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Text production reproduction and appropriation within the abbaco tradition: A case study

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I Introduction

In this paper we present a chapter on algebra from an abbaco treatise on arithmetic. The abbaco tradition of teaching arithmetic and algebraic problem solving is situated between two major works of the Italian Middle Ages: the *Liber Abbaci* of Fibonacci (1202) and the *Summa di Arithmetica et Geometria* of Lucca Pacioli (1494). Peculiar of abbaco texts is their strong similarities and coherence within that period of almost three centuries. We will argue that this feature stems from the way texts were produced and appropriated. Problems and problem solving play a central function in abbaco treatises and the way problems were 'invented' and adapted determines this process of text production and appropriation. We will illustrate this with one chapter from a family of several manuscript copies of a single treatise, providing a critical edition and an English translation. We will also discuss the relation with other abbaco texts before and after the creation of our text. With the possible exception of Høyrup's recent book on the abbaco tradition, who calls his transcription semi-critical, all publications of abbaco texts have been based on a single manuscript though several copies are usually available. We believe that a critical edition in line with the Latin scholarly tradition provides us with the necessary insights in the way the production of abbaco texts functioned.

II Abbacus algebra: a brief characterization

With some exceptions, algebraic practice was completely absent from the scholarly tradition or university curriculum before the mid-sixteenth century.[1] It took until the late seventeenth century before algebra became taught at universities. Instead,

[1]There are some lectures or publications where the scholarly tradition displays a knowledge of algebra. One such case is the *Quadripartitum numerorum* of Jean de Murs (1343) [15], which provides evidence that algebra, as known from Fibonacci's *Liber abbaci*, was studied in scholarly circles in Paris. However, it is unlikely that it was ever taught within the *quadrivium*.

algebra flourished within the vernacular tradition of the abbacus schools in Italian cities during the fourteenth and fifteenth century. We call this the abbacus or abbaco tradition, spelled as in the *Liber abbaci* of Fibonacci (1202) to distinguish it from the abacus as a calculating tool. The abbacists practised calculation with hindu-arabic numerals as opposed to calculation using material means such as the *tavola* or the abacus. The abbaco masters were hardly known before the first transcriptions of their manuscript treatises by Gino Arrighi during the 1960s and 1970s. It is only with Warren van Egmond's extensive catalogue of manuscripts that we have a fairly complete picture of the extent of texts from this tradition [27]. Abbacus masters earned a living from teaching commercial arithmetic to sons of merchants and artisans, renting rooms and occasionally surveying assignments [9]. For the sake of prestige and also out of genuine interest many of them wrote long treatises on arithmetic and algebra in which they solved hundreds of problems. Such manuscripts were often illustrated and presented as gifts to patrons and important merchants. Van Egmond's catalogue lists about 250 extant abbaco manuscripts kept in libraries all over the world, many dealing with algebra. The seemingly evident narrative that the tradition was initiated with Fibonacci's book is currently challenged by Jens Høyrup [10][11]. Although the first abbaco manuscript dealing with algebra dates from 1307, there is evidence that the tradition existed at Fibonacci's time. Furthermore it seems that it originated from the Provençe (south of France) and Catalan regions (north of Spain) [10].

Abbacus algebra is all about problem solving. Most of the folios of these sometimes hefty collections deal with arithmetical and algebraic solutions to a large number of problems. In these treatises the introduction – if there is one – explains the rules of algebra, exceptionally with a geometrical demonstration. The earliest treatises within the abbaco tradition already expand on the six rules of Arabic algebra, but later *maestri d'abbaco* extend the list to several more types, resulting in the rather preposterous list of 198 equation types of Maestro Dardi [28] accompanied by problems to illustrate each of them. Later treatises occasionally discuss addition and multiplication of polynomials as an introduction to algebra. But that is as far as it goes for the theory. The bulk of the text is pure problem solving. There is a remarkable consistency in the structure, style and rhetoric of abbaco texts during the two centuries of their existence. Practically every text dealing with algebra follows the same rigid structure which can be divided into six parts:

1. *problem enunciation*: in a first section the problem text is provided and a question is posed. Usually problems are set in a practical context.
2. *choice of the rhetorical unknown*: every solution start with the sentence "pose that <some unknown quantity of the problem> equals <some quantity of> the *cosa*" (the rhetorical unknown). Often a clever choice of the unknown or a power of the unknown is an important step in the solution of the problem. Most

abbaco texts deal with a single unknown, though there are some exceptions [14]. However, a straightforward translation of all unknown quantities of the problem into symbolic form is a practice which is established in Europe only during the eighteenth century.

3. *manipulation of polynomials*: using the unknown, the problem text is formulated in terms of coequal polynomials and manipulated in such a way that these are kept equal. The vernacular terms *ristorare* and later *ragguagliare* are used for both the restoration and opposition operations, known from Arabic algebra [13].
4. *reduction to a canonical form*: the purpose of manipulating the polynomials is to reduce them to a form in which a standard rule applies. This marks the end of the analytical part of the reasoning.
5. *applying a rule*: usually the rule is reformulated and literally applied. Typically it includes the normalization of the equation by dividing it by the coefