articulum vero in tertia ab ipso. <**30.4**> Verbi gratia, cum multiplicaveris 30 in quater millies m. m. millium et quotiens iterare volueris, figuram que est 3 multiplica in figuram que est 4 et proveniunt 12. Digitum ergo qui est duo pone in differentia secunda post 4[180] et articulum in tertia post 4[181] hoc modo: 4,000,000,000,000, qui fiunt centies vigies millies .m.m. millium quater: 120,000,000,000,000. <**30.5**> Cum multiplicaveris aliquem centenorum in aliquem millenorum sepe iteratorum multiplica figuram in figuram et digitum si provenerit pone in differentia tertia a multiplicante, articulum vero in quarta ab eo. <**30.6**> Verbi gratia, cum multiplicaveris ducenta in quinquies millies .m.m.m. quater, multiplica duo in quinque et, quoniam articulus provenit, ponatur in quarta differentia a quinque hoc modo: 1,005,000,000,000,000.

<**31.1**> Cum autem volueris scire de qua differentia sint digiti vel articuli \vel centeni/[182] sepe iteratorum millium, vide quotiens iteratur mille et in tres multiplica[183] numerum quo iterantur et productum inde retine. Si autem voluisti scire de differentia digitorum iteratorum millium ut bis vel ter vel quater usque ad novies millies .m. millium, quotiens repetere volueris mille, de qua differentia sint semper adde unum primo producto retento et a numero qui inde excrescit denominatur differentia de qua sunt digiti repetitorum millium. <**31.2**> Verbi gratia, si volueris scire de qua differentia sunt ter millies .m. m. millium, numerum quo iteratur mille, sicut hic est quattuor, multiplica in tres et fiunt 12. Quibus adde unum et fiunt 13. Tredecima ergo est differentia predictorum. <**31.3**> Si autem voluisti scire de qua differentia sint articuli sepe iteratorum millium, ut decies vel vigies millies millium et quotiens mille repetere volueris, priori producto retento semper adde \duo/[184] et a numero qui inde excrescit denominatur differentia de qua sunt articuli sepe iteratorum mil/12va/lium. <**31.4**> Verbi gratia, si volueris scire de qua differentia

[179] que P

[180] 3 PN

[181] 3 PN

[182] P adds 'vel centeni' in the margin; N gives 'vel centum' in the text.

[183] PN reverse the phrases: 'in tres multiplica' and 'vide quotiens iteratur mille et'

[184] P adds 'duo' in the margin

sint suprapositi articuli sepe iteratorum millium, ut quinquies millies .m.m. mille quater, numerum quo iteratur mille, sicut hic est quattuor, multiplica in tres et fiunt 12. Quibus adde duo et fiunt 14. De quartadecima ergo differentia sunt articuli suprapositorum millium iteratorum. <**31.5**> Si autem voluisti scire de qua differentia sint centeni millium iteratorum, priori producto semper adde tres et a numero qui inde excrescit denominatur differentia centenorum iteratorum millium. <**31.6**> Verbi gratia, si volueris scire de qua differentia sint centies vel ducenties et huiusmodi millies mille et quotiens iterare volueris mille, numerum quo hic iteratur mille, scilicet duo, multiplica in 3 et fiunt 6. Quibus adde tres et fiunt novem. Nona est ergo differentia de qua sunt predicti centeni iteratorum millium.

<**32.1**> Cum volueris multiplicare quodlibet millies vel decies vel centies millies millium sepe iteratorum in aliud[185] quodlibet ex illis, reiecta iteratione de millies a multiplicato et multiplicante, ea que de utroque remanent multiplica inter se et productum inde retine. Deinde aggrega numeros iterationis utriusque et summam inde excrescentem pone sub prius producto et quod inde fit est numerus qui ex ductu unius in alterum provenit. <**32.2**> Verbi gratia, cum volueris multiplicare digitos millenorum inter se, ut ter millies mille in septies millies .m. m. mille, pretermissis numeris iterationis utriusque, scilicet duo et quattuor, in multiplicato etenim bis numeratur mille et in multiplicante quater, remanent tantum figure utriusque scilicet 3 et septem. Quarum altera multiplicata in alteram fiunt 21. Deinde aggrega ipsos numeros iterationis utriusque scilicet duo et quattuor, et fiunt sex. Quos pone sub producto prius, scilicet 21, hoc modo: $\boxed{\begin{array}{c}21\\6\end{array}}$

Et dices quia 20 et una vicibus millies m. m. m. m. mille sexies proveniunt ex ductu unius predictorum in alterum. Per suprapositam igitur regulam si numerum iterationis multiplicaveris in tres, fient 18. Quibus addito uno fiunt 19. Digitus ergo suprapositus qui est un[i]us erit in nonadecima differentia et articuli 20 in vicesima.

<**32.3**> Cum autem centenos iteratorum millium inter se multiplicare volueris, ut quingenta[186] millies millia in tre[s]centa millies .m.m. millium quater, reiecto numero iterationis utriusque qui est duo et quattuor, in multiplicato etenim bis iterabatur mille et in multiplicante quater, remanent de multiplicato 50<0> et de multiplicante 300. Que in se ducta fiunt centum quinquaginta millia. Quos retine. Deinde aggrega numeros iterationis utriusque, scilicet duo et quattuor, et fiunt sex. Quos pone sub prioribus et significabitur summa que ex ductu unius suprapositorum in alterum fit, scilicet centies quinquagies millies .m.m.m.m.m.m. septies, hoc modo: 15<0,000,000,000,000,000,000,000>. Per priorem ergo regulam quinquaginta millia

[185]alium N

[186]quinquaginta PN

erunt in vicesima tertia[187] erunt differentia et centum in vicesima quarta. <**32.4**> Similiter etiam fit cum multiplicaveris digitos iteratorum millium in decenos iteratorum millium et centenos et econverso. Similiter etiam fit cum multiplicaveris decenos iteratorum millium inter se, vel in centenos.

<**33.1**> Cum autem habueris aliquam differentiam et volueris scire quis numerus est illa, numerum a quo denominatur differentia divide per 3. Si autem de divisione nichil remanserit, illa differentia centenorum millium totiens vel totiens iteratorum erit. <**33.2**> Si autem volueris scire numerum iterationis, id est quotiens iteratur, ab eo quod de divisione exit, extrahe unum et quod remanserit numerus iterationis erit illorum centenorum millium qui illius differentie sunt. <**33.3**> Verbi gratia, si habueris duodecimam differentiam et volueris scire quis numerus est illa, divide duodecim a quo denominatur dif/12vb/ferentia per tres et exibunt 4. A quibus extracto uno remane<n>t 3 et quoniam de divisione nichil remansit, duodecima differentia centenorum millies millies m. ter iteratorum erit. <**33.4**> Si autem de divisione remanserit duo, illa differentia erit decenorum millium totiens iteratorum quotus fuerit numerus qui de divisione exit. <**33.5**> Verbi gratia, cum habueris undecimam differentiam et volueris scire quis numerus est illa, divide undecim per tres et exibunt 3 integri et remanebunt duo. Undecima ergo differentia est decenorum millium ter iteratorum. <**33.6**> Si autem de divisione remanserit unum, illa differentia erit digitorum millium totiens iteratorum quotus est numerus qui de divisione exit. <**33.7**> Verbi gratia, si habueris decimam differentiam et volueris scire quis numerus est [in] illa, divide decem per tres et exibunt de divisione 3 remanente uno. Decima ergo differentia est millium ter iteratorum.

<**34.1**> Numerum quem quis occultatum[188] in corde suo tenet ipso non significante sic invenies. Primum precipe ut ipsum numerum triplicet. Deinde triplicationis summa<m> in duo dividat. Postea an pares sint partes interroga. Si autem impares fuerint, unum retine et ut maiorem partem triplicet iterum precipe et summam in duo dividat. Que si interrogatus responderit esse imparia, tu duo retine. Que cum priore uno aggrega<ta> tria fiunt. Deinde ut de ipsa parte maiore novem reiciat precipe et alios iterum novem et sic donec non remaneat unde novem reiciat. De unoquoque autem novem tu quattuor accipe. Qui aggregati cum primis sunt numerus quem occultavit. Aut si de parte maiore novem eicere non potuerit, primi tres erunt numerus occultus. Aut si utraque divisio fuerit per paria, tu nichil accipies, set quaternarii unus vel plures sumpti de novenario uno vel pluribus, numerus occultus erunt.[189] Quotiens autem divisio fuerit per imparia, de prima accipe xnxm[190] de se-

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[187] PN add a tironian et with 'm' suprascript.

[188] occultum N

[189] erit N

[190] *i.e. unum*

cunda dxp.[191] <**34.2**> Verbi gratia, sit binarius numerus quem occultat. Triplicatus autem efficit 6. Senarius vero in duo equa dividitur. Cuius iterum altera<utra> pars triplicata[192] efficit novem, qui dividitur in duo inequalia. Unde quia secunda divisio est, duo retineo. Et quia de parte maiore novem non possunt reici dxp quos retknxk sxnt n;.;s[193] pccxlt;.;s.[194] Et hpc (hoc) f[195] (est) quod proppsxkm;.;s.[196]

<**34.3**> Item 113c53 4cc5lt1nt3[197] quot solidos habeant, dic ut de tuis <nummis> tot singulos vel tot binos vel ternos et huiusmodi quot volueris accipiat. Et de omnibus tuis nummis unum aliquid ut pote gallinam unam vel aliquid huius modi[198] emat. Deinde de omnibus suis solidis secundum idem pretium quot potuerit gallinas emat. Tu ergo tunc divide solidum in numerum quem sibi dedisti, videlicet in unum vel duo vel tres. Et exeunti de divisione numero adde unum et quod exit addito uno est numerus eorum que[199] emuntur. <**34.4**> Verbi gratia, ponamus quod occultet quinque solidos. Acceptis de meis totidem nummis quot sunt solidi, scilicet 5, gallina una ematur[200] et secundum idem pretium de suis solidis duodecim galine emuntur. Divide ergo solidum in 12 nummos in numerum quem sibi dedisti, scilicet unum, exibunt 12 qui est numerus galinarum emptarum. Si vero binos dedisses galline sex essent, aut si ternos, essent quattuor et sic de ceteris.

<**34.5**> Si aliqua duo equalia occultantur et de uno eorum duo accepta alteri addantur et de augmentato equale residuo addatur, necessario quattuor remanebunt, aut si tres necessario sex et ita semper remanet duplum eius quod primo accipitur. <**34.6**> Verbi gratia, si occultantur 5 solidi in una manu et quinque in alia, duos acceptos de una appone aliis quinque et fiunt 7 in una, remanentibus tribus in alia. Cum ergo de septem acceperis equale residuo, id est tribus, necessario quattuor remanent.

F

<**35.1**> Queritur cur non omnes[201] vel plurimos numeros propriis nominibus designamus /13ra/, vel cur non semper per adiectionem novorum set post decem per

[191] *i.e. duo*

[192] triplica P, triplicata N

[193] n;.;s has an abbreviation mark above it in PN, indicating that it is short for 'numerus'

[194] pccxt;.;s P, pccxle;.;s N

[195] 'f' has a line over it in P; N reads it as 's'

[196] *i.e. duo quos retinui sunt numerus occultus. Et hoc est quod proposuimus.*

[197] *i.e. alicui occultanti*

[198] alquid hoc modo PN

[199] quem P, qui N

[200] emat P

[201] omnis PN

repetitionem priorum[202] semper numeramus. <**35.2**> Ad quod dicitur quia non fuit possibile ut omnes numeri propria nomina haberent, idcirco quod numerorum in infinitum crescit multitudo, nominum autem in qualibet lingua infinita non potest esse inventio. <**35.3**> Cum enim in omni lingua certa et terminata sint[203] instrumenta et eorum definite naturaliter modulationes, quibus vox articulata formatur et unde[204] litterarum figure apud omnes gentes et earum varie set diffinite sunt secundum ordinem preponendi et postponendi ad representanda rerum omnium nomina compositiones, necessario omnes numeri cum[205] cum sint infiniti, nomina non potuerunt nec debuerunt habere singuli, precipue cum et homines in omni pene re numeris utentes nimis impedirentur, si in numerationibus suis infinitam numeralium nominum multitudinem in promptu semper habere numerandi necessitate cogerentur. <**35.4**> Idcirco[206] necesse fuit infinitam numerorum progressionem certis limitibus terminare, paucis nominibus illos[207] designare, ne cogeretur homo in numerando per novas additiones tam numerorum quam nominum semper procedere, set per repetitionem priorum brevem quantamlibet summam paucis nominibus possit comprehendere.[208] <**35.5**> Unde cum omnes numeros habere nomina fuerit[209] impossibile et aliquos necesse, ratio exegit natura predicante ut ex omnibus numeris soli 12 nomina haberent: tres limites, videlicet denarius centenarius et millenarius et novem primi numeri ab uno usque ad novem infra decem constituti.

<**36.1**> Quam rationem novenarius pre aliis omnibus numeris proprio privilegio merito vindicavit, ut pote continens in se omnes pene species numerorum et numeralium proportionum. <**36.2**> In ternario etenim quamvis deo dicato predicta ratio consistere non debuit, quia sibi deerat primus perfectus, qui est senarius; set nec propter hoc in senario, quia deerat ei primus cubus qui est octonarius; set nec ideo in octonario, quia deerat ei prima vera superficies, que est in novenario. <**36.3**> Ex hac ergo plenitudine virtutum novenarius promeruit ut in se ratio numerandi et numeros appellandi consisteret, ultra quam nisi tres tantum limites, nullus numerus proprium nomen haberet. <**36.4**> Nimirum cum ad instar novenarii tam celestia quam terrestria, tam corpora quam species formata et ordinata esse videantur; novem enim sunt spere celestium corporum, novem etiam sunt ordines celestium spirituum, novem etiam complexiones omnium corporum. <**36.5**> Novem igitur

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[202]primorum P

[203]*Liber mahameleth* adds 'loquendi'

[204]*Liber mahameleth* adds 'et'

[205]omnes numeri cum] idcirco cum numeri *Liber mahameleth*

[206]unde *Liber mahameleth*

[207]*Liber mahameleth* omits 'illos'

[208]*Liber mahameleth* omits 'set...comprehendere'

[209]Unde cum...fuerit] quoniam et...fuit *Liber mahameleth*

debuerunt esse compositiones numerorum in quibus solis tota consisteret infinitas numerorum, sicut ex complexionibus novem universitas corporum. <**36.6**> Sicut enim in complexionibus una est equalis et altera inequalis, una vero tantum temperata, sic et in numeris unus est par, alius impar et inter omnes sola est unitas ex nulla parte sibi dissimilis, semper eadem, semper equalis.

<**37.1**>Sic creature a similitudine sui creatoris qualicumque modo non recederent[210] dum intra illum numerum se continerent,[211] quia primo impari in se multiplicato generatur qui post unitatem deo solus consecratus est, quia numero deus impare gaudet. <**37.2**> Unde et soli tres limites preter novenarium inter alios nomina sortiti sunt, ut per hoc videlicet trinitatis que[212] verus limes est omnium, A et Ω principium et finis, qualemcumque similitudinem teneant et a radice novenarii numquam recedant. <**37.3**> Idcirco igitur ratio postulavit ut, quia universitas rerum intra novenarium continetur, similiter et numerorum infinitas intra novenarium coartaretur novem et nominibus designaretur et novem figuris representaretur. <**37.4**> Omne enim exemplum similitudinem sui retinet exemplaris; alioquin non esset alterum alterius exemplar vel exemplum, et quia ut predictum est pene omnia condita sunt ad instar novenarii, ipsa quoque numerorum infinitas rationabiliter debuit sub novenario coartari,[213] ut numerus etiam ab ea forma non discederet ad quam creator cuncta componeret et quam a numero rerum universitas mutuaret.

<**38.1**> Unde et homines primevam naturam imitantes non nisi solis novem numeris nomina imposuerunt et ad omnes representandos[214] non nisi novem figuras adinvenerunt. <**38.2**> Set quia quedam species numeri adhuc /13rb/ deerat quam novenarius intra se non continebat, scilicet numerus superfluus, qui primus est duodenarius, ideo post novenarium tribus tantum limitibus nomina sunt imposita,[215] ut novenarius cum radice sua, scilicet ternario, omnes dignitates et proprietates numeri intra se contineret et nichil proprietatis nichil misterii in numeris possit inveniri quod in toto novenario cum radice sua non videtur contineri. <**38.3**> Cum igitur non omnes nec plurimi set pauci numeri propriis nominibus necessario fuerant designandi,[216] propter predictas causas novem tantum numeris et tribus limitibus nomina sunt indita, ut per commoditatem paucitatis humanis usibus melius deservirent et rerum occulta misteria quibuscumque signis exprimerent et a nature rationibus non discederent. <**38.4**> De his <h>actenus.

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[210]recederet PN

[211]continent PN

[212]qui PN

[213]coaritari P

[214]representandas P

[215]impositam P

[216]P adds 'et'

<**39.1**> Unitas est origo et <prima> pars numeri; omnis enim numerus naturaliter ex unitatibus constat et ipsa omnem numerum natura precedit quoniam simplex est. <**39.2**> Et quia simplex est, ideo per multiplicationem sui nichil nisi id per quod multiplicatur generare potest, quod non fit in aliis qui simplices non sunt; ex cuiuslibet enim numeri multiplicatione in se vel in alium necesse est alium provenire diversum. <**39.3**> Unitas autem per se multiplicata non generat nisi se; semel enim unum unum est; per quemcumque enim numerum multiplicaveris non nisi ipsum per quem multiplicas efficis, et quia nullus ex ea generatur nisi ille in quem prius ipsa multiplicatur, idcirco in principio cum nichil esset cui ipsa adiungi posset ad generationem primi numeri, necesse fuit ipsam in se congeminari et a se quodam[217] modo alterari, ut ex se ipsa et ex se altera quasi ex diversis posset aliquid generari. <**39.4**> Et hec est prima numeri generatio que apparet in binario; unde et principium alteritatis dicitur, quoniam ex unitate alterata genitus est. Ideo etiam sibi soli et nulli alii contingit quod ex sui in se multiplicatione itidem quod ex aggregatione provenit; non enim constat ex numero. <**39.5**> Et quoniam preter binarium adhuc non erat nisi unitas, ideo ipsa[218] binario tamquam vir femine iungitur, ex quorum copula ternarius nascitur, qui post unitatem primus impar et masculus vocatur.

<**40.1**> Numerus etenim par femina dicitur quasi mollis eo quod facile solvitur, set masculus impar quasi fortis indivisibilis. <**40.2**> Unitas autem nec par nec impar est actu; unde unitas in se nec femina nec masculus est actu, set potestate utrumque. <**40.3**> Unde quando cum femina iungitur inde masculus, scilicet impar, generatur; quando vero cum masculo coit, feminam quia parem gignit. <**40.4**> Unde ex prima generatione unitatis non nisi femina nascitur, scilicet par, quia binarius. <**40.5**> Decebat enim ut unitas in procreatione prime sobolis non nisi vice viri, scilicet dignioris uteretur et ex ea quasi viro femina nasceretur; prima etenim femina ex viro non primus vir ex femina. <**40.6**> Unde in secundo gradu quoniam unitas femine, scilicet binario, iungitur, ternarius qui est masculus generatur; in tertio vero gradu unitas coniungitur masculo et femina procedit, scilicet quaternarius. Similiter in ceteris usque in infinitum. <**40.7**> Unde unitas nec debuit esse par nec impar, quia si par tantum esset, quando paribus iungeretur, sicut ex coniunctione duarum feminarum, nichil procrearetur; si vero impar tantum esset imparibus iuncta, tamquam masculus cum masculo nichil procrearet. <**40.8**> Unde necesse fuit ut neutrum esset actu, set potestate utrumque, ut cum secundum utriusque sexus potestatem[219] omnibus nascentibus vicissim iungeretur, fecunda numerorum soboles in infinitum propagaretur.

<**41.1**> Set quia numerorum prima et naturalis generatio secundum predictum

[217] quoddam P

[218] ipsa in P, ipsam N

[219] potestate P

modum videbatur sine fine multiplicari, placuit postmodum diligentie quorumdam hominum eam ad instar humane generationis quibusdam certis gradibus et limitibus terminari. <**41.2**> Hominum etenim sicut et numerorum generatio ab uno secundum sexum geminato per masculum et feminam descendens in infinitum progreditur. <**41.3**> Set humana cura postmodum gradus et limites adinvenit, quibus cognationes inter homines designavit, ut licet ab uno se omnes eque descendisse cognoscerent, /13va/ tamen propter assignatos gradus alii ad alios potius pertinere cognationis gratia non dubitarent et de uno genere esse dicerentur quicumque sub eisdem cognationis gradibus invenirentur. <**41.4**> Similiter et in numeris post naturalem eorum compositionem et essentiam, humana industria radices, nodos et limites, sicut in hominibus truncos et gradus adinvenit et numerorum generationes per novenarios distinxit, ut numeri qui ex eodem limite nascerentur, usque ad nonum gradum, omnes uno cognationis nomine communi ad aliorum differentiam vocarentur; qui autem aliquem novenarium excederent, ad aliam omnino cognationem pertinere se se cognoscerent. <**41.5**> Unde ad distinguendas huiusmodi numerorum cognationes humana adinventio quosdam appellavit digitos, quosdam articulos, quosdam vero compositos, illos autem ex quibus omnes isti nascuntur vocavit limites, quasi singularum generationum primos parentes. <**41.6**> Illos enim quos in prima creatione per aggregationem sui unitas genuerat, usque ad novem, digitos, quia ab unitate primogenitos vocari instituit, ut[220] hic primus novenarius digitorum sive unitatum novenarius diceretur, cuius novenarii primi unitas limes et primus esset, ut pote quos primum ex se unitas genuisset. <**41.7**> Post hunc autem sequitur secundus novenarius qui est decenorum sive articulorum, et huius novenarii sicut et primi limes unitas est, set decupla primi. <**41.8**> Post hunc vero novenarium decenorum sequitur tertius novenarius centenorum, cuius quoque limes unitas est, set decupla secundi. <**41.9**> Post hunc autem tertium sequitur quartus novenarius millenorum, cuius quoque limes unitas est, set decupla tertii. Et sic usque ad infinitum.

<**42.1**> Et quia omnes numeri ab unitate sunt geniti, merito ipsa etiam constituta[221] est limes omnium novenariorum pro varietate positionum, videlicet ut que ex se species omnium genuerat numerorum, eadem etiam limes esset limitum pro diversitate locorum. <**42.2**> Unde in principio omnium generationum prima et limes ponitur, ut ex hoc cunctorum mater esse comprobetur. <**42.3**> Unde fit ut unitas sicut in prima creatione natura primus limes per aggregationem sui cum ipsis genuerat digitos, sic etiam in secunda institutione placuit ut ipsa eadem omnis limes aggregata primis generet compositos, multiplicata per primos procreet articulos. <**42.4**> Digiti ergo sunt dicti numeri qui ab unitate usque ad novem naturaliter sunt geniti; articuli vero qui per multiplicationem primorum a ceteris limitibus gen-

[220] et PN

[221] etiam constituta] inconstituta P

erantur; compositi vero numeri dicuntur qui ex digitis et limitibus sive articulis simul iunctis nascuntur, dicti compositi tamquam ex diversis generibus procreati. <**42.5**> Unde et a quibus substantiam sortiuntur, eorum etiam[222] proprietatem secuntur. <**42.6**> Cum enim dicitur 12 vel 23 vel centum viginti, ex digito et limite vel et[223] ex articulo compositi sunt. <**42.7**> Set quod est in eis de limite vel articulo in vi limitis vel articuli sumitur, scilicet pro decem vel pro 20 vel pro c; quod autem de novenario digitorum est, pro tot unitatibus sumitur quot in ipso contineri videntur. <**42.8**> Omnes itaque novenarii ad instar prioris ordinati sunt, unde singuli habent unitates limites, habent binarios suos, habent ternarios suos et sic usque ad novem consequenter singulos, sicut subiecta dispositio declarat.

Differentia centies millies millenorum	Differentia decies millies milenorum	Differentia millies milenorum	Differentia centies milenorum	Differentia decies millenorum	Differentia millenorum	Differentia centenorum	Differentia decenorum	Differentia unitatum sive digitorum
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

<**42.9**> /13vb/ Sicut enim in primo limite bis unum[224] faciebat binarium unitatum, ita in secundo limite bis decem efficit binarium decenorum qui est 20 et in tertio limite bis centum binarium centenorum qui est ducenti; et sic in singulis per singulos, usque ad novem. <**42.10**> Et quia ex numeris nichil nisi per aggregationem aut multiplicationem primi novenarii nascitur, idcirco in omnibus in se iteratur et omnibus prior esse comprobatur, quia ante omnes genitus naturalem institutionem adhuc servare videtur. <**42.11**> Unde etiam ipsa unitas, que mater est omnium, in quocumque limite fuerit sive per aggregationem sive per multiplicationem iuxta numerum primogenitorum non nisi novem tantum numeros gignit.

<**43.1**> Set quia post novem naturali ordine decem sequitur et ipsa semper post novem nisi in primo limite humana institutione posita invenitur, idcirco necesse est ut per unitatem post primum novenarium positam decem significentur et sic ipsa ex natura loci in denarium genita secundus limes fiat decenorum, sicut prius simpliciter limes fuerat unitatum, ut eadem esset mater articulorum sive compositorum quam constabat matrem etiam fuisse digitorum. <**43.2**> Et quia post novem semper decem naturaliter sequitur in quo loco semper unitas ponitur, ideo post novenarium decenorum sequitur iterum unitas, tertius limes qui est centenorum;[225] et sic sem-

[222] N adds 'et'

[223] N omits

[224] unus N

[225] decenorum PN

per post quemlibet novenarium unitas sequitur limes sequentium. <**43.3**> Quoniam autem omnis limes excepto primo post precedentem novenarium sequitur, ideo ipse factus denarius precedentis limitis semper decuplus invenitur, quia ipse post quemcumque novenarium fuerit ex decuplatione precedentis limitis nascitur. <**43.4**> Et quia omnes articuli ex multiplicatione sui limitis per primos nascuntur, necesse est, ne de genere videantur, ut suorum limitum regulam sequantur, videlicet ut sicut limites decupli sunt precedentium limitum, ita et qui[226] ex eorum multiplicatione numeri nascuntur, precedentium numerorum decupli similiter inveniantur. <**43.5**> Sicut enim secundus limes decuplus est primi, ita et articuli decenorum decupli sunt digitorum; et sicut limes tertius decuplus est[227] secundi, ita et articuli centenorum decupli sunt ad articulos decenorum. <**43.6**> Sic semper sequentes limites, articuli, compositi qui interiacent, decupli sunt precedentium limitum, articulorum, compositorum singuli singulorum.

<**44.1**> Omnes itaque limites et articuli et compositi sicut et digiti sub novenario sunt constituti, ita ut primus novenarius sit digitorum, secundus articulorum, tertius compositorum et sic ceteri huiusmodi. <**44.2**> Sic ergo placuit ut omnis numerus in novenarium quasi in ultimum gradum sui generis terminaretur et post novenarium unitas precedentis limitis decupla, quia post novem decima, omnium novenariorum limes constitueretur. <**44.3**> Et sic per generationes suas a limitibus tamquam a progenitoribus suis descendens numerorum fecunda progenies tota per novenos gradus distincta in infinitum extenditur. <**44.4**> Sic novenarius principatum tenet in omnibus infinita restringens, restricta distinguens, qui tamen a limite incipit et limite terminatur, ut non ipse auctor rerum, set in animo auctoris rerum exemplar fuisse ostendatur; unde[228] ipse a ternario in se multiplicato generatur. <**44.5**> Qui enim cuncta condidit ipsum quoque fecit ad cuius exemplar cetera formavit; omnia enim deus fecit in numero, pondere et mensura. <**44.6**> Unde et ipsum numerum si factus est, ad numerum fecit, ut numerus leges numeri non excederet, ad cuius formam cetera componi deberent. <**44.7**> Set numerus ad quem numerus creatus est, sic quidem increatus est.

G

<**45.1**> Multiplicationis sunt octo species et totidem divisionis. Aut enim multiplicamus integros per integros, aut fractiones per fractiones, aut fractiones per integros, aut integros per fractiones, aut fractiones et integros per integros, aut fractiones et integros per fractiones, aut integros per integros et fractiones, aut fractiones per integros et fracti/14ra/ones. <**45.2**> Quotiens per vel impar numerus parem vel par imparem multiplicat, par provenit. Si vero impar, imparem, impar exit.

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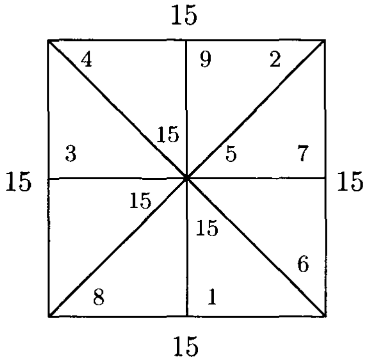
[226] quia P

[227] P omits

[228] N adds 'et'

<**46.1**> Divide minuta per minuta vel secunda per secunda vel tertia per tertia vel quarta per quarta vel quinta per quinta vel sexta per sexta; quicquid provenerit erunt gradus, quandoque[229] unumquodque istorum numerorum multiplicatum <est> in gradus quicquid provenerit erit de genere eiusdem fractionis. <**46.2**> Et si minuta diviserint secunda vel secunda tertia vel quarta quinta vel quinta sexta vel tertia sexta, quicquid exierit de divisionibus erunt denominata a fractionibus maioribus.[230]



[229]quorumque] quare quia (?) N

[230]P omits paragraph 46

The Toledan *Regule*. Translation

A

Here begin the rules and first, concerning addition
<**1.1**> When the numbers are placed in their natural order, every number is half the sum of the two numbers surrounding it on either side,[1] as far as 1. <**1.2**> E.g., 5 is half the sum of 4 and 6; similarly, <it is half> the sum of 3 and 7; similarly, of 2 and 8; similarly, of 1 and 9. The same applies in all cases.

<**2.1**> Likewise, every number multiplied by itself produces the same total as results from the multiplication of the surrounding numbers by each other plus the multiplication by each other of the differences between the surrounding numbers and the middle number. <**2.2**> For so much results from 5 times 5 as from 4 times 6 plus the multiplication of the differences between them and 5, i.e., 1 in each case. When these are multiplied by each other, they only make 1. Also as much as what results from 3 times 7 or 2 times 8 or 1 times 9, always plus the multiplication of the differences between the end points and the middle. <**2.3**> Likewise, every number up to 10,[2] when multiplied by itself, produces as much as the first two surrounding numbers multiplied by each other when 1 is taken from the multiplication of the middle number.

<**3.1**> If you wish to know how great a sum results from adding the numbers following each other naturally from 1, multiply that number at which you ended, if it is even, by its half and add the half itself and this will be the sum which results from them. <**3.2**> E.g., put in order 1, 2, 3, 4, 5, 6, 7, 8. Multiply that number at which you ended, i.e., 8, by its half, i.e., by 4, and you will get 32. Add to this 4 and this will be the sum which resulted from their addition, i.e., 36. <**3.3**> Or multiply the following odd number by half the same even number and you will have the sum. <**3.4**> But if you end at an odd number, e.g., 1, 2, 3, 4, 5, 6, 7, multiply that number (i.e., 7) by its larger part (i.e., 4; for 7 consists of 3 and 4) and 4 times 7 makes 28, which is the sum of the above-mentioned numbers. <**3.5**> Or multiply the same odd number by half the following even number. <**3.6**> But if you do not start from 1, but count continuously, you should proceed as if you had started from 1 and you should add all the numbers which are below that from which you started; that sum should be subtracted from the product; the remainder will be the total.

<**4.1**> If you wish to know what sum will result from even numbers alone added together, when they follow each other in a natural order from 2, consider how many numbers away from the first even number is the number at which you have ended

[1] I.e., with the same differences from the given number.

[2] 'up to 10' is not necessary.

and multiply the following number by what this is denoted by and you will get the sum. <**4.2**> E.g., 2, 4, 6, 8, 10. The last is 10 and it is fifth from the first even number (i.e., 2). 6 follows 5. 6 then should be multiplied by 5 (by which 10 is denoted) and 5 times 6 (or the reverse) will result in 30 and this will be the sum of the above-mentioned numbers. <**4.3**> Or multiply half the even number at which you ended by half the next following even number and you will have the sum. <**4.4**> But if you count continuously, missing out the odd numbers, not starting from 2, you should proceed as if you had started from 2 and you should add all the even numbers which are below that from which you started; that sum should be subtracted from the product and the remainder will be the total.

<**5.1**> If you wish to know what sum will result from odd numbers added together, when they follow each other in a natural order from 1, multiply by itself the number by which is denoted the last number at which you ended and you will get the sum. <**5.2**> E.g., 1, 3, 5, 7, 9. The last is 9 and it is the fifth in order. The 5, therefore, by which it is denoted, should be multiplied by itself and you will get 25. For 5 times 5 makes 25 and this was the sum of the above-mentioned numbers. <**5.3**> Or from the closest even number which is after the odd number at which you ended, subtract <its> half; multiply the subtracted part by itself and you will have the sum. <**5.4**> If, however, you do not start from 1, you should proceed as if you had started from 1 and you should add all the odd numbers which are below that from which you started; that sum should be subtracted from the result; the remainder will be the total.

<**6.1**> If you wish to know what sum will result from any number of doubles[3] added together, when they follow each other in a natural order from the first, multiply the last number at which you wish to end by the first even number, i.e., 2, and the total that results from their multiplication, when the first even number is subtracted, is gathered from adding the above-mentioned numbers. <**6.2**> E.g., 2, 4, 8, 16, 32. Multiply the last number, which is 32, by 2 and you will get 64. Subtract <the number> from which you started and what will result is the sum of the above-mentioned numbers, i.e., 62. <**6.3**> Or in another way: the last of however many doubles following each other naturally from the first double, when doubled, is the sum of all of them added together, if the first even number is subtracted.[4] <**6.4**> Or in another way, more generally: if you wish to know what sum will result from any doubles added together, double the last number at which you ended and subtract the first number from which you started and what results from the doubling of the last with the subtraction of the first is the sum that you require. <**6.5**> Or if you wish

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[3]'doubles' here is a technical word which represents 2^n.

[4]It would make more sense if this were 'added', so that this procedure balances the previous procedure, rather than reproducing it.

to start from 1, you should add it to the double of the last <with the 2 subtracted> and you will have the sum of the preceding numbers.

<**6.6**> If you wish to know what sum will result from any triples[5] added together, divide the last number at which you ended and triple its smaller part and what results from its tripling is the sum of the preceding numbers.[6] <**6.7**> E.g., 3, 9, 27, 81, 243. The smaller half[7] of the last number, when tripled, makes 363. The preceding triples added together make this total.

<**7.1**> If you wish to know what sum will result from adding all the squares of any numbers when they follow each other naturally from 1, consider first what sum results from adding the numbers themselves. Then multiply two thirds of the number, however many are following each other naturally by the above-mentioned sum, with the addition of one third of the sum and what results will be the sum of the squares of the numbers themselves. <**7.2**> E.g., let the <number of> numbers following each other naturally be four, i.e., 1, 2, 3, 4, which, when added together, make 10. <The number of> their squares are four, i.e., 1, 4, 9 and 16. If you wish to know what sum will result from adding these squares, multiply two thirds of 4, however many that is, <namely> 3 minus a third, by the above-mentioned sum of numbers, which was 10 and with the addition of one third <of ten> this becomes 30. And this total results if you should add the above-mentioned squares, i.e., 1, 4, 9, 16.

<**8.1**> In every case of numbers exceeding each other by an equal difference, if they consist of an odd number <of numbers>, the middle number when doubled produces as much as the extremes (and the extremes of the extremes) added together. E.g., 2, 4, 6, 8, 10, 12, 14. <**8.2**> In every case of numbers distant from each other by an equal ratio, if they are an odd number <of numbers>, the middle number when multiplied by itself produces as much as the two surrounding numbers (and the numbers surrounding the surrounding numbers) multiplied by each other, as far as 1.[8] E.g., 2, 4, 8, 16, 32. <**8.3**> But if they are an even number <of numbers>, the two middle numbers, multiplied by each other, <produce> as much as the two extremes (and the extremes of the extremes) multiplied by each other. E.g., 2, 4, 8, 16, 32, 64, 128, 256. <**8.4**> But if they are an even[9] number <of numbers>, the two middle numbers, added to each other, produce as much as the two extremes (and the extremes of the extremes as far as 1).[10]

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[5]'triples' here is a technical word which represents 3^n.

[6]The text argues the case when the triples follow each other naturally from the first triple.

[7]The 'smaller half' is the same as 'the smaller part'.

[8]'As far as the first term' would be expected.

[9]The text has 'odd'.

[10]The proper place of <8.4> should be immediately after <8.1>, because what is discussed in <8.4>

On the multiplication of digits by each other

<**9.1**> Every number below 10, when multiplied by itself, produces the total of its denomination times 10, when the multiplication of its difference from 10 by itself is subtracted. <**9.2**> E.g., 6 times 6 should be said to be 60, which is the denomination from 6 multiplied by 10. The difference between 6 and 10 is 4, which, when multiplied by 6, makes 24. When this 24 is subtracted from 60, 36 remains, which is the total which 6 times 6 produces. And this applies for all numbers from one to ten.[11] <**9.3**> But if you wish to multiply a larger number by a smaller one or vice versa, multiply the difference between the larger number and ten by the smaller number and subtract <the result of> this multiplication from the denomination made from the smaller number and what remains is the total which results from the multiplication of different numbers. <**9.4**> E.g., when you multiply 5 by 7, say that it becomes 50, which is the denomination from 5 which was the smaller number there. But the difference between the larger number, i.e., 7, and 10 is 3. When 3 is multiplied by the smaller number, i.e., 5, 15 results. When this is subtracted from 50, 35 remains and this is the total produced from 5 times 7 or vice versa. <**9.5**> Or in another way: let the denomination be made from the larger number and the difference between the smaller number and 10 be multiplied by the larger and <the result> subtracted from the total of the first denomination <is the multiplication of different numbers>. <**9.6**> E.g., 5 times 7 should be said to become 70, which is denominated from 7 and this is the larger number in that multiplication. But the difference between the smaller number, i.e., 5, and 10 is 5 which, when multiplied by the larger, i.e., 7, makes 35. When this is subtracted from 70, which was the denomination from the larger number, i.e., 7, multiplied by 10, there remains 35 and this is the total produced by the multiplication of the one number by the other, <i.e.,> 5 times 7 or vice versa: 7 times 5.

<**10.1**> In every case of three numbers in the same ratio, if you multiply the first by the third, as much results from their multiplication as from the multiplication of the middle number by itself. If the first and middle number are known, but the third alone is unknown, multiply the middle number by itself and divide what results by the first number and what results from the division will be the third. <**10.2**> Or if the first number alone is unknown, multiply the middle number by itself and divide by the third and the first will result. <**10.3**> Or if the middle number alone is unknown, multiply the first by the third and the root of what results is the middle

and <8.1> is about the arithmetical progression, while <8.2> and <8.3> discuss the geometrical progression. <8.1> discusses the case when the number of the terms is odd, while <8.4> discusses the case when the number of the terms is even (assuming our emendation of the manuscripts' 'odd' to 'even' is correct).

[11] 10 can only be retained instead of 9, if 0 is considered as a number. A reference mark of three dots in a triangular shape appears here in P.

number, since the middle number multiplied by itself produces as much as the two extremes, which you can observe easily in the above-mentioned numbers.

<**11.1**> If, then, any four numbers are in proportion, i.e., so that the ratio of the first to the second is the same as that of the third to the fourth, then as much will result from the multiplication of the first by the fourth as from the multiplication of the second by the third. For in these four terms the first and fourth are companions, as are the second and third. Hence, generally: in all <numbers>, whichever is unknown, divide either of the other two by the companion of the unknown and multiply what results by the companion of the dividend[12] and the unknown term will result. Or when any of them are unknown, the product from the other two should be divided by the companion of the unknown and the unknown will result. <**11.2**> Hence, if three numbers are known and only the fourth is unknown, multiply the second by the third and divide what results from this by the first and the result will be the fourth. <**11.3**> Or if the first alone is unknown, multiply the second by the third and divide <the result> by the fourth and the first will result. <**11.4**> Or if the second alone is unknown, multiply the first by the fourth and divide <the result> by the third and the second will result. <**11.5**> Or if the third is unknown, multiply the first by the fourth and divide <the result> by the second and the third will result.

<**11.6**> E.g., if 10 measures are sold for 30 crowns, then 6 crowns are owed for 2 measures. Here four numbers are in proportion, i.e., (1st) 10 measures, (2nd) 30 crowns, (3rd) 2 measures, (4th) 6 crowns. For the ratio of 10 measures to 30 crowns, which is their price, is the same as the ratio of 2 measures to 6 crowns, which is their price. When, then, you multiply the first number, which is 10 measures, by the fourth number, which is 6 crowns, the result is 60. Similarly so much will result from the multiplication of the second number, which is 30 crowns, by the third number which is 2 measures.

<**11.7**> When, therefore, anyone, concealing from you the fourth number, which is 6 crowns, should say: "When 10 measures are sold for 30 crowns, how much is owed for 2 measures?" Multiply, then, 30 crowns, which is the second number, by 2 measures, which is the third number and divide what results by 10 measures, which is the first number and the fourth will result, i.e., 6 crowns, which is owed for 2 measures. <**11.8**> Similarly, if, when concealing the first, which is 10 measures, he should say: "2 measures are sold for 6 crowns, how many measures will be had for 30 crowns?" Multiply, then, the second number, which is 30 crowns, by the 2 measures, which is the third number and divide what results from this by 6 <crowns>, which is the fourth number, and the first will result, which is, namely, 10 <measures> — the

[12] MS has 'divisor'.

amount given for 30 <crowns>. <**11.9**> Similarly, if, when concealing the second number, which is 30 crowns, he should say: "Since I paid 6 crowns for 2 measures, how many crowns should I pay for 10 measures?" Multiply, then, the first number, which is 10 measures, by the fourth number, which is 6 crowns, and divide what results by 2 measures, which is the third number, and the second number will result, i.e., 30 crowns, which is owed for 10 measures. <**11.10**> Similarly, if, when concealing the third <number>, which is 2 measures, he should say: "Since 10 measures are given for 30 crowns, how many measures will they give for 6 crowns?" Multiply, then, the first number, which is 10 measures, by the fourth number, which is 6 crowns and divide <the result> by the second, which is 30 crowns, and the third number will result, which is 2 measures.

<**11.11**> In these questions one should note very carefully what is called 'first' and what is called 'second', i.e., whether the thing[13] or the price is named first. Whatever is named first is repeated in the third place and whatever is named second is <placed> fourth, whether it is revealed or concealed.

<**12.1**> If you do not know the third or the fourth number, but only their sum is given to you, when you wish to find them, add the first and the second and let their addition be the second number for you, in respect to the first known number. Then multiply the first number by the first sum and divide what results from this by the second sum and what results from the division will be the third number. <**12.2**> Similarly, for finding the fourth, multiply the second number by the first sum and divide what results by the second sum and what results will be the fourth number. <**12.3**> Similarly also vice versa: if you do not know the first and the second, knowing their sum, you will find them through the third and fourth according to the above-mentioned rule. <**12.4**> E.g., let there be four numbers, the first 2, the second 4, the third 3, the fourth 6. If, then, the third and fourth are unknown (i.e., 3 and 6), but their sum is shown to you (which is 9), when you wish to find them, add the first and second (i.e., 2 and 4) and you will get 6. Then multiply the first (which is 2) by the first sum (which is 9) and 18 will result. Divide this by the second sum (which is 6) and the third number will result (which is 3). A similar procedure can be used for others, as the figure below makes clear.

6 3	4 2
9	6

<**12.5**> But if the third and fourth are unknown, but what remains from the subtraction of the smaller of them from the larger is known to you, if you wish to find them, subtract the first from the second or vice versa (always the smaller from the larger) and call what remains the 'second subtraction', in respect to the first that was given. Then multiply the first number by the first subtraction and divide what results from this by the second subtraction and what results from this will be the

[13] The 'thing' (*res*) here is the 'measure' of whatever item is being sold.

third number. <**12.6**> Similarly, to find the fourth number, multiply the second number by the first subtraction and divide what results by the second subtraction and what results will be the fourth number. <**12.7**> Similarly also vice versa: if you do not known the first and second, when their subtraction has been shown to you, you will find them through the third and fourth according to the above-mentioned rule, as is clear in the same numerical example. <**12.8**> For if you do not know the third and fourth (i.e., 3 and 6) but what remains from the subtraction of the one from the other is shown to you (which is 3), subtract the first from the second or vice versa (always the smaller from the larger) and 2 remains, which is the second subtraction. Then multiply the first number by the first subtraction and divide what results by the second subtraction and what results from this will be the third number. A similar procedure can be used for others, as the figure below makes clear.

6 3	4 2
3	2

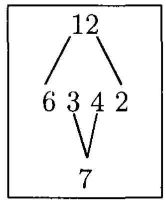
<**12.9**> Similarly also you will find this same thing if you adapt this to things and their prices.

<**12.10**> But if the third and fourth alone are unknown to you, but their multiplication only is given to you, you multiply the first by that multiplication and divide what results from that by the second <number> and the root of what results from the division will be the third number. <**12.11**> Similarly, to find the fourth, multiply the second by that multiplication and divide what results from that by the first number and the root of what results from the division will be the fourth number. <**12.12**> E.g., if you do not know the third and fourth of the above-mentioned numbers (i.e., 3 and 6), but their multiplication alone is given to you (which is 18), then multiply the first (which is 2) by that multiplication and 36 will result. Divide this 36 by the second number (which is 4) and 9 will result from the division, whose root (i.e., 3) is the third number. <**12.13**> Similarly, to find the fourth, multiply the second (i.e., 4) by that multiplication and 72 will result. Divide this by the first number and 36 will result, whose root is 6 and this is the fourth number, as the figure below makes clear.

6 3	4 2
18	8

<**12.14**> But if the two middle numbers are unknown, but their sum alone is given to you, you multiply the first by the fourth and put aside what results from this. Then divide the sum into two parts such that the one multiplied by the other produces the multiplication of the first by the fourth and these parts will be the unknown middle numbers. <**12.15**> E.g., if you do not know 4 and 3, but their sum is given to you (i.e., 7), multiply the first (which is 2) by the fourth (which is 6) and you will get 12. Then divide the given sum (which is 7) into parts such that when multiplied by each other they produce 12 and these parts will be the unknown

middle numbers (i.e., 3 and 4), as the figure below makes clear.



degree	minute	second	3rd	4th	5th	6th	7th	8th	9th
minute	second	3rd	4th	5th	6th	7th	8th	9th	10th
second	3rd	4th	5th	6th	7th	8th	9th	10th	11th
3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th
4th	5th	6th	7th	8th	9th	10th	11th	12th	13th
5th	6th	7th	8th	9th	10th	11th	12th	13th	14th
6th	7th	8th	9th	10th	11th	12th	13th	14th	15th
7th	8th	9th	10th	11th	12th	13th	14th	15th	16th
8th	9th	10th	11th	12th	13th	14th	15th	16th	17th
9th	10th	11th	12th	13th	14th	15th	16th	17th	18th.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

B

<**13.1**> Every number up to 10, when multiplied by itself, produces as much as the first two numbers on each side multiplied by each other, if 1 is taken from the multiplication of the middle number.[14] <**13.2**> Or in another way, more generally: every number multiplied by itself, produces as much as the two extremes (and the extremes of the extremes as far as 1), but with the multiplication of the differences between the number and the extremes added.[15] <**13.3**> For 5 multiplied by itself produces as much as 4 times 6 with the multiplication of the differences by each other, which are two 1s, and as much as 3 times 7 with the multiplication of the differences by each other, which are two twos, and as much as 2 times 8 with the

[14] This rule is the same as <2.3>.

[15] This rule is the same as <2.1>.

multiplication of the differences by each other, which are two threes, and so on as far as 1.[16] <**13.4**> Every number produces as much when multiplied by itself as the two numbers multiplied by themselves which are distant from it by an equal ratio.[17]

<**14.1**> Every number multiplied by itself produces as much as its two parts, if each of them is multiplied by itself and one of them is multiplied by the other twice. <**14.2**> Every number multiplied by another produces as much as when it is multiplied by all the parts of that number. <**14.3**> When any number multiplies another, it produces as much as if the same number multiplied the limit, when the product of the difference between the multiplied and the limit and the multiplier is subtracted.

<**15.1**> Every number to be divided by another is either equal or larger or smaller than the other number. If it is equal, then single units in the dividend result for each of the units of the divisors. <**15.2**> If it is larger, then however many times the divisor comes in the dividend, so many whole units result for each of (the units of) the divisors. <**15.3**> But if anything is left over, it will be divided into fractions. <**15.4**> Thus, as the smaller number is such a part or such a number of parts of the larger number, such a part or such parts should be given to each (of the units) of the divisors. <**15.5**> When you add fractions to fractions, if the number of the fractions is the same as that of their denomination, then a whole unit arises from the addition. <**15.6**> E.g., from three thirds or four fourths one whole unit is produced. <**15.7**> But if the number of the fractions is less than that of their denomination, then, whatever ratio the number of fractions has to the number of their denomination, the fractions have the same ratio to the whole unit. <**15.8**> E.g., six twelfths have the same relation to the whole unit as six to twelve. They are its half. <**15.9**> But if it is larger, then however many times it is larger, so many whole units result from the fractions added together. <**15.10**> E.g., six thirds restore two whole units, since the number of the fraction contains the number of the denomination twice. <**15.11**> But if it contains it a certain number of times plus a part or some parts of it, then however many times it contains it, so many whole units result from the fractions added together plus such a part or as many parts of one whole unit as the number remaining is part or parts of the number denumerating the fractions. <**15.12**> E.g., eight thirds: 8 contains 3 twice plus two thirds of it.

<**15.13**> But if you wish to know what relation a part of any part has to the whole unit, multiply by themselves the numbers by which the fractions are denominated and the part of the part will be related to the whole unit in the way that 1 is related to that product. <**15.14**> E.g., a third part of one quarter is a twelfth part of

[16]This rule is the same as <2.2>.

[17]This rule is equivalent to <10.1>.

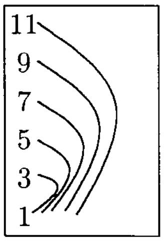
one whole unit. For 3 times 4 becomes 12. <**15.15**> If you wish to know how the parts of any part are related to the whole unit, multiply by themselves the numbers by which the fractions are denominated and those added fractions <denominated by the product> will be related to the whole unit in the way that the number <of fractions> you added is related to the number already resulting. <**15.16**> E.g., two third parts of one quarter are a sixth part of one whole unit. For 3 times 4 are 12, whose sixth part is 2.

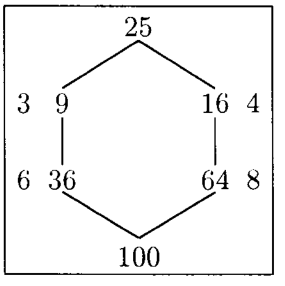
<**15.17**> But if <two>fractions are denominated by two different numbers, then by whatever ratio the greater number is related to the smaller, in such a way the fraction denominated by the smaller number is related to the fraction denominated by the larger. <**15.18**> E.g., a third part of anything contains two sixths of it. For 6 contains two threes. <**15.19**> If fractions of different numbers or different quantities are denominated by the same number, as the whole units are related to each other, so the fractions too and *vice versa.* <**15.20**> For just as 12 is related to 9, so a third part of 12 is related to a third of 9 and *vice versa.* <**15.21**> But if you wish to add any number of fractions denominated by different names, add the numbers by which the fractions are denominated and by the total <times> resulting from this add the fraction denominated by the number which arises from the multiplication of the numbers denominating the fractions. <**15.22**> For if you wish to know what the third and the fourth part make when added together, add the numbers by which the fractions are denominated, i.e., 3 and 4, and they become 7. By this 7 <times> add the fractions denominated by the number which arises from the multiplication of the numbers denominating the fractions, i.e., 12 (for 3 times 4 is 12). The third part and the fourth part of something, therefore, are seven twelfth-parts. What they constitute when added has been shown above.

<**16.1**> ...Or multiply it by itself and add it to the multiplication arising from it and divide this total into two equal <parts> and that half is the whole sum of that and of all the numbers below it.[18] <**16.2**> If you wish to know what sum results when numbers exceeding each other by an equal distance are added together, if those to be added consist of an odd number <of numbers>, consider how many they are and multiply the middle of those to be added by that number and the total is the sum. <**16.3**> E.g., let 3, 4, 5 be added together. Multiply the middle number (i.e., 4) by 3, which is how many they are and the result will be the sum the given numbers produce when added together. <**16.4**> Or let 3, 5, 7, 9, 11 be added together. Multiply the middle number (i.e., 7) by 5, for so many numbers are to be added, and the result will be the sum of the given numbers. <**16.5**> Or let 2, 5, 8 be added. Multiply the middle number (i.e., 5) by 3 and the result will be their sum. <**16.6**>

[18] In discussing the arithmetical progression, this rule is more general than those in <3.1> to <3.5> because the common difference here is not limited to 1, as is shown by the following examples.

But if the numbers to be added consist of an even number <of numbers>, join a larger number belonging to the same series to the earlier numbers. Then consider how many they are and multiply the middle of them by the whole number and when you have taken away what you had joined to the numbers to be added, the sum of the added numbers is produced. <**16.7**> E.g., 2, 4, 6, 8 should be added. Join the next number in the series (i.e., 10). Multiply its middle number (i.e., 6) by 5, since that is the number of the numbers to be added. The result will be 30. Take away from this total the 10 which you had joined and the 20 that remains is the sum of the added numbers. <**16.8**> If any odd number is added to all the odd numbers below it plus 1, the sum which arises will be a square number.[19]



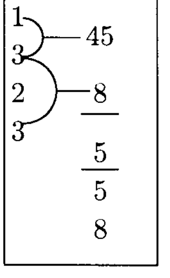


<**17.1**> If the squares of two numbers taken together are a square number, it is necessary that the squares of any two numbers related to each other by the same proportion, when taken together, are a square number.[20] <**17.2**> If any number of quantities are proportioned to a quantity by different, but known, ratios and, when taken together, make a known sum, <the problem is> to find the value of the same first quantity. <**17.3**> E.g., let a be a quantity and let it also be the case that b, c and d are proportioned to a by known ratios. Let it also be the case that b, c and d taken together make up the quantity g and it is known how much g is. It is, then, proposed that one should also find how much a is. <**17.4**> E.g., Socrates is understood to have twice as many pennies as Plato plus two third parts more than those which Plato has and the total that Socrates has, when taken together, is 15 pennies. The problem is to find how much Plato has. I take, then, the number to which the two numbers are proportioned by the above-mentioned ratios. This is 3. For 6 is twice as large as it and 2 is two third parts of it. I add these two numbers, 6 and 2 and 8 results. Having considered, therefore, what ratio 15 has to 8, I pronounce that Plato has the same ratio to the three pennies. For 15 contains

[19] A figure is added in the margin of PQ without any explanation. But we can easily see that this figure is showing the examples of the rule. It can be expressed as: $1+3=2^2; 1+3+5=3^2; 1+3+5+7=4^2; 1+3+5+7+9=5^2; 1+3+5+7+9+11=6^2$.

[20] The figure is in the margin and there is no corresponding explanation about it. It is easy to see that the figure shows an example of the rule. The figure can be explained as follows: since $3^2+4^2=5^2$ and $6:3=8:4=2$, $6^2+8^2=10^2=100$.

the whole of 8 plus seven eighths of it. Similarly 5 pennies and an obol and a fourth part of an obol contain 3 pennies and seven eighths of them. I say, then, that Plato has 5 pennies, an obol and a fourth part of an obol. For twice this amount and two third parts of this are 15 pennies. For if you take 5 pennies, an obol and the fourth part of an obol twice, there will be 11 pennies and the fourth part of one penny. If you take two thirds of the same 5 pennies, one obol and a fourth part of an obol, they will make 3 pennies and three quarters of one penny. <**17.5**> Understand this in the following way: take two third parts of 9 obols (i.e., 6 obols) and they are 3 pennies. But the remaining penny and fourth part of an obol are nine eighth parts of a penny. Since two third parts (i.e., six eighths of a penny) are three quarter parts of one penny, if you add these to the earlier 11 pennies and a fourth part of a penny and 3 pennies, they are 15 pennies. <**17.6**> Or in another way, more simply: multiply the number of the denomination of the parts (i.e., 3) by 15 and the result is 45. Then multiply the same 3 by <two and> two thirds and the result is 8. For 3 <multiplied> by 2 becomes 6 and when two thirds <of three> have been added, it becomes 8. Then divide the first product (i.e., 45) by the last product (i.e., 8) and the result is 5 and five eighths, in this way:

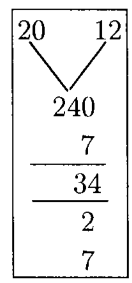


<**17.7**> When the question is: "How many smaller things are in a certain number of larger things?" you should multiply the larger things by the number of smaller things which are in one of the larger ones and the number which arises will show how many smaller things are in so many larger things. <**17.8**> But if the question is: "How many larger things <are> in a certain number of smaller things?" you will divide the smaller things by the number of smaller things which are in one of the larger things and the number which results from the division will show how many larger things are in so many smaller things. <**17.9**> E.g., a shilling is less than a pound. Therefore, if the question is: "How many shillings are in 100 pounds?" find out how many smaller things are in one larger thing. You should, therefore, multiply the number of the larger things (i.e., pounds), i.e., 100, by the number of smaller things (i.e., shillings) which are in one of the larger things (i.e., one pound), i.e., by 20 (for 20 shillings make one pound) and 2,000 results. Know, then, that so many shillings, i.e., 2,000, are in 100 pounds. <**17.10**> Again, let the question be: "How many pounds are in 24,000 pennies?" The question is asked, then: "How many greater numbers are in a number of smaller things?" because pennies are less than pounds. You will divide, then, the number of the smaller things (i.e., pennies), namely 24,000, by the number of the smaller things, i.e., pennies, which are in one of the larger (i.e., in one pound), i.e., by 240 (for so many pennies are in one pound) and 100 will result. Know, then, that so many pounds (i.e., 100) are in 24.000 pennies.

<**17.11**> When the sum of parts of any denomination of a whole is known, <the problem is> to know the whole itself. First, add the numbers denominating the

given parts. Then, multiply one by the other and thus you will have four terms: namely the sum of the given parts, the sum of the numbers denominating the parts and the product of these and the fourth is the whole which is unknown. For the ratio of the whole to the sum of the proposed parts is the same as that of the product of the numbers denominating the parts to the sum arising from their addition. The sum of the proposed parts, therefore, should be multiplied by the product of the numbers denominating the same parts and the product from this[21] is divided by the sum of the numbers denominating them and the total, which is unknown, will result, by the above-mentioned rule of four numbers in proportion: <namely> if the first was unknown, multiply the second by the third and divide <the result> by the fourth and the first will result.

<17.12> E.g., let the third and a quarter of my money be 20 pennies. The problem, therefore, is to find how much is the total amount of money. When the denominators (i.e., 3 and 4) are added, they become 7. Multiply one of them by the other and the product will be 12. You have, then, four terms. First multiply the second by the third, i.e., 20 by 12, and 240 will result. Divide this by 7 and 34 and two sevenths will result. This number should be put into the terms of its genus, i.e., in the pennies, shillings or pounds, in which the proposed parts had been. And this is the total sought.[22] **<17.13>** When three numbers are in proportion, if the first is multiplied by the third the square of the middle number will result.[23] **<17.14>** When several men contribute different sums of money for making a profit, if you wish to known how great a part of the profit which arises from the whole amount comes to each one of them by right, add the portions which they had put down and multiply the contribution of whomever you wish by the total profit. Then divide the product of the multiplication by the sum and what results from the division will be the portion of him whose contribution you have multiplied. **<17.15>** Or the reverse: divide the contribution by the sum and multiply what results by the total profit and what results from this will be his portion. You will follow a similar procedure for each of the others in turn. **<17.16>** E.g., three merchants have contributed their money for making a profit, one 6 shillings, another 8, another 12, which altogether

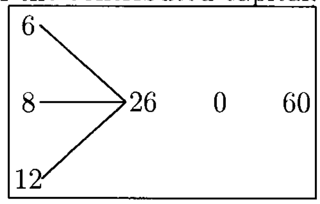


[21]P has added a '7' here, which has probably strayed into the text from the figure in the margin (see next note). In Q the three numbers from the figure —20 (written '30') 12 and 7—have been inserted here.

[22]In P there a defective figure has been added in the margin. Although only 20 and 12 of the first line and the last result $34\frac{2}{7}$ are given in the margin, we can see from the figure of <17.6> that this figure is to show the relevant calculation. Below $34\frac{2}{7}$ is an explanation in the margin: 'This number contains the whole of 20 plus five sevenths of it, just as $\frac{12}{7}$ <of it>', which means $34\frac{2}{7} = 20 + 20 \cdot \frac{5}{7} = 20 \cdot \frac{12}{7}$.

[23]This rule is a repetition of the first sentence of rule <10.1>.

make 26. From this they have made a profit of 60. If you wish, then, to know how much of the profit comes to each one of them according to the quantity of the money contributed, add together the contributions of all of them and they become 26. Then multiply individually the contribution that each has made by the total profit. Then divide what results from the multiplication by the total of the contributed capital, i.e., 26, and what results from the division is what is owed to him whose contribution you multiplied. You will proceed in this way for each person individually. There are, then, these four terms–i.e., the contribution of each one of them, 26, the third is unknown, the fourth is 60–which are proportional according to the above-mentioned rule. Multiply the first by the fourth, i.e., the contribution of each one of them by 60 and divide the product by the second, i.e., 26, and the third will result, i.e., the portion that comes to the man whose contribution you made the first term.[24]



<17.17> If you wish to know how much of any definite sum that is owed to many men comes to some of them, multiply the number of those men about whom you want information, by the sum itself and divide what results from the multiplication by the whole number of the many men and what results is what is due to them. **<17.18>** E.g., 24 pennies are owed to 8 men and you wish to know how much comes to 3 of them. Multiply those 3 by 24 and divide what results from the multiplication by 8 and you will see what will come to them. And according to this rule likewise you will prove how much will come to the other 5, i.e., by multiplying those 5 by 24 and by dividing what results from their multiplication by 8. There are, therefore, these three terms, 8, 24, 3 and the fourth is unknown, i.e., how much is owed to 3. By the above-mentioned rule, then, multiply the second by the third and divide the product of that by the first, i.e., 8 and the fourth (the unknown) will result, i.e., the amount owed to three men. For as 8 is related to 3, so is 24 related to the amount owed to three men. Therefore it contains it twice plus two thirds of it.

8	24	3	0

C

<18.1> Excerpts from the book which is called 'Jabr Muqabala'. A certain three-part division by opposites is made. For the question is either <1> "Which *thing*[25] with so many times its root makes a number?" or <2> "Which *thing* with such a

[24] In the figures of <17.16>, <17.17> and <19.2>, '0' is adopted to represent the symbol of the unknown. Robert Kaplan analyses the tradition of this expression in ancient Greek and Indian mathematics (Robert Kaplan, *The Nothing That Is*, London, 1999, pp. 57-67).

[25] In translating 'res' (the unknown) literally as 'thing', we have printed 'thing' in italics. In MS O 'res' is replaced by 'quadratus' ('square').

number makes the root so many times?" or <3> "Which root taken so many times with such a number makes the *thing*?"

<**18.2**> <1> The question is: "Which *thing* with ten of its roots (i.e., with its root taken 10 times), makes 39?" To find this multiply half <the number of> the above-mentioned roots by themselves, add what results from this to the earlier number, take the root of what arises from this and subtract from this root half <the number of> the above-mentioned roots; what remains from this is the root of the *thing*. If you multiply the root by itself, the *thing* which you seek results. <**18.3**> E.g., since 10 roots had been proposed above, if you multiply half them (which is 5) by themselves, you make 25. Add these to the above-mentioned number (which is 39) and you will make 64, whose root is 8. If you subtract from this root (i.e., 8) half <the number of> the roots (which is 5), there remains 3, which is the root of the *thing*, it being 9. This, with 10 of its roots (i.e., 10 threes) makes 39. Therefore 9 is the *thing* which is sought.

<**18.4**> (2) Likewise, what is the *thing* which, with 9 added to it, makes 6 of its roots? To find this multiply half <the number of> the roots by themselves, take away the above-mentioned number from what results and subtract the root of what remains from half <the number of> the roots. What remains will be the root of the *thing* which you seek. <**18.5**> E.g., 6 roots have been proposed. Multiply its half (which is 3) by itself and you will make 9. From this 9 take the above-mentioned number (i.e., 9) and nothing remains. Subtract the root of this which remains (i.e., nothing), which similarly is nothing, from half <the number of> the roots (which is 3). Because you have subtracted nothing from 3, 3 remains, which is the root of the *thing* which you seek, i.e., of 9, which with 9 added to it becomes 18, which is 6 roots of 9, i.e., 6 times 3, this 3 being the root of 9.[26]

<**18.6**> (3) Likewise, what are the roots which with 4 added to them make their *thing*? To find this multiply half <the number of> the roots by themselves, add what results from this to the above-mentioned number and add the root of what arises from it to half <the number of> the roots and what arises from this is the root of the *thing* which you seek. <**18.7**> E.g., 3 roots have been proposed. Its half is 1 and a half, which when multiplied by itself makes 2 and a quarter. Add these to the earlier number (which is 4) and you will make 6 and a quarter. The root of this is 2 and a half. Add this root to half <the number of> the roots, which is 1 and a half and they will become 4. This is the root of the *thing* which you seek, i.e., of 16. Three of its roots (i.e., 3 fours or four three times) with 4 added to them make this 16.

[26] This also occurs in Ibn Turk, English trans. p. 100-101. (Aydin Sayili gives no source).

D

<**19.1**> If you wish to divide any given quantity into any given ratios, or to divide an undivided number in the same way as a divided one, first the ratios of the proposed divisions should be arranged in their terms, then the terms should be added up and the sum should be placed first. When the number of the ratio is in a ratio to what is sought, it should be placed second. The given quantity should be placed third. An empty space should be placed fourth. The given quantity is multiplied by the number of the ratio and the product from that should be divided by the sum of the terms themselves and what is sought will emerge through the rule of four <numbers> in proportion. If the fourth is unknown, the third should be multiplied by the second, the result should be divided by the first and the fourth will emerge. If you wish to know the others through one known number, you will have the multiples by multiplying it by the number denominating the ratios. One calls 'multiples' those which contain a number a certain number of times. One calls 'submultiples' those which are contained <by another number> a certain number of times. If any number remains after division, it will be the number of fractions of the quantity, denominated in the division by the divisor. <**19.2**> E.g., we are asked to divide 40 shillings between 4 men in such a way that the second has 4 times what the first has, but the third has 5 times the second, the fourth 3 times the third.

85	first	1		40 0
	second	4	four	
	third	20	five	
	fourth	60	three	

4, therefore, should be multiplied by 1 and 4 results. Let the first term be 1, the second 4. Likewise, 4 should be multiplied by 5 and 20 results. The third term, therefore, will be 20. Likewise, 20 should be multiplied by 3 and the result is 60. The fourth term, therefore, will be 60. When the terms are added, 85 results. Let 85 be put first. Then, if you want to know what comes to the second <man>, place the second term second.[27] Likewise, if you want to know what comes to the third, place the third term second. Likewise, for each one of them.

<**19.3**> So, place the second term (i.e., 4) second. Place the given quantity third, the empty space fourth. Let the given quantity (i.e., 40) be multiplied by 4 and the result is 160. Let 160 be divided by 85 and there emerge 1 and 75 85ths. This number should be arranged in the denominations of its genus–pennies, shillings or pounds–in which the given quantity was proposed. This is the answer. <**19.4**> If you wish to know through this what comes to the first (man), divide the given number, i.e., one whole (i.e., 12 pennies which are 1 shilling) by 4 and the result is 3 pennies. Likewise, divide the numerator of the fractions (i.e., 75) by the same 4 and the result is 18 85ths and three quarters of an 85^{th} and this is the portion of the

[27] I.e., in the second place in the rule of four.

first (man), i.e., 3 pennies and 18 85ths plus three quarters of an 85$^{\text{th}}$ (of a shilling). <**19.5**> Likewise, if you wish to know the others, i.e., its multiples, through the number already known, multiply the known number, i.e., 1 and 75 85ths by the number denominating the third ratio, e.g., by 5 and the result is 9 and 35 85ths shillings and this is the portion of the third (man). (One does) the same with each multiple. <**19.6**> But if you wish to know the submultiples, divide the already known number by the number denominating the ratio. <**19.7**> For proving this add whole numbers to whole numbers.

<**20.1**> If you wish to continue any given ratios to any least term, multiply the first of the given ratios by the least term, multiply the product by the next (ratio) and proceed in this way, multiplying the result of the following (product) by the following (ratio) and the result from multiplying the penultimate following (product) by the last following (ratio) will be the first of the numbers to be continued. Having done this, multiply the second of the given ratios by the same term and multiply the product by the following (ratio) and proceed in multiplying in the above-mentioned way and the result from multiplying the penultimate following (product) by the last following (ratio) will be the second of the numbers to be continued. Likewise multiply the third of the given ratios by the same least term and multiply the result by the following ratio and proceed in multiplying in the above-mentioned way and the result from multiplying the penultimate following (product) by the last following (ratio) will be the third of the numbers to be continued. Likewise concerning each one. <**20.2**> E.g., let the given ratios be 4, 3, 2, 1 and let the least term be 6. Let 4 be multiplied by 6 and the product will be 24. Multiply this by 3 and the product will be 72. Let this be multiplied by 2 and the result will be 144. If this is multiplied by 1, the same number will result. This, therefore (i.e., 144), will be the first of the numbers to be continued. <**20.3**> Likewise let 3 be multiplied by 6 and the result will be 18. Multiply this by 2 and 36 becomes the second number to be continued. <**20.4**> Likewise multiply 2 by 6 and 12 results. 12, therefore, will be the third of the numbers to be continued. <**20.5**> Likewise multiply 1 by 6 and 6 results. This, then, will be the fourth of the numbers to be continued.

<**21.1**> If you wish to find the root of any given square, multiply the square by any other square and take the root of the product and divide the root by the root of the square by which you multiplied the given square. What results will be the root sought. <**21.2**> E.g., if you wish to find the root of 2 and a quarter, multiply that by another square, which could, for instance, be 4 and 9 results. For twice 4 becomes 8 and a quarter times 4 becomes one — hence 9. The root of this 9 is 3. Divide this by the root of 4 (which is 2); 1 and a half will result. In this way: [28]

1
3
2

[28] In this figure, 1 is the result of a quarter times 4, 3 is the square root of 9, and 2 is the square

<**21.3**> If you wish to find the root of the nearest square, multiply the third given number by any square and proceed in the above-mentioned way.

<**22.1**> If you wish to multiply an article by an article or a digit by a digit or a composite number by a composite number, multiply the numeral by the numeral. Then add the numbers denominating the columns (decimal places), subtract 1 from the sum and the following column, denominated by what remains, grows so many times as the number of units grow from the multiplication of the numerals, and the following column[29] will grow as many times as the number of tens grow.

<**23.1**> When you multiply one number by another, consider what part one of them is of another article or limit and take the same part from the other number. Multiply that part by that article or limit and the product from that is what results from the multiplication of the one by the other. <**23.2**> E.g., if you multiply 32 by 25, whatever part 25 is of the limit 100 (i.e., a quarter), take the same part of 32, i.e., a quarter, which is 8. Multiply this 8 by 100 and the product from this is what comes from the multiplication of 32 by 25. The proof: just as 8 is to 32, so 25 is to 100. As much arises, then, from the multiplication of the two middle numbers by each other as from the multiplication of the two extremes, according to the above-mentioned rule of four numbers in proportion, and the same is true in all cases. <**23.3**> One can do a similar thing with the article: e.g., 25 is half 50. But half 32 is 16. Multiply this by 50 and the same result is reached as from the multiplication of 32 by 25.
$\boxed{8\ \ 32\ 25\,100}$

<**23.4**> When you multiply together two composite numbers consisting of the same or different digits but the same article or limit, such as 16 by 18 and the like, multiply the digit by the digit and the article by the article and add the products from this. Then add the digit to the digit and multiply the sum by the article or limit and add the product from this to the first sum and the total sum is the result which the multiplication of one composite number by another produces. <**23.5**> E.g., let us suppose that 16 is to be multiplied by 18. The digit, then, should be multiplied by the digit (i.e., 6 by 8) and the result is 48. Then the article by the article (i.e., 10 by 10) and the result is 100. Add these two products together and they become 148. Then add the digit to the digit and the result is 14. Multiply this by the article (i.e., 10) and the result is 140. Add this to the first sum (which was 148) and the result is 288. This is the total which arises from the multiplication of 16 by 18.

<**24.1**> When you wish to multiply the roots of any numbers, multiply the numbers by each other and the root of the product is the product resulting from the multiplication of one root by another. <**24.2**> E.g., if you wish to multiply the root

root of 4.

[29] The Latin text gives 'the beginning of the following column' in both cases.

of 10 and 40, multiply 10 by 40 and 400 results. The root of this 400 is 20. This 20 is the number produced from the multiplication of the root of 10 by the root of 40 through the rule of three numbers related to each other proportionally, of which, when the second is related to the third in the same way as the first is to the second, then whatever results from the multiplication of the middle one by itself, so much results from the multiplication of the two extremes, in this way: $\boxed{10\ 20\ 40}$

<**25.1**> If you wish to know how old a man is who, if he lives as long as he has already lived and the same amount again and half that amount and half of half that amount, completes 100 years, add what is given and divide the total which is completed by the sum and the result is his age. <**25.2**> E.g., when it is proposed that <he lives> as long as he has lived and the same amount again and half that amount and half of half that amount, all these added together make 4 less a quarter, which are 15 quarters. If you divide 100 (turned first into quarters) by these 15 quarters, the result is 26 and two thirds. When these are taken 4 <minus a quarter> times, they complete 100 and this is the (length of time) he has lived.

E

<**26.1**> ...Or another way: since every number is either a digit or an article or a limit or composite, then however many times a number is multiplied by a number, either a digit is multiplied by a digit or by an article or by a limit or by a composite number or *vice versa*, or a composite number (is multiplied) by a composite number or an article or a limit or a digit or *vice versa*. <**26.2**> When you wish to multiply an article by an article, multiply the numeral by the numeral. Then consider which places the articles themselves belong to and add the numbers by which their places are denoted. Take 1 from the sum and in the place named by the remaining number put the product from the multiplication of the numerals if it is only a digit. If, however, it is only an article, <put> it in the <place> following it. But if it is a digit and an article, the digit should be put in the place denoted by the remaining number, but the article should be put in the following place. What is indicated there will be the total that results from the multiplication of one article by another. <**26.3**> E.g., if you wish to multiply 20 by 70, multiply together the numerals by which they are represented (i.e., 2 by 7) and the result is 14. But because both belong to the second place (which is of the tens), add the numbers denoting the place of each of them, i.e., 2 and 2–for the second <place> is denoted by '2'–and the result is 4. Subtract 1 from this 4 and 3 remains. By this 3 is denoted the third place, which is of the hundreds. Because the result of the numerals multiplied by each other had been 14, which is a digit and an article, put the digit (i.e., 4) in the same place (i.e., the third place) and the article (i.e., 10)[30] in the following place

[30] '1' would be more correct.

which is the fourth and you will have 1400. This is the total that results from the multiplication of one article by another (i.e., 20 by 70). One should proceed in a similar way if a digit is multiplied by an article or a limit or a composite number and *vice versa*. <**26.4**> When you wish to multiply a composite number by a composite number, you will observe the above-mentioned rule, with this addition: that each of the higher numerals should be multiplied by each of the lower numerals, i.e., the digit by the digit and the article and the article by the digit and the article, <and> however many they are, each of the higher numerals <should be multiplied> by all the lower numerals. <**26.5**> E.g., when you wish to multiply 23 by 64, multiply the higher digit (i.e., 3) by the lower digit (i.e., 4) and the result is 12. According to the earlier rule, then the digit will be in the first place, the article in the second. Then multiply the same digit (i.e., 3) by the numeral of the lower article (which is 6) and the result is 18. When the numbers denoting the places are added (i.e., 1 and 2, for the article is in the second place, the digit in the first), the result is 3. When 1 is subtracted from this, 2 remains; this denotes the second place. So put the digit (i.e., 8) in the second place and 1 in the following place (i.e., the third). Then you will multiply the numeral of the higher article (which is 2) by the lower digit (which is 4) and the result is 8. When the numbers denoting the places (i.e., 1 and 2) are added the result is 3. When 1 is subtracted from this, 2 remains; this denotes the second place. Therefore, the digit (i.e., 8) should be put in the second place. Then you will multiply the numeral of the higher article <(i.e., 2)> by the numeral of the lower article (i.e., 6) and the result is 12. Since each belongs to the second place, when the numerals denoting the places are added together (i.e., 2 and 2) the result is 4. When 1 is subtracted from this, 3 remains; this denotes the third place. So place the digit (i.e., 2) in the third and the article < (i.e., 1)> in the following (i.e., fourth) place and the result looks like this:

		1	2
	1	8	
		8	
1	2		

<**26.6**> When these are arranged like this, add them up and the result is 1,472. This is the total that results from the multiplication of 23 by 64. <**26.7**> In this same rule one also learns how a composite number should be multiplied by an article, a limit or a digit.

<**27.1**> When you multiply a digit by a digit the result will be either a digit only or a ten only or a digit with a ten once or several times or a ten many times. <**27.2**> When you multiply any digit by any article which is below 100, multiply the numeral by the numeral and however many units there are in the digit which arises, there will be so many tens. However many tens there are in the article which arises, there will be so many hundreds. <**27.3**> E.g., when you wish to multiply 7 by 70, multiply the numeral by the numeral and 49 results. There are 9 units in the digit. So, there will be this many tens: i.e., 90. In the article there are four times ten; so there will be this many hundreds, i.e., 400. 490, therefore, is the total that results from

the multiplication of these numbers by each other. A similar procedure is used in all other cases. <**27.4**> When you multiply a digit by any hundred below 1,000, multiply the numeral by the numeral and however many units there are in the digit (if it results), so many hundreds there will be. But however many tens there are in the article (if it results), so many thousands there will be. <**27.5**> E.g., if you multiply 3 by 900, multiply the numeral by the numeral and the result is 27. There are 7 units in the digit and 2 tens in the article. Therefore 2,700 is the sum which results from their multiplication by each other. A similar procedure is used in all other cases.

<**28.1**> When you multiply any article by another which is below 100, multiply the numeral by the numeral and however many units there are in the digit which results, so many hundreds there will be. But however many tens there are in the article, so many thousands there will be. <**28.2**> E.g., if you multiply 30 by 70, multiply the numeral by the numeral and the result is 21. There are 2 tens in the article and one unit in the digit. From the multiplication, therefore, of the earlier numbers there results 2,100. A similar procedure is used in all other cases. <**28.3**> When you multiply any article which is below 100 by any hundred which is below 1,000, multiply the numeral by the numeral and however many units there are in the digit, so many thousands there will be. But however many tens there are in the article, so many ten thousands there will be. <**28.4**> E.g., if you multiply 30 by 500, multiply the numeral by the numeral and the result is 15. Since there are 5 units in the digit, there will be 5 thousands. But there is only one ten in the article, <so there will be one ten thousand>. From the multiplication, therefore, of the earlier numbers there results fifteen thousand, like this: 15,000. A similar procedure is used in all other cases.

<**29.1**> When you multiply any hundred which is below 1,000 by another of them, multiply the numeral by the numeral and however many units there are in the digit which results, so many ten thousands there will be. But however many tens there are in the article, so many hundred thousands there will be. <**29.2**> E.g., when you multiply 300 by 500, multiply the numeral by the numeral and the result is 15. There are 5 units in the digit and one ten in the article. From the multiplication, therefore, of the above-mentioned numbers there results 150,000, which is one hundred and fifty thousand.

<**30.1**> When you multiply any digit by any article belonging to the thousands, such as ten or twenty thousand and so on, or repeated thousands, such as ten thousand thousand and however much you wish to repeat the thousand, multiply the numeral by the numeral and put the digit (if it results) in the place of the multiplier and the article in the following place. <**30.2**> E.g., if you multiply 6 by 30,000, multiply the numeral by the numeral and the result is 18. The digit (which is 8) should

be put in the same place as the multiplier (which is 3) and the article (which is 1) should be put in the following place, in this way: 180,000 and there results one hundred and eighty thousand. <**30.3**> But when you multiply any article by any oft repeated thousands, like ten or twenty thousand thousand thousand (however many times you wish to repeat the thousand), multiply the numeral by the numeral and put the digit (if it results) in the second place away from the multiplier, but the article in the third place from it. <**30.4**> E.g., when you multiply 30 by four thousand thousand thousand thousand (and however many times you wish to repeat it), multiply the numeral (which is 3) by the numeral (which is 4) and the result is 12. Therefore place the digit (which is 2) in the second place from the 4, and the article in the third place from the 4, in this way: 4,000,000,000,000 becomes one hundred and twenty thousand thousand thousand thousand (i.e., 4 thousands): 120,000,000,000,000. <**30.5**> When you multiply any hundred by any oft repeated thousands, multiply the numeral by the numeral and place the digit (if it results) in the third place from the multiplier, but the article in the fourth place from it. <**30.6**> E.g., when you multiply 200 by five thousand thousand thousand thousand (i.e., 4 thousands), multiply 2 by 5 and, since an article results, it should be put in the fourth place from the 5, in this way: 1,005,000,000,000,000.[31]

<**31.1**> When you wish to know in which place are the digits, articles, or hundreds of oft repeated thousands, consider how many times the thousand is repeated, multiply by 3[32] the number of times it is repeated and keep in mind the product of this. If you wished to know in which place are the digits of the repeated thousands, e.g., two, three or four and up to nine thousand thousand thousand (however many times you wish to repeat the thousand), always add 1 to the first result kept in mind and the place belonging to the digits of the repeated thousands is denoted by the number that arises from this. <**31.2**> E.g., if you wish to know in which place is three thousand thousand thousand thousand, multiply by 3 the number of times the thousand is repeated (i.e., 4 in this case) and the result is 12. Add 1 to this and the result is 13. The place of the above-mentioned is therefore the thirteenth. <**31.3**> If you wished to know in which place are the articles of oft repeated thousands (e.g., ten or twenty thousand thousand, however many times you wish to repeat the thousand), always add 2 to the earlier result kept in mind and the place belonging to the articles of the oft repeated thousands is denoted by the number that arises from this. <**31.4**> E.g., if you wish to know in which place are the above-mentioned articles of oft repeated thousands, such as fifty thousand thousand thousand thousand (i.e., 4 thousands), multiply by 3 the number of times the thousand is repeated (i.e., 4 in this case) and the result is 12. Add 2 to this and the result is 14. The place of the articles

[31] The scribe has shown an interim calculation. In the final result the '5' should be replaced by a '0'.

[32] In the Latin 'multiply by 3' precedes 'consider how many times the thousand is repeated'.

of the above-mentioned repeated thousands is therefore the fourteenth. <**31.5**> If you wished to know in which place are the hundreds of repeated thousands, always add 3 to the earlier result and the place belonging to the hundreds of the repeated thousands is denoted by the number that arises from this. <**31.6**> E.g., if you wish to know in which place is a hundred or two hundred or some other hundred thousand thousand (however many times you wish to repeat the thousand), multiply by 3 the number of times the thousand is repeated (i.e., twice) and the result is 6. Add 3 to this and the result is 9. The place of abovementioned hundreds of repeated thousands is therefore the ninth.

<**32.1**> When you wish to multiply any thousand or ten or hundred thousand of oft repeated thousands by any other thousand, putting aside the repetition of thousands in the multiplied and the multiplier, multiply by each other what remains in each of them and keep in mind what results from this. Then add the number of repeats of each and place the total arising from this under what resulted earlier and what arises from this is the number which results from the multiplication of one by the other. <**32.2**> E.g., when you wish to multiply digits of thousands[33] by each other, such as three thousand thousand by seven thousand thousand thousand thousand, having put aside the number of repeats of <the thousands of> each (i.e., 2 and 4, for in the multiplied the thousand is counted twice, in the multiplier 4 times) and there remain only the numerals of each of them (i.e., 3 and 7). When one of these is multiplied by the other the result is 21. Then add the number of repeats of each of them (i.e., 2 and 4) and the result is 6. Put this under what resulted earlier (i.e., 21) in this way: $\boxed{\begin{array}{c}21\\6\end{array}}$ You will say that 21 times a thousand thousand thousand thousand thousand thousand (i.e., 6 thousands) results from the multiplication of one of the above-mentioned numbers by the other. According to the above-mentioned rule, therefore, if you multiply the number of repeats by 3, 18 will result. When 1 is added, 19 results. The above-mentioned digit therefore (which is 1) was in the nineteenth place and the article, 2,[34] in the twentieth. <**32.3**> When you wish to multiply hundreds of repeated thousands by each other, such as five hundred[35] thousand thousand by three hundred thousand thousand thousand thousand (i.e., 4 thousands), having put aside the number of repeats of <the thousands of> each (i.e., 2 and 4, for in the multiplied the thousand was repeated twice, in the multiplier 4 times) there remain 50<0> in the multiplied and 300 in the multiplier. When these are multiplied by each other the result is 150 thousand. Keep this in mind. Then add the number of repeats of each of them (i.e., 2 and 4) and the result is 6. Put this under the earlier numbers and the total will be indicated, which results from the multiplication of one

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[33] A different word is used for 'thousands' here: 'millena' instead of 'milia'.

[34] Instead of writing '2' the scribes, thinking of the value of the numerals, have written '20'.

[35] The MSS gives 'fifty'.

of the above-mentioned numbers by the other, i.e., one hundred and fifty thousand thousand thousand thousand thousand thousand thousand (i.e., 7 thousands), in this way: 150,000,000,000,000,000,000,000. According to the previous rule, therefore, 5 will be in the twenty-third place and 1[36] in the twenty-fourth. <**32.4**> The procedure is similar also when you multiply digits of repeated thousands by tens or hundreds of repeated thousands and *vice versa*. The procedure is similar also when you multiply tens of repeated thousands by each other or by hundreds.

<**33.1**> When you have a place and want to know what number it is, divide the number by which the place is denoted by 3. If there is no remainder from the division, that place will belong to a hundred thousand repeated a certain number of times. <**33.2**> If you wish to know the number of repeats (i.e., how many times <the thousand> is repeated) take 1 from what results from the division and what remains will be the number of repeats of that hundred thousand which belong to that place. <**33.3**> E.g., if you have the twelfth place and wish to know what number that is, divide that 12 by which the place is denoted by 3 and 4 will result. When 1 is subtracted from this, 3 remains and since nothing has remained from the division, the twelfth place will belong to a hundred thousand thousand thousand (<i.e.,> repeated 3 times). <**33.4**> But if the remainder from the division is 2, that place will belong to a ten thousand repeated as many times as the number which results from the division. <**33.5**> E.g., when you have the eleventh place and wish to know what number that is, divide 11 by 3 and 3 whole numbers will result and the remainder will be 2. The eleventh place, therefore, belongs to a ten thousand repeated 3 times. <**33.6**> But if the remainder from the division is one, that place will belong to a digit thousand repeated as many times as the number which results from the division. <**33.7**> E.g., if you have the tenth place and wish to know what number that is, divide 10 by 3 and 3 will result from the division with 1 as the remainder. The tenth place, therefore, belongs to a thousand repeated three times.

<**34.1**> You will find in this way the number which someone holds concealed in his heart, without him indicating it. First, tell him to triple the number. Then, let him divide the result of the tripling into two parts. After this ask whether the parts are equal. If they are unequal, keep 1 in mind and tell him again to triple the larger part and divide the result into two parts.[37] If, when asked, he replies that the parts are unequal, keep 2 in mind. When this is added to the earlier 1, 3 results. Then, tell him to subtract 9 from the larger part and again another 9 and continue in this

[36]Instead of writing '5' and '1' the scribes, thinking of the value of the numerals, have written '50,000' and '100'.

[37]The two parts are either equal or their difference is 1. This belongs to the field of number theory such as that later developed by Fermat.

way until there does not remain anything from which 9 can be subtracted. For each 9 take 4. The sum of these fours, together with the first <numbers kept in mind> is the number which he concealed. Or if he is not able to subtract 9 from the larger part, the first 3 will be the concealed number. Or if each division results in equal numbers, you will take nothing, but the single or many fours taken from the single or many nines will be the concealed number. However many times the division results in unequal numbers, take 1 from the first (unequal) division and 2 from the second (unequal) division. <**34.2**> E.g., let 2 be the number that he conceals. When tripled it makes 6. But 6 is divided into two equal parts. When either part is again tripled it makes 9, which is divided into two unequal parts. Hence, because it is the second division, I keep 2 in mind. And because 9 cannot be subtracted from the larger part, the 2 which I kept in mind is the concealed number. And this is what we proposed to do.[38]

<**34.3**> Likewise, tell someone who is concealing[39] the number of shillings he has, that he should take from your <pennies> for each <shilling> a single <penny> or 2 <pennies> or 3 <pennies> or any such amount that you wish. With all your pennies he should buy something, such as a chicken or something else like this. Then with all his shillings he should buy as many chickens as he can for the same price. Divide, then, the shilling by the number which you gave him (i.e., by 1 or 2 or 3 <or such like>). Add 1 to the number resulting from the division and what results when 1 is added is the number of those things that are bought. <**34.4**> E.g., let us suppose that he is concealing 5 shillings. Having taken from my pennies as many pennies as there are shillings (i.e., 5), let him buy one chicken and at the same price 12 chickens are bought with his shillings. Divide, then, the shilling into the 12 <pennies> <and divide this> by the number that you gave him (i.e., 1 <each>); 12 will result, which is the number of chickens that were bought. If you had given him 2 for each <shilling>, there would be 6 chickens, if 3 for each <shilling>, there would have been 4 and similarly for the others.

<**34.5**> If any two equal amounts are concealed and 2 taken from one of them is added to the other and from the augmented one something equal to the remainder is subtracted,[40] necessarily 4 will remain, or if 3, necessarily 6 and thus there will always remain the double of what is taken first. <**34.6**> E.g., if 5 shillings are concealed in one hand and 5 in the other, place 2 taken from one hand with the other 5 and they become 7 in one hand, while 3 remain in the other. When then you

[38] In these last two sentences the scribe has used a code whereby each vowel is represented by the letter of the alphabet that follows (here I = k, o = p and u = x).

[39] Here a different code is used, whereby the five vowels are represented by the first five numerals (a = 1, e = 2, i = 3, o = 4 and u = 5).

[40] MS gives 'added'.

take from the 7 what is equal to the remainder (i.e., 3), necessarily 4 <shillings> remain.

F

<**35.1**> It is asked why we do not designate all or most numbers with their own names, or why we always count, not by adding new names, but, after ten, by the repetition of the first ones. <**35.2**>The response to this is that it was not possible for all numbers to have their own names, because the mass of numbers grows to infinity, but the invention of names cannot in any language be infinite.[41] <**35.3**> For, since in every language the instruments <of speech[42] > are certain and limited and so are their naturally determined variations (by which the articulated spoken word is formed and from which arise the shapes of letters among all people and the compositions of these letters, which are diverse but defined by placement before or after <each other> for representing the names of all things), necessarily, although all numbers <taken together> are infinite, each neither could nor ought to have had its own name, especially since men as well, using numbers in almost every action, would be excessively hindered if in their counting they were always forced by the necessity of counting to have ready to hand an infinite multitude of names for numbers. <**35.4**> For this reason it was necessary to limit the infinite progression of numbers by means of definite limits <and> to designate them with <only a> few names, so that a person would not always be forced to proceed in counting by new additions of both numbers and names, but could embrace any sum with a few names through the brief repetition of earlier names.<**35.5**> Hence, since it was impossible for all numbers to have names, but necessary for some,[43] reason demanded – by the prescription of nature – that only 12 from all numbers have names: three limits, i.e., the ten, the hundred and the thousand and the nine first numbers established under ten from one to nine.

<**36.1**> The nine more than all the other numbers justified this rationale by a rightful privilege, inasmuch as it contained in itself almost all species of numbers and of numerical ratios. <**36.2**> For this rationale could not consist in the three, even though it is dedicated to God, because it lacked the first perfect number, which

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[41] Cf. *Liber mahameleth* 3.10-11: numerus crescit in infinitum. Unde singuli numeri non potuerunt propriis nominibus designari. From this point until 35.5 (et aliquos necesse) the text follows the *Liber mahameleth* (ed. Vlasschaert, 3.11-24) almost *verbatim*.

[42] Reading <*loquendi*> *instrumenta* with the *Liber mahamaleth*.

[43] Here the agreement between the *Liber mahameleth* and this text ends. The *Liber mahameleth* continues with 'et quoniam necesse erat eos inter se multiplicari, idcirco dispositi sunt per ordines sive differentias' ('and since it was necessary that they be multiplied by each other, they were arranged in orders or differences, <i.e., places>').

is the six; but neither for this reason could it consist in the six, because it lacked the first cube, which is the eight; but neither for that reason in the eight, because it lacked the first true plane, which is in the nine. <**36.3**> Thus it was from this plenitude of virtues that the nine deserved to have the rationale of counting and of naming numbers in itself, beyond which no number except the three limits would have its own name. <**36.4**> And of course both celestial and earthly things, both bodies and spirits seem to have been formed and put in order according to the model of the nine; for nine are the spheres of celestial bodies, nine are the orders of celestial spirits and nine are the temperaments of all bodies. <**36.5**> It was, therefore, incumbent that there be nine compositions of number in which alone the whole infinity of numbers would consist, just as the universe of bodies from nine temperaments. <**36.6**> For just as in temperaments one is equal and another unequal, but one alone is in the middle (tempered), likewise in numbers also one is even, another odd and among them all only the unit is not unlike itself in any part, is always the same, always equal.

<**37.1**> In this manner creatures will only avoid departing from whatever likeness <they have> to their Creator while they contain themselves within that number, because, when the first odd number is multiplied by itself, that number comes about which, after the unit, alone is consecrated to God (because "God rejoices in the odd number[44] "). <**37.2**> Hence also, beside the nine, among the other <numbers> only the three limits received names, so that by this of course they might maintain some likeness to the Trinity, which is the true limit of all things, "the alpha and the omega, the beginning and the end";[45] and so that they might never depart from the root of the nine. <**37.3**> Because of this, then, reason demanded that, because the universe of things is contained within the nine, in the same manner the infinity of numbers should be restrained within the nine and be designated by nine names and represented by nine figures. <**37.4**> For every copy retains the likeness of its paradigm; otherwise the one would not be the copy or paradigm of the other, and because, as was said, almost all things were founded on the model of the nine, it was necessary that the very infinity of numbers itself be restrained under the nine, so that number also would not depart from that shape according to which the Creator composed all things and which the totality of things borrowed from number.

<**38.1**> Hence men as well, imitating primeval nature, gave names to only nine numbers and invented only nine shapes to represent all numbers. <**38.2**> But because there was still a species of number lacking, which the nine did not contain

[44]Virgil, *Eclogues* 8.73-5: "I draw these triple threads with their three different colours around you and thrice I lead this effigy around these altars; god rejoices in the odd number."

[45]*Revelation* 1:8: "I am Alpha and Omega, the beginning and the ending, saith the Lord, which is and which was and which is to come, the Almighty."

within itself—i.e., the "superabundant" number, the first <of> which is the twelve—for that reason, after the nine, names were given only to the three limits, so that the nine with its root, i.e., the three, contained all the dignities and properties of number within itself and no property, no mystery could be found in numbers which does not appear to be contained in the whole nine with its root. <**38.3**> Since, therefore, neither all nor most numbers, but only a few were necessarily to be designated by their own names, for the reasons already stated names were granted only to the nine numbers and the three limits: that they might better serve human needs through their conveniently small number, express the hidden secrets of things through some sort of symbols and not depart from the rules of nature. <**38.4**> So much for these things.

<**39.1**> The unit is <both> the origin and the first part of number;[46] for every number is naturally made up of units and the unit, since it is simple, precedes every number by nature. <**39.2**> Furthermore, because it is simple, it can generate[47] nothing by multiplication of itself except that by which it is multiplied, something which do not occur in the others which are not simple; for from the multiplication of any number by itself, just as <from its multiplication> by any other, it is necessary for a different number to emerge. <**39.3**> When the unit is multiplied by itself, however, it generates only itself; for one times one is one. For by whatever number you multiply it, you make nothing except that number by which you multiply it and because no number is generated from it except that by which it is previously multiplied, for that reason, since in the beginning there was nothing to which it could be joined for the generation of the first number, it was necessary that it itself be doubled in itself and made different from itself in some fashion, so that, taking it both as itself and as something different from itself, as if from different things, it was possible for something to be generated. <**39.4**> This is the first generation of number, which appears in the two; for this reason the two is also called the principle of difference, because it was born from the unit when the latter was made different. For this reason as well, it is only the case with the two and with no other that from its multiplication by itself the same arises as from its addition <to itself>; for it does not consist of number. <**39.5**> And since, beside the two, there was not yet anything except the unit, for this reason the unit is joined with the two like a man to a female; from the joining of these is born the three, which, after the unit, first receives the names "odd" and "male".

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[46]It would seem necessary to add "prima" (the "first" part) because of the parallel passage in the *Liber Alchorismi* (Allard 1992: 63): unitas est origo et prima pars numeri. Omnis enim numerus ex ea componitur...; see Lampe 2005: 14.

[47]Compounds of *genero* are translated as "generate (generation)" throughout, although at times "produce" would be closer to modern mathematical usage.

<**40.1**> For even numbers are called female as if soft because they are easily divided;[48] but odd numbers <are called> male, as if strong and indivisible. <**40.2**> The unit, however, is neither even nor odd in actuality; hence, the unit per se is neither female nor male in actuality, but potentially both. <**40.3**> Hence when it is joined with the female, then a male, i.e., an odd number, is generated; when it comes together with a male, however, it begets a female, because <it begets> an equal number. <**40.4**> Thus from the first generation of the unit only the female is born (i.e., an equal number), because <it is> the two. <**40.5**> For it was fitting that, in the procreation of its first offspring, the unit only play the role of the man, i.e., of the worthier <gender> and that from the unit, as if from the male, the female be born. For the first female <was born> from a man and not the first man from a woman.[49] <**40.6**> Thus at the second level, inasmuch as the unit is joined to the female (i.e., the two), the three, which is male, is generated. Then, at the third level, the unit is conjoined with the male and a female—i.e., the four—comes forth; similarly in the rest *ad infinitum.* <**40.7**> Thus it was necessary that the unit be neither even nor odd. For, if it were only even, when it was joined to even numbers, nothing would be engendered, just as in the conjunction of two females. If, on the other hand, it were only odd, when it was joined to odd numbers, it would engender nothing, just as a male with a male. <**40.8**> Thus it was necessary that it be neither in actuality, but both in potentiality, so that when it was joined in turn with all <numbers>, being born according to the power of each sex, the fecund offspring of numbers would be propagated *ad infinitum.*

<**41.1**> But because the first and natural generation of numbers, in the fashion previously stated, seemed to be multiplied without end, it pleased the diligence of certain men afterwards that it should be limited by certain definite levels and limits according to the model of human generation. <**41.2**> For the generation of men, like that of numbers, progresses from one individual being doubled in gender and descending by male and female *ad infinitum.* <**41.3**> Later, however, humans invented levels and limits through their labor, by which they designated relationships among people, so that, although they knew that they were all equally descended from one man, yet they did not doubt, because of the assigned levels, that some belong more to others due to <these> relationship<s>; and all those found to be included in the same levels of relation were said to be from one family. <**41.4**> Similarly in the case of numbers too, following their natural composition and essence,

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[48]This idea depends on the similarity of *mulier* ("woman") to *mollis* ("soft") and the etymology on this basis in Isidore (*Etym.* XI.2.17).

[49]cp. *Gen.* 2:23: "And Adam said, This is now bone of my bones and flesh of my flesh: she shall be called Woman, because she was taken out of Man."

humans invented through their industry roots, nodes[50] and limits, just as it invented trunks (?) and levels in humans and they distinguished the generations of number by the nines, so that all numbers up to the ninth level which are born from the same limit are called by a single common name <indicating> their relationship and distinguishing them from others; those which exceed any given nine, on the other hand, recognize that they belong to an entirely different relationship. <**41.5**> Thus, to distinguish among this sort of relationships among numbers, humans, by their inventiveness, called some 'digits', others 'articles',[51] and others 'composites', but they called those from which all these are born the limits, like the first parents of each generation. <**41.6**> For they established that those which the unit had engendered in the first creation by the aggregation of itself–those up to nine–should be called 'digits', inasmuch as they are the first-born of the unit, so that this first nine was called the nine of digits or of units, since the first limit of this first nine was the unit, given that the unit had first engendered these from itself. <**41.7**> After this follows the second nine, which is that of the tens or of articles and the limit of this nine, as for the first, is a unit, but ten times greater than the first <nine>. <**41.8**> After this nine of tens, there follows the third nine, that of hundreds; the limit of this is also a unit, but ten times greater than that in the second <nine>. <**41.9**> After this third nine follows the fourth nine, that of thousands; its limit is also a unit, but ten times that of the third. And so on *ad infinitum.*

<**42.1**> Because all numbers were engendered by the unit, this was rightly established as the limit of all the nines, according to the variety of their positions, namely so that it came about that what had engendered all numerical species from itself would also be the limit of limits according to the diversity of their places. <**42.2**> Hence the unit is set first at the beginning of all generations as their limit, so that from this it is shown to be the mother of all. <**42.3**> Hence arises the fact that, just as the unit, which by nature is the first limit, had engendered the digits in the first creation by aggregation of itself with them, so in the second establishment it seemed good that this very same unit, as each limit, when added to the first < establishment> should generate the composites and when multiplied by the first < establishment > should produce the articles. <**42.4**> Those numbers have, therefore, been called digits, which were naturally engendered from the unit up to nine; those <have been called> articles, which are generated from the other limits by the multiplication of the first < establishment >; those numbers are called composites, which are born from digits and limits or articles linked together (for they are called "composite" as if created from different kinds). <**42.5**> Hence they follow the property of those very things from which they are allotted their substance. <**42.6**> For when we say 12 or 23 or 120, these are composites <created out of> digit<s> and

[50] The nodes are presumably the "articuli" referred to in earlier parts of this text and <41.5> below.

[51] Literally articulus, or "joint" (as in the finger joints).

boundar<ies> or else also from article<s>. <**42.7**> But whatever limit or article is in them is taken according to the power of limit or article, i.e., for ten, 20, or 100; whatever is from the nine of digits, however, is taken for as many units as are seen to be contained in it. <**42.8**> Thus, all the nines are organized according to the model of the one preceding, so that each has a unit as its limit, has a two, a three and thus obtains each up to nine, as the table below indicates.

Difference of hundred-millions	Difference of ten-millions	Difference of millions	Difference of hundred-thousands	Difference of ten-thousands	Difference of thousands	Difference of hundreds	Difference of tens	Difference of units or digits
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

<**42.9**> For just as in the first limit twice one made the two of units, so in the second limit twice ten makes the two of tens, which is 20 and in the third limit, twice one hundred <makes> the two of hundreds, which is two hundred and so on in each by each up to nine. <**42.10**> And because nothing is born from numbers except by either aggregation or multiplication of the first nine, <the nine> is repeated in itself in all numbers; and it is shown to be prior to all because, having been born before all others, it appears still to preserve its natural establishment. <**42.11**> For this reason the unit itself, which is the mother of all <numbers>, only engenders nine numbers in whatever limit it appears, whether through aggregation or multiplication according to the number of the first-born <numbers>.

<**43.1**> But because ten follows after nine in the natural order and the unit is always, except in the first limit, found set after nine by human establishment, for this reason it is necessary that ten be signified by a unit placed after the first nine, and thus the unit, having become a ten by the nature of its location, becomes the limit of the tens, just as earlier it had simply been the limit of the units, so that the mother of the articles or composites is the same as what was clearly the mother of the digits. <**43.2**> And because ten naturally always follows nine and a unit is always placed in that position, for this reason another unit follows after the nine of tens, which is the third limit — that of the hundreds, and thus a unit follows any nine whatsoever as the limit of those which follow. <**43.3**> Since moreover every limit—except the first—follows after the preceding nine, for this reason, when it becomes a ten, it is always found to be ten times the preceding limit, because, no matter what nine it comes after, it is born from the preceding limit multiplied by ten. <**43.4**> And because all articles are born through the multiplication of their limit by the first <nine>, it is necessary, in order that they not appear irregular, for them to follow the rule of their limits, namely, inasmuch as limits are ten times the

limits preceding them and those numbers that are born from the multiplication of these <limits>, those numbers must be found to be ten times the numbers preceding them. <**43.5**> For just as the second limit is ten times the first, so the articles of the tens are ten times the articles of the digits and just as the third limit is ten times the second, so the articles of the hundreds as well are multiplied by ten by comparison to the articles of the tens. <**43.6**> Thus the limits, articles and composites (which lie between them) are each ten times those preceding limits, articles and composites that they follow.

<**44.1**> And thus all limits and articles and composites, just like digits, are established in subordination to the nine, such that the first nine is that of digits, the second, of articles, the third, of composites and so on for the rest of this sort. <**44.2**> For thus it seemed good that every number be terminated in a nine as if in the ultimate level of its kind and that a unit ten times the preceding limit—because <it is> the tenth <element> after nine—should be established after the nine, as the limit for every nine. <**44.3**> Thus the entire fecund progeny of numbers is extended infinitely, separated into sets of nine and descending through its generations from its limits as if from its forebears. <**44.4**> Thus the nine holds the position of leadership in all things, restricting what is infinite, marking distinctions in what is restricted, but in such a way that it begins from and finds its terminus in a limit, so that it is shown to have been, not itself the maker of the world, but rather the paradigm for the world in the mind of the Maker; hence it is generated from the three multiplied by itself. <**44.5**> For he who made all things made that as well according to which, as a paradigm, he formed the rest; for God made all things in number, weight and measure.[52] <**44.6**> Hence if number too was made, He made it according to number, so that number did not exceed the laws of number, since other things were to be composed according to its form. <**44.7**> But the number according to which number was created is in this manner indeed uncreated.

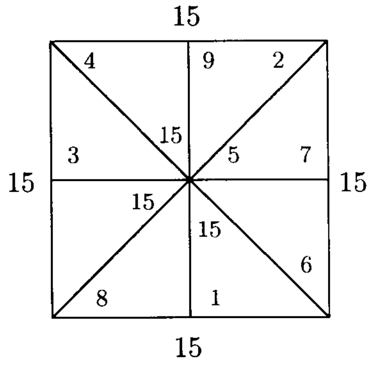
G

<**45.1**> There are eight species of multiplication and the same number of division. For either we multiply whole units by whole units, or fractions by fractions, or fractions by whole units, or whole units by fractions, or fractions and whole units by whole units, or fractions and whole units by fractions, or whole units by whole units and fractions, or fractions by whole units and fractions. <**45.2**> Whenever an even or odd number multiplies an even number, or an even number multiplies an odd number, an even number results. But if an odd number multiplies an odd number, an odd number arises.

<**46.1**> Divide minutes by minutes or seconds by seconds or thirds by thirds or

[52]cf. *Wisdom of Solomon* 11:21: "You have disposed all things by measure, number and weight."

fourths by fourths or fifths by fifths or sixths by sixths, and whatever results will be degrees. And when each of these numbers is multiplied by degrees, whatever results will be of the same kind as the fraction. <**46.2**> If minutes divide seconds or seconds thirds or fourths fifths or fifths sixths or thirds sixths, whatever results from the divisions will be denominated by the larger fractions.



Mathematical translation and notes

A

<1> **Rule of the mean value**

<1.1> $n = \frac{1}{2}[(n-m)+(n+m)],\ 1 \leq m < n.$

<1.2> $5 = \frac{1}{2}(4+6) = \frac{1}{2}(3+7) = \frac{1}{2}(2+8) = \frac{1}{2}(1+9).$

<2> **The difference of squares**

<2.1> $n^2 = (n-m)(n+m)+m^2.$

<2.2> $5^2 = 4\times 6+1^2 = 3\times 7+2^2 = 2\times 8+3^2 = 1\times 9+4^2.$

<2.3> $n^2-1 = (n-1)(n+1).$

<3> **The sum of arithmetical progressions of continuous numbers**

<3.1> If n is even, $\sum_{i=1}^{n} i = \frac{n}{2}\cdot n + \frac{n}{2}.$

<3.2> $\sum_{i=1}^{8} i = \frac{8}{2}\times 8 + \frac{8}{2} = 36.$

<3.3> If n is even, $\sum_{i=1}^{n} i = \frac{n}{2}(n+1).$

<3.4> If n is odd, $\sum_{i=1}^{n} i =$ (the larger part[1] of n) $\cdot\, n$;

$$\sum_{i=1}^{7} i = 4\times 7 = 28.$$

<3.5> If n is odd, $\sum_{i=1}^{n} i = \frac{n+1}{2}\cdot n.$

<3.6> $\sum_{i=m}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{m-1} i.$[2]

<4> **The sum of arithmetical progressions of continuous even numbers**

<4.1> $\sum_{i=1}^{n} 2i = n(n+1).$

<4.2> $\sum_{i=1}^{5} 2i = 5\times 6 = 30.$

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[1]'The larger part' and 'the smaller part' of an odd number are two technical words which frequently occur in the text. Sometimes we find in their place 'the larger half' and 'the smaller half' respectively. Let an odd number be denoted by n, then the larger part of it is $\frac{n+1}{2}$, as the rule of <3.5> shows; the the smaller part of it is $\frac{n-1}{2}$.

[2]In discussing the sum of the arithmetical progression of continuous numbers, the cases when the number of the terms is even and odd are considered respectively, The general case is discussed in the rule <16.1>.

<4.3> $\sum_{i=1}^{n} 2i = \frac{2n+2}{2} \cdot \frac{2n}{2}$.

<4.4> $\sum_{i=m}^{n} 2i = \sum_{i=1}^{n} 2i - \sum_{i=1}^{m-1} 2i$.

<5> The sum of arithmetical progressions of continuous odd numbers

<5.1> $\sum_{i=1}^{n}(2i-1) = n^2$.

<5.2> $\sum_{i=1}^{5}(2i-1) = 5^2 = 25$.

<5.3> $\sum_{i=1}^{n}(2i-1) = (2n - \frac{2n}{2})^2$.

<5.4> $\sum_{i=m}^{n}(2i-1) = \sum_{i=1}^{n}(2i-1) - \sum_{i=1}^{m-1}(2i-1)$.

<6> The sum of geometrical progressionss of doubles and triples

<6.1> $\sum_{i=1}^{n} 2^i = 2^n \times 2 - 2$.

<6.2> $\sum_{i=1}^{5} 2^i = 32 \times 2 - 2 = 62$.

<6.3> $2^n \times 2 - 2 = \sum_{i=1}^{n} 2^i$.

<6.4> $\sum_{i=m}^{n} 2^i = 2^n \times 2 - 2^m$.[3]

<6.5> $1 + \sum_{i=1}^{n} 2^i = 1 - (2^n \times 2 - 2)$.

<6.6> $\sum_{i=1}^{n} 3^i = \frac{3^n - 1}{2} \times 3$.

<6.7> $\sum_{i=1}^{n} 3^i = \frac{243 - 1}{2} \times 3 = 363$.[4]

<7> The sum of squares

<7.1> $\sum_{i=1}^{n} i^2 = (\sum_{i=1}^{n} i) \cdot \frac{2}{3}n + (\sum_{i=1}^{n} i) \cdot \frac{1}{3}$.

<7.2> $\sum_{i=1}^{4} i^2 = (\sum_{i=1}^{4} i)(\frac{2}{3} \times 4) + (\sum_{i=1}^{4} i) \cdot \frac{1}{3} = 10(3 - \frac{1}{3}) + 10 \cdot \frac{1}{3} = 30$.

[3]This is the general formulae of the sum of the continued 'doubles' whether the first term is 2 or not.

[4]Although the text produces correct results for the sum of the continued 'doubles' and 'triples', we cannot see any generalisation of them to the other numbers.

<8> Relation between the middle number(s) and the extremes of arithmetic and geometric progressions

<8.1> If n is odd, $a_{i+1} = a_i + d$ $(i = 1, 2, ..., n-1)$, $m = \dfrac{n+1}{2}$, then $2a_m = a_{m-j} + a_{m+j}$ $(j = 1, 2, ..., m-1)$.

<8.2> If n is odd, $a_{i+1} = q \cdot a_i$ $(i = 1, 2, ..., n-1)$, $m = \dfrac{n+1}{2}$, then $a_m^2 = a_{m-j} \cdot a_{m+j}$ $(j = 1, 2, ..., m-1)$.

<8.3> If n is even, $a_{i+1} = q \cdot a_i$ $(i = 1, 2, ..., n-1)$, $m = \dfrac{n}{2}$, then $a_m \cdot a_{m+1} = a_{m-j} \cdot a_{(m+1)+j}$ $(j = 1, 2, ..., m-1)$.[5]

<8.4> If n is even, $a_{i+1} = a_i + d$ $(i = 1, 2, ..., n-1)$, $m = \dfrac{n}{2}$, then $a_m + a_{m+1} = a_{m-j} + a_{(m+1)+j}$ $(j = 1, 2, ..., m-1)$.[6]

<9> Multiplication of digits

<9.1> If $a < 10$, $a^2 = 10a - a(10 - a)$.

<9.2> $6^2 = 10 \times 6 - 6(10 - 6) = 36$.

<9.3> If $a < b < 10$, $ba = 10a - a(10 - b)$.

<9.4> $7 \times 5 = 50 - 5(10 - 7) = 35$.

<9.5> If $a < b < 10$, $ab = 10b - b(10 - a)$.

<9.6> $5 \times 7 = 70 - 7(10 - 5) = 35$.[7]

<10> The Rule of three numbers in proportion

<10.1> If $a : b = b : c$, then $ca = b^2$, $c = \dfrac{b^2}{a}$.

<10.2> If $a : b = b : c$, then $a = \dfrac{b^2}{c}$.

<10.3> If $a : b = b : c$, then $b = \sqrt{ca}$.

<11> The Rule of four numbers in proportion

<11.1> If $a : b = c : d$, then $da = cb$.

(a) If $x_1 : x_2 = x_3 : x_4$, $x_4 x_1 = x_3 x_2 = A$, then $x_i = \dfrac{A}{x_{5-i}}$.

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[5] Rules <8.2> and <8.3> discuss the relation between the middle number(s) and the extremes of geometric progression. <8.2> discusses the case when the number of the terms is odd, and <8.3> when the number is even. The general rule is stated in <13.4>.

[6] Rules <8.1> and <8.4> discuss the relation between the middle number(s) and the extremes of arithmetical progression. <8.1> discusses the case when the number of the terms is odd, and <8.4> when the number is even. However, the general rule $2a_i = a_{i-1} + a_{i+1}$ is not discussed.

[7] Rules <9.1>, <9.3> and <9.5> are also discussed in Cashel I (pp. 18-19). Apart from the difference of the terminology, Cashel I gives different examples for the cases as well. It is worth noting that Cashel I gives a synthetic statement of the rules <9.3> and <9.5>, and the formulae can be expressed as $ab = 10a - b(10 - b)$, $a < b < 10$ which should be $ab = 10a - a(10 - b)$, $a < b < 10$. The rule <14.3> can be considered as the generalized case of the above three rules.

(b) If $x_1 : x_2 = x_3 : x_4$, then $x_i = \dfrac{x_j}{x_{5-i}} \cdot x_{5-j}$.

<11.2> If $a : b = c : d$, then $d = \dfrac{cb}{a}$.

<11.3> If $a : b = c : d$, then $a = \dfrac{cb}{d}$.

<11.4> If $a : b = c : d$, then $b = \dfrac{da}{c}$.

<11.5> If $a : b = c : d$, then $c = \dfrac{da}{b}$.

<11.6> since $10 : 30 = 2 : 6$, $6 \times 10 = 2 \times 30 = 60$.

<11.7> If $10 : 30 = 2 : x$, then $x = \dfrac{2 \times 30}{10} = 6$.

<11.8> If $x : 30 = 2 : 6$, then $x = \dfrac{2 \times 30}{6} = 10$.

<11.9> If $10 : x = 2 : 6$, then $x = \dfrac{6 \times 10}{2} = 30$.

<11.10> If $10 : 30 = x : 6$, then $x = \dfrac{6 \times 10}{30} = 2$.

<11.11> If $a : b = c : d$, then a, c and b, d represent the same things respectively.[8]

<12> Application of the rule of four

<12.1> If $a : b = c : d$, then $c = \dfrac{(c+d)a}{a+b}$.[9]

<12.2> If $a : b = c : d$, then $d = \dfrac{(c+d)b}{a+b}$.[10]

<12.3> If $a : b = c : d$, then $a = \dfrac{(a+b)c}{c+d}$, $b = \dfrac{(a+b)d}{c+d}$.[11]

<12.4> If $2 : 4 = c : d$, $c + d = 9$, then $c = \dfrac{9 \times 2}{2+4} = 3$.

<12.5> If $a : b = c : d$, $c < d$, then $c = \dfrac{(d-c)a}{b-a}$.

If $a : b = c : d$, $c > d$, then $c = \dfrac{(c-d)a}{a-b}$.[12]

<12.6> If $a : b = c : d$, $c < d$, then $d = \dfrac{(d-c)b}{b-a}$.

[8]Here the text discusses the categories of the four terms of a proportion. The category of the first term must be the same as that of the third term, as must that of the second and fourth term. In theoretical mathematics, these are unnecessary to mention. But here it is set down as a general rule, which shows the evident practical tendency of the proportional theory in the twelfth century.

[9]This rule is based on the following: if $da = cb$, then $da + ca = cb + ca$.

[10]This rule is based on the following: if $da = cb$, then $da + db = cb + db$.

[11]This rule is based on the following: if $ad = bc$, then $ad + ac = bc + ac$, $ad + bd = bc + bd$. Rules <12.1>-<12.3> could be considered as applications of the proportion by addition theorem.

[12]This rule is based on the following: if $cb = da$, then $cb - ca = da - ca$ when $c < d$, $ca - cb = ca - da$ when $c > d$.

If $a:b=c:d$, $c>d$, then $d=\dfrac{(c-d)b}{a-b}$.[13]

<12.7> If $a:b=c:d$, $a<b$, then $a=\dfrac{(b-a)c}{d-c}$, $b=\dfrac{(b-a)d}{d-c}$.

If $a:b=c:d$, $a>b$, then $a=\dfrac{(a-b)c}{c-d}$, $b=\dfrac{(a-b)d}{c-d}$.[14]

<12.8> If $2:4=c:d$, $d-c=3$, then $c=\dfrac{3\times 2}{4-2}=3$.

<12.9> The rules are the same when the categories of the terms of a proportion are considered.[15]

<12.10> If $a:b=c:d$, then $c=\sqrt{\dfrac{(cd)a}{b}}$.[16]

<12.11> If $a:b=c:d$, then $d=\sqrt{\dfrac{(cd)b}{a}}$.[17]

<12.12> If $2:4=c:d$, $cd=18$, then $c=\sqrt{\dfrac{18\times 2}{4}}=3$.

<12.13> If $2:4=c:d$, $cd=18$, then $d=\sqrt{\dfrac{18\times 4}{2}}=6$.

<12.14> If $a:b=c:d$, $b+c=p=m+n$, then m, n which satisfy $\begin{cases} m+n=p \\ mn=ad \end{cases}$ are b, c.[18]

<12.15> If $2:b=c:6$, $b+c=7=m+n$, the solutions for m, n which satisfy $\begin{cases} m+n=7 \\ mn=6\times 2=12 \end{cases}$ are 3 and 4, and they are b, c respectively.[19]

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[13]This rule is based on the following: if $da=cb$, then $db-da=db-cb$ when $c<d$, $da-db=cb-db$ when $c>d$.

[14]This rule is based on the following: if $ad=bc$, then $ad-ac=bc-ac$ and $bd-bc=bd-ad$ when $a<b$, $ac-ad=ac-bc$ and $bc-bd=ad-bd$ when $a>b$. Rules <12.5>-<12.7> could be considered as applications of the proportion by addition and subtraction theorem.

[15]Here again the categories of the four terms of the proportion are discussed. See the note on <11.11>.

[16]This rule is based on the following: if $ad=bc$, then $(ad)c=(bc)c$, or $(cd)a=bc^2$.

[17]This rule is based on the following: if $ad=bc$, then $(ad)d=(bc)d$, or $(cd)b=ad^2$. Rules <12.10> and <12.11> could be considered as applications of the equal ratios theorem.

[18]The two conditions for m, n are in the form of a Babylonian quadratic equation. When it is reduced into one variable equation, it turns into $m^2+(ad)=pm$ or $n^2+(ad)=pn$. To get the values for m, n, however, from the following example we can see that this text does not give the method. In fact, because $\begin{cases} b+c=p \\ bc=(ad) \end{cases}$, the values for b, c must be those for m, n. So this rule belongs to proportional theory rather than quadratic theory.

What is more, the following example does not consider the other case when the values for b, c are 4 and 3 respectively, i.e. when the proportion is $2:3=4:6$.

[19]Rules <12.14> and <12.15> are based on the rule of three and connect the proportional theory

B

<13> Relation between the middle number and its extremes
<13.1> $x^2 - 1 = (x-1)(x+1)$.
<13.2> $x^2 = (x-a)(x+a) + a^2$.[20]
<13.3> $5^2 = 4 \times 6 + 1^2 = 3 \times 7 + 2^2 = 2 \times 8 + 3^2 = \cdots = 1 \times 9 + 4^2$.
<13.4> If $a_{i+1} = q \cdot a_i$ $(i = 1, 2, ..., n-1)$, then $a_{j+1}^2 = a_j \cdot a_{j+2}$ $(j = 1, 2, ..., n-2)$.[21]

<14> Multiplication of a sum
<14.1> $(a+b)^2 = a^2 + b^2 + 2ab$.[22]
<14.2> $(\sum_{i=1}^{n} a_i)a = \sum_{i=1}^{n}(a_i \cdot a)$.[23]
<14.3> If $10^{n-1} < b < 10^n$, then $ab = a \times 10^n - a(10^n - b)$.[24]

<15> Calculation of fractions[25]
<15.1> If $a = b$, then $a \div b = 1$.
<15.2> If $a > b$, $a = nb$, then $a \div b = n$.
<15.3> If $a > b$, $a = nb + c$, $1 \leq c < b$, then $a \div b = n + \frac{c}{b}$.
<15.4> If $a < b$, then $a \div b = \frac{a}{b}$.
<15.5> $a \cdot \frac{1}{a} = a$.
<15.6> $3 \times \frac{1}{3} = 4 \times \frac{1}{4} = 1$.
<15.7> If $a < b$, then $a \cdot \frac{1}{b} = \frac{a}{b}$.

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with the quadratic theory.

[20]Although the rules <13.1> and <13.2> here are essentially repetitions of the rules <2.3> and <2.1> respectively, they discuss the topic in a different way. First, <13.1> is the same as <2.2>, and <13.2> is the same as <2.1>. The rule <2.2> is a special case of the general rule of <2.1>, so there is some disharmony in their arrangement. Here the text takes the inverse sequence and emphasizes that <13.2> is more general than <13.1> which is more logical.

[21]Although rules <8.2> and <8.3> have discussed this topic of geometric progression, they are in fact rather special cases of <13.4>. <8.2> and <8.3> apply only to the middle term(s) of all of the terms whereas <13.4> applies to any medium terms of geometric progression. So this rule is much more general and essential than the former two ones.

[22]This is the well-known formula of the square of the sum. In al-Khwarizmi (p. 24), there is a similar rule expressed by a number and an unknown, i.e., $(10+x)^2 = 100 + 20x + x^2$.

[23]This rule exhibits the distributive law of multiplication.

[24]Here the use of 'limit' is not the same as in other contexts. It is any number with only one numeral and one or more zeros. So here 'limit' seems to include 10, which is not the case in most other contexts.

[25]Multiplication, division and addition of the fractions are discussed in these rules. However, subtraction is not discussed.

<15.8> $6 \times \frac{1}{12} = \frac{6}{12} = \frac{1}{2}$.

<15.9> If $a = nb$, then $a \cdot \frac{1}{b} = n$.

<15.10> $6 \times \frac{1}{3} = (3 \times 2) \times \frac{1}{3} = 2$.

<15.11> If $a = nb + c$, $c < b$, then $a \cdot \frac{1}{b} = n + \frac{c}{b}$.

<15.12> $8 \times \frac{1}{3} = 2\frac{2}{3}$.

<15.13> $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$.

<15.14> $\frac{1}{3} \times \frac{1}{4} = \frac{1}{3 \times 4} = \frac{1}{12}$.

<15.15> $\frac{c}{a} \cdot \frac{1}{b} = \frac{c}{ab}$.

<15.16> $\frac{2}{3} \times \frac{1}{4} = \frac{2}{3 \times 4} = \frac{1}{6}$.

<15.17> If $a < b$, then $\frac{1}{a} : \frac{1}{b} = b : a$.

<15.18> $\frac{1}{3} : \frac{1}{6} = 6 : 3 = 2$.

<15.19> $\frac{b}{a} : \frac{c}{a} = b : c$.

<15.20> $12 : 9 = \frac{12}{3} : \frac{9}{3}$.

<15.21> $\frac{1}{a} + \frac{1}{b} = (a + b) \cdot \frac{1}{ab}$.

<15.22> $\frac{1}{3} + \frac{1}{4} = (3 + 4) \cdot \frac{1}{3 \times 4} = \frac{7}{12}$.

<16> Sum of arithmetic progressions

<16.1> $\sum_{i=1}^{n} i = \frac{n^2 + n}{2}$.

<16.2> If n is odd, $a_{i+1} = a_i + d$ $(i = 1, 2, ..., n-1)$, $m = \frac{n+1}{2}$, then $\sum_{i=1}^{n} a_i = n \cdot a_m$.

<16.3> $3 + 4 + 5 = 3 \times 4$.

<16.4> $3 + 5 + 7 + 9 + 11 = 5 \times 7$.

<16.5> $2 + 5 + 8 = 3 \times 5$.

<16.6> If n is even, $a_{i+1} = a_i + d$ $(i = 1, 2, ..., n-1)$, $m = \frac{n}{2} + 1$, then $\sum_{i=1}^{n} a_i = (n+1)a_m - a_{n+1}$.[26]

<16.7> $2 + 4 + 6 + 8 = (2 + 4 + 6 + 8 + 10) - 10 = 5 \times 6 - 10 = 20$.

<16.8> $1 + \sum_{i=1}^{n} (2_i - 1) = n^2$.[27]

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[26] This rule is the deduction from rule <16.2>.

[27] The rule is equivalent to, but not the same as, rule <5.1>. In this rule, 1 is not considered as odd which is consistent with part F below, while in rule <5.1>, 1 is considered as an odd number.

<17> Calculation of rations

<17.1> If $a^2 + b^2 = c^2$, $a_1 : a = b_1 : b = c_1 : c = p$ then $a_1^2 + b_1^2 = c_1^2$.[28]

<17.2> If $a_i = p_i a$ $(i = 1, 2, ..., n)$, then $a = \dfrac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} p_i}$.

<17.3> If $b = p_b \cdot a$, $c = p_c \cdot a$, $d = p_d \cdot a$, $b + c + d = g$, then $a = \dfrac{g}{p_b + p_c + p_d}$.[29]

<17.4> Let S, P represent the money of Socrates and Plato.

$S = 2\frac{2}{3}P = 15$; $2\frac{2}{3} = \frac{2 \times 3 + 2}{3} = \frac{8}{3}$;

$15 : 8 = P : 3 = (5\,pennies + 1\frac{1}{4}\,obols) : 3\,pennies$,

so $P = 5\,pennies + 1\frac{1}{4}\,obols$;

$(5\,pennies + 1\frac{1}{4}\,obols) \times 2\frac{2}{3} = 2(5\,pennies + 1\frac{1}{4}\,obols) + \frac{2}{3}(5\,pennies + 1\frac{1}{4}\,obols)$

$= (11\frac{1}{4} + 3\frac{3}{4})\,pennies = 15\,pennies$.

<17.5> $\frac{2}{3}(9\,obols) = 6\,obols = 3\,pennies$; $\frac{9}{8} \times \frac{2}{3} = \frac{6}{8} = \frac{3}{4}$; $11\frac{1}{4} + 3 + \frac{3}{4} = 15$.

<17.6> Or if $2\frac{2}{3}P = 15$, then $P = \dfrac{15 \times 3}{2 \times 3 + 2} = 5\frac{5}{8}$.

<17.7> If $b > a$, then $\dfrac{cb}{a} = \dfrac{b}{a} \cdot c$.

<17.8> if $b < a$, then $\dfrac{cb}{a} = \dfrac{c}{a/b}$.

<17.9> $\dfrac{100 \times 1\,pound}{1\,shilling} = \dfrac{1\,pound}{1\,shilling} \times 100 = 20 \times 100 = 2000$.

<17.10> $\dfrac{24,000 \times 1\,penny}{1\,pound} = \dfrac{24,000}{1\,pound/1\,penny} = \dfrac{24,000}{240} = 100$.

<17.11> If $(\frac{1}{a} + \frac{1}{b})c = d$, then $c : d = (ab) : (a + b)$, $c = \dfrac{(ab)d}{a + b}$.

<17.12> If $(\frac{1}{3} + \frac{1}{4})c = 20$; $\dfrac{3 + 4}{3 \times 4} = \dfrac{7}{12}$; then $c : 20 = 12 : 7$, $c = \dfrac{12 \times 20}{7} = 34\frac{2}{7}$.[30]

<17.13> If $\frac{a}{b} = \frac{b}{c}$, then $ac = b^2$.[31]

<17.14> If $b_1 : a_1 = b_2 : a_2 = \cdots = b_n : a_n$, then $b_i = \dfrac{(\sum_{i=1}^{n} b_i)a_i}{\sum_{i=1}^{n} a_i}$ $(i = 1, 2, ..., n)$.

<17.15> If $b_1 : a_1 = b_2 : a_2 = \cdots = b_n : a_n$, then $b_i = \dfrac{a_i}{\sum_{i=1}^{n} a_i}(\sum_{i=1}^{n} b_i)$ $(i =$

There is a similar rule in Cashel I (p. 19). Unlike rules <5.1> and <16.8>, it uses the 'large half' to express the number of the terms, but, as in rule <5.1>, 1 is considered as an odd number.

[28] This rule is an application of the famous Pythagorean theorem to the proportional theory.

[29] Rules <17.2> and <17.3> could be considered applications of the distributive law of multiplication.

[30] Note in the margin: $c = (1 + \frac{5}{7})20 = \frac{12}{7} \times 20$.

[31] This rule is a repetition of the rule in the first sentence of <10.1>.

$1, 2, ..., n)$.[32]

<17.16> If $a_1 = 6$, $a_2 = 8$, $a_3 = 12$, $a_1 + a_2 + a_3 = 26$, $b_1 + b_2 + b_3 = 60$, then $a_i : 26 = b_i : 60$, $b_i = \dfrac{60a_i}{26}$ $(i = 1, 2, 3)$.

<17.17> If $a : b = c : x$, then $x = \dfrac{bc}{a}$.

<17.18> If $8 : 24 = 3 : x$, then $x = \frac{24\times 3}{8}$; if $8 : 24 = 5 : x$, then $x = \dfrac{24 \times 5}{8}$.

C

<18> Solutions for quadratic equations

<18.1> The three compound quadratic equations are:
(1) $x^2 + ax = c$; (2) $x^2 + c = ax$; (3) $ax + c = x^2$.[33]

<18.2> (1) If $x^2 + ax = c$, $x = \sqrt{\left(\dfrac{a}{2}\right)^2 + c} - \dfrac{a}{2}$.

<18.3> If $x^2 + 10x = 39$, $x = \sqrt{\left(\dfrac{10}{2}\right)^2 + 39} - \dfrac{10}{2} = 3$.[34]

<18.4> (2) If $x^2 + c = ax$, $x = \dfrac{a}{2} - \sqrt{\left(\dfrac{a}{2}\right)^2 - c}$.[35]

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[32] Rules <17.14> and <17.15> are based on the proportion by addition theorem.

[33] In this text, "the thing" refers to the square of the root. Modern readers may think it easier to express "the thing" as an unknown x. But in ancient Arabic mathematics, 'the thing' can refer not only to the root of a square, such as in Khwarizmi, but also to a square itself, such as in Abu Kamil.

[34] This example exists in many Arabic algebraic texts such as in Khwarizmi (p.8) and and Abu Kamil (pp.30-32) with different terminologies for the square.

[35] The statement of the procedure of this rule is similar to rules <18.2> and <18.6>. Strictly speaking, the solution is not complete. The correct formula expressing the solutions of this type of quadratic equation should be $x = \frac{a}{2} \pm \sqrt{(\frac{a}{2})^2 - c}$.

<18.5> If $x^2 + 9 = 6x$,[36] $x = \frac{6}{2} - \sqrt{\left(\frac{6}{2}\right)^2 - 9} = 3 - \sqrt{0} = 3 - 0 = 3$.[37]

<18.6> (3) If $ax + c = x^2$, $x = \frac{a}{2} + \sqrt{c + \left(\frac{a}{2}\right)^2}$.

<18.7> If $3x + 4 = x^2$, $x = \frac{3}{2} + \sqrt{4 + \left(\frac{3}{2}\right)^2} = 4$.[38]

D

To divide a number by given ratios

<19.1> If $\sum_{i=1}^{n} x_i = x$, $x_1 : a_1 = x_2 : a_2 = \cdots = x_n : a_n$, then

$\sum_{i=1}^{n} a_i : a_i = x : x_i$, $x_i = (\frac{x}{\sum_{i=1}^{n} a_i})a_i$.

For $x_i = (\frac{x}{\sum_{i=1}^{n} a_i})a_i$, if the residue of xa_i is r, and the integer part of the quotient is s, then $x_i = (\frac{x}{\sum_{i=1}^{n} a_i})a_i = s + (\frac{1}{\sum_{i=1}^{n} a_i})r$.

<19.2> If $x = 40$, $n = 4$, $x_2 = 4x_1$, $x_3 = 5x_2$, $x_4 = 3x_3$;

$a_1 = 1$, $a_2 = 1 \times 4 = 4$, $a_3 = 4 \times 5 = 20$, $a_4 = 20 \times 3 = 60$, $\sum_{i=1}^{4} a_i = 85$;

then $85 : a_i = 40 : x_i$.[39]

<19.3> $85 : 4 = 40 : x_2$, $x_2 = \frac{(4 \times 40)}{85} = 1\frac{75}{85}$.

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[36]This example exists in Ibn Turk (p.101, p.166), who does not give arithmetical procedures for the solution. The equation $x^2 + c = ax$ generally has two solutions $x = \frac{a}{2} \pm \sqrt{(\frac{a}{2})^2 - c}$ when $(\frac{a}{2})^2 > c$; in the special case when $(\frac{a}{2})^2 = c$ there is one solution; and when $(\frac{a}{2})^2 < c$ there is no solution for it. These had already been discussed by Arabic mathematicians such as Khwarizmi and Abu Kamil who influenced medieval European mathematics greatly. The statement of the general rule for this equation only considers one solution $x = \frac{a}{2} - \sqrt{(\frac{a}{2})^2 - c}$ which is connected with subtraction. But the example $x^2 + 9 = 6x$ is rather special because it belongs to the case of $(\frac{a}{2})^2 = c$, and in such a case the solutions from addition and subtraction are the same, i.e. there is only one solution for the equation. So, although only subtraction is considered in the example, the result is still right.

[37]The calculation of the square root of 0 and the subtraction by 0 are novelties in ancient Arabic mathematics. From this calculation, we may see some influences of Indian mathematics on this text.

[38]This example is the same as that of Khwarizmi (pp. 12-13). The order of the three types of composite quadratic equations is the same as that of Khwarizmi and the examples of the first case and the third case are also the same as those in Khwarizmi. So there is a strong tendency to think that Part C of this text should be based on Khwarizmi or in his tradition.

[39]The diagram in the text shows that 85 is the first term of the proportional expression, 40 is the third, the empty place symbol '0' is the fourth, which represents the number corresponding to the

<19.4> $x_1 = \frac{x_2}{4} = \frac{1\frac{75}{85}\, shillings}{4} = \frac{12\, pennies + \frac{75}{85} shillings}{4}$
$= 3\, pennies + (\frac{18}{85} + \frac{1}{85} \cdot \frac{3}{4})\, shilling$
<19.5> $x_3 = 5x_2 = 5 \times 1\frac{75}{85} = 9\frac{35}{85}$.
<19.6> If $a = mb$, $b = \frac{a}{m}$.
<19.7> $160 \times 5 = 85 \times 9 + 35$.

<20> Multiplication of continuous ratios

<20.1> If $x_i = a_i \cdot x_{i+1}$ $(i = 1, 2, ..., n-1)$ and $x_n = a_n \cdot x_0$, then
$x_1 = x_0 \cdot a_1 \cdot a_2 \cdot ... \cdot a_{n-1} \cdot a_n$, $x_2 = x_0 \cdot a_2 \cdot a_3 \cdot ... \cdot a_{n-1} \cdot a_n$,
$x_3 = x_0 \cdot a_3 \cdot a_4 \cdot ... \cdot a_{n-1} \cdot a_n$, $x_i = x_0 \cdot a_i \cdot a_{i+1} \cdot ... \cdot a_{n-1} \cdot a_n$.
<20.2> If $a_1 = 4$, $a_2 = 3$, $a_3 = 2$, $a_4 = 1$, $x_0 = 6$, then $x_1 = 6 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 144$.
<20.3> $x_2 = 6 \cdot 3 \cdot 2 = 36$.
<20.4> $x_3 = 6 \cdot 2 = 12$.
<20.5> $x_4 = 6 \cdot 1 = 6$.

<21> To find the root

<21.1> $\sqrt{a} = \frac{\sqrt{b \cdot a}}{\sqrt{b}}$.[40]
<21.2> $\sqrt{2\frac{1}{4}} = \frac{\sqrt{4 \times 2\frac{1}{4}}}{\sqrt{4}} = 1\frac{1}{2}$.
<21.3> $\sqrt{2} = \frac{\sqrt{a \cdot 2}}{\sqrt{a}}$.[41]

<22> Multiplication of the columns

<22.1> If $a, b < 10$, then $(a \times 10^{n-1})(b \times 10^{m-1}) = (a \cdot b) \times 10^{n+m-2}$.[42]

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second term, and either of the ratios, i.e. 1, 4, 20 and 60 for the four men respectively, can be the second term. So the figure denotes in fact four proportional expressions. It is interesting to point out that the empty place or the symbol '0' can take four different values. Let the second term be denoted as x, then $85 : x = 40 : 0$, or $0 = \frac{40x}{85}$. It will be much clearer if we substitute y for the symbol '0' so that we have $y = \frac{40x}{85}$, a special "function" in the modern sense. Of course, it was not considered as a function at that time, although we now acknowledge it as belonging to a certain function. However, such practices are very important in the formation of the general concept of function.

[40]There is a similar rule in Khwarizmi (pp. 30-31) which is illustrated by several numerical examples.

[41]We are not sure if this explanation corresponds to the text exactly. The text here is not very clear. "The nearest square" and "the third number" may refer to the same number 2 which is in the diagram.

[42]Let a and b be two numerals belonging to two numbers respectively; their places are n and m respectively, or the number denoting a's column is n and that denoting b's column is m. The text says the numbers denoting the digit and article of the product of a and b are $n + m - 1$ and $n + m$

<23> **Simplified multiplication**

<23.1> If $c < 10$, $\dfrac{a}{c \times 10^n} = \dfrac{1}{d}$, then $ab = (c \times 10^n)\dfrac{b}{d}$.

<23.2> $25 \times 32 = 100(32 \times \dfrac{25}{100}) = 100 \times 8$, $8 : 32 = 25 : 100$.

<23.3> $25 \times 32 = 50(32 \times \dfrac{25}{50}) = 50 \times 16$.[43]

<23.4> If $a, b, c < 10$, then

$(c \times 10^n + a)(c \times 10^n + b) = (c \times 10^n)(c \times 10^n) + (c \times 10^n)(a + b) + a \cdot b$.

<23.5> $18 \times 16 = 10 \times 10 + 10(8 + 6) + 8 \times 6 = 288$.

<24> **Multiplication of roots**

<24.1> $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$.[44]

<24.2> $\sqrt{40} \times \sqrt{10} = \sqrt{40 \times 10} = 20$.

<25> **Calculation of ratios**

<25.1> $x + x + x + \dfrac{1}{2}x + \dfrac{1}{4}x = 100$.

<25.2> $1 + 1 + 1 + \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{15}{4}$, $\dfrac{100}{\frac{15}{4}} = \dfrac{\frac{400}{4}}{\frac{15}{4}} = 26\dfrac{2}{3}$.

E

<26> **To determine the places of the digit and the article of the product of two numerals**

<26.1> The text says there are four types of numbers, i.e. the digit, the article, the limit and the composite. The digit, article and limit in this text have two usages. On the one hand, the digit may represent any number which contains only one numeral, i.e. any of the integers from 1 to 9; the article may represent any number which contains one numeral and one zero, i.e. any of the tens from 10 to 90; and the limit may represent any number that contains one numeral and two or more zeros, i.e. $a \times 10^n$, a is a numeral and $n \geq 2$. On the other hand, they may also represent the column of the digit (the first column), the

respectively.

To explain this, we see that a represents the number $a \times 10^{n-1}$, and b represents $b \times 10^{m-1}$. So the product of a and b represents $(a \times 10^{n-1})(b \times 10^{m-1}) = (a \cdot b) \times 10^{n+m-2}$.

This means that the place of the digit of $a \cdot b$ is $n + m - 1$, and the place of the article of $a \cdot b$ which is 1 more than that of the digit is $n + m$.

The statement of the text is a general rule which is consistent with the modern rule of multiplication of exponents with base 10.

[43]This rule is still taught and emphasized very much in modern junior schools and is vivid in our everyday life.

[44]This rule is equivalent to <21.1>.

column of the ten (the second column) and any other column (any of the rest columns) respectively. The composite number contains at least two numerals. These four types of numbers exhaust the positive integers. So it is based on the numbers of numerals and zeros contained in a number that the text gives the four types of numbers.

<26.2> For the case of when two articles are multiplied, let a, b be the numerals. First calculate the product $a \cdot b$, then determine the places of the digit and the article of $a \cdot b$. The places of a, b are both 2, so the place of the digit of $a \cdot b$ is $2+2-1=3$, and that of the article of $a \cdot b$ is $2+2=4$. So the contexts are equivalent to the formula $(a \times 10)(b \times 10) = (a \cdot b) \times 100$.

<26.3> $70 \times 20 = (7 \times 2) \times 100 = 1,400$.

<26.4> For the multiplication of two composite numbers, multiply all the numerals of the multiplicand by those of the multiplier, determine the place of each numeral of the products, and then take their sum.

<26.5-26.6> $64 \times 23 = 4 \times 3 + 6 \times 3 \times 10 + 4 \times 2 + 6 \times 2 = 12 + 180 + 80 + 1200 = 1,472$.

<26.7> By means of this rule and its examples, the rule of multiplication of the columns (<22>) is explained again in a much clearer way.

<27> **Multiplication of the digit**

<27.1> The text states different results of the multiplication of two digits.

<27.2> If $a, b < 10$, then $(a \times 10) \cdot b = (a \cdot b) \times 10$.

<27.3> $70 \times 7 = (7 \times 7) \times 10 = 490$.

<27.4> $(a \times 100) \cdot b = (a \cdot b) \times 100$.

<27.5> $900 \times 3 = (9 \times 3) \times 100 = 2,700$.

<28> **Multiplication of the article**

<28.1> If $a, b < 10$, then $(a \times 10)(b \times 10) = (a \cdot b) \times 100$.

<28.2> $70 \times 30 = (7 \times 3) \times 100 = 2,100$.

<28.3> $(a \times 100)(b \times 10) = (a \cdot b) \times 1,000$.

<28.4> $500 \times 30 = (5 \times 3) \times 1000 = 15,000$.

<29> **Multiplication of the hundred**

<29.1> $(a \times 100)(b \times 100) = (a \cdot b) \times 10,000$.

<29.2> $500 \times 300 = (5 \times 3) \times 10,000 = 150,000$.

<30> **Multiplication involved with one repeated thousand**

<30.1> $(a \times 10^n) \cdot b = (a \cdot b) \times 10^n$ (a, b are numerals and $n \geq 3$).

<30.2> $30,000 \times 6 = (3 \times 6) \times 10^4 = 180,000$.

<30.3> $(a \times 10^n)(b \times 10) = (a \cdot b) \times 10^{n+1}$ (a, b are numerals and $n \geq 3$).

<30.4> $4,000,000,000,000 \times 30 = (4 \times 3) \times 10^{12+1} = 120,000,000,000,000$.

<30.5> $(a \times 10^n)(b \times 100) = (a \cdot b) \times 10^{n+2}$ (a, b are numerals and $n \geq 3$).

<30.6> $50,000,000,000,000 \times 200 = (5 \times 2) \times 10^{13+2} = 10,000,000,000,000,000$.

<31> The column number of the numeral of a repeated thousand

<31.1> For $a \times 10^{3n}$ ($a < 10$), the column number of a is $3n + 1$.

<31.2> For $3 \times 10^{3\times4}$, the column number of 3 is $3 \times 4 + 1 = 13$.

<31.3> For $(a \times 10) \times 10^{3n}$ ($a < 10$), the column number of a is $3n + 2$.

<31.4> For $50 \times 10^{3\times4}$, the column number of 5 is $3 \times 4 + 2 = 14$.

<31.5> For $(a \times 100) \times 10^{3n}$ ($a < 10$), the column number of a is $3n + 3$.

<31.6> For $(a \times 100) \times 10^{3\times2}$, the column number of a is $3 \times 2 + 3 = 9$.

<32> Multiplication of two repeated thousands

<32.1> $(a \times 10^p \times 10^{3n})(b \times 10^q \times 10^{3m}) = [(a \times 10^p) \times (b \times 10^q)] \times 10^{3(m+n)}$ $(p, q = 0, 1, 2)$.

<32.2> $(a \times 10^{3n})(b \times 10^{3m}) = (a \times b) \times 10^{3(m+n)}$.
$(3 \times 10^{3\times2})(7 \times 10^{3\times4}) = (3 \times 7) \times 10^{3(2+4)} = 21 \times 10^{18}$.

<32.3> $(a \times 100 \times 10^{3n})(b \times 100 \times 10^{3m}) = [(a \times 100) \times (b \times 100)] \times 10^{3(m+n)}$.
$(300 \times 10^{3\times4})(500 \times 10^{3\times2}) = (300 \times 500) \times 10^{3(4+2)} = 150,000 \times 10^{18} = 15 \times 10^{22}$.

<32.4> The text mentions briefly the case when $p = 1$ and $q = 1, 2$ in the formula of <32.1>.

<33> The name of the column of the numeral of a repeated thousand

<33.1-33.2> If $n = 3l$ (n is the column number), then $10^{n-1} = 100 \times 10^{3m}$ $(m = l - 1)$.

<33.3> If $n = 12$, $l = \frac{12}{3} = 4$ and $m = 4 - 1 = 3$, then $10^{11} = 100 \times 10^{3\times3}$.

<33.4> If $n = 3l + 2$ (n is the column number), then $10^{n-1} = 100 \times 10^{3l}$.

<33.5> If $n = 11$ and $l = 3$, then $10^{10} = 10 \times 10^{3\times3}$.

<33.6> If $n = 3l + 1$ (n is the column number), then $10^{n-1} = 10^{3l}$.

<33.7> If $n = 10$ and $l = 3$, then $10^9 = 10^{3\times3}$.[45]

<34> Calculation of the concealed number

<34.1> Let n be the concealed number.
If $3n = 2m + 1$, $3(m - 1) = 2l - 1$, $l - 1 = 9p - q$ $(q < 9)$, then $n = 4p + 3$.
If $3n = 2m + 1$, $3(m + 1) = 2l - 1$, $l + 1 < 9p$, then $n = 3$.
If $3n = 2m$, $3m = 2l$, $l = 9p$, then $n = 4p$.
If $3n = 2m + 1$, $3(m + 1) = 2l$, $l = 9p + q$ $(q < 9)$, then $n = 4p + 1$.
If $3n = 2m$, $3m = 2l + 1$, $l + 1 = 9p + q$ $(q < 9)$, then $n = 4p + 2$.
If $3n = 2m$, $3m = 2l + 1$, $l + 1 = q < 9$, then $n = 2$.[46]

[45] Although there is no concept of exponent in this manuscript, the rules of the multiplication of two numbers show the equivalent computation to the multiplication of exponents with base 10, especially those rules of <22> and <30>-<33>.

[46] The text gives the first three rules in detail. With the fourth and fifth rule, the text does not give detailed discussions, probably to avoid repetitions, but only mentions the operations on the times of 4.

<34.2> If $n = 2$, $3 \times 2 = 6$, $3 \times \dfrac{6}{2} = 9$, $\dfrac{9+1}{2} < 9$, then $n = 2$.[47]

<34.3> If $\dfrac{ax}{1} = \dfrac{12x}{b} = \dfrac{ax + 12x}{c}$, then $c = \dfrac{12}{a} + 1$.

<34.4> If $x = 5$, $a = 1$, then $b = \dfrac{12}{1} = 12$; If $x = 5$, $a = 2$, then $b = 6$; If $x = 5$, $a = 3$, then $b = 4$.[48]

<34.5> $(x + m) - (x - m) = 2m$.[49]

<34.6> $(5 + 2) - (5 - 2) = 4$.

These rules were arrived at most probably by means of such considerations as follows.

Every number n can be expressed by either $n = 4p$ $(p > 0)$, or $n = 4p + 1$ $(p \geq 0)$, or $n = 4p + 2$ $(p \geq 0)$, or $n = 4p + 3$ $(p \geq 0)$. We consider the four forms one by one.

If $n = 4p$, $3n = 2(6p) = 2m$, $3m = 2(9p) = 2l$, then $l = 9p$. This corresponds to the third rule.

If $n = 4p + 1$, $3n = 2(6p + 1) + 1 = 2m + 1$, $3(m + 1) = 2(9p + 3) = 2l$, then $l = 9p + 3$. This corresponds to the fourth rule.

If $n = 4p + 2$, $3n = 2(6p + 3) = 2m$, $3m = 2(9p + 4) + 1 = 2l + 1$, then $l + 1 = 9p + 5$. This corresponds to the fifth and the sixth rules.

If $n = 4m + 3$, $3n = 12m + 9$, $3 \times \frac{(12m+9)+1}{2} = 18m + 15$, then $\frac{(18m+15)+1}{2} = 9m + 8$. This corresponds to the first and the second rules.

[47] The formula for expressing the statement of the text looks strange. This is because of the brevity of the expressions in this text. In fact, it is a statement by a third person of a game which is performed by two other persons. The game goes like this:

A: I am concealing a number in my mind. (The number is 2 which A knows and B does not.) Can you work out what it is?

B: Sure. You triple your number first and tell me whether the resulting number is even or odd.

A: (A will get 6 as the result.) It is even.

B: Then you divide the result into two equal parts, and take the triple of either of them. After that, you tell me again whether the result is even or odd.

A: (This time A will get 9.) It is odd.

B: Now you take the larger part of the odd number, i.e. take the number which is the nearest to the half of the odd number and is larger than this half. Having done this, tell me how many nines are contained in the larger part.

A: (A will get the larger part of 9 as 5.) Your question is absurd. In fact the larger part is less than 9.

B: So the number that you asked me to seek is 2.

[48] This problem is different from the above one. In this case, it is unnecessary to know the value of the concealed number. It is the same with the following problem.

[49] Rule <35.5> may be considered as a kind of practical application of the rule <1.1>.

F

The section from <35.1> to <44.7> (= F) is a philosophical discussion concerning the construction of numbers. The meanings of the digit, article, limit and composite are explained. In contrast to the previous five parts, the meaning of the 'limit' of a number here refers to the start of the corresponding place value (column), for example, the 'limit' of digits is 1; the limit of articles is 10 and so on, while the 'limit' of a number in the first five parts of this manuscript refers to the start of the next corresponding place value (column), and the places of the 'limit' are at least 3. For example, the limit of 25 is 100, the limit of 250 is 1000 and so on.
Sentences <39.5> and <40.2> state that 1 (the unit) is neither even nor odd. The rule <16.8> in B is consistent with this, i.e., 1 is not considered as an odd number; while in the rules from <5.1> to <5.4> in A, 1 is considered odd. Therefore, A and B are not based on the same source, whereas B and F may be.

G

<45> Multiplication of whole units and fractions

<45.1> Let u denote whole units, and f denote fractions, the rule states 8 species of multiplication and division with the former listed as following:
1 $u \cdot u$, 2 $f \cdot f$, 3 uf, 4 fu, 5 $u(f+u)$, 6 $f(f+u)$, 7 $(u+f)u$, 8 $(u+f)f$.

<45.2> $n(2i) = 2k$, $(2i+1)(2k+1) = 2j+1$.

<46> Multiplication and division of fractions

<46.1> In sexagesimal numeration the result of the division of two fractions of the same kind is expressed in degrees, but the multiplication of a fraction and a degree is expressed according to the kind of fraction.

<46.2> When a fraction is divided by another fraction, the result is denominated by a higher fraction.

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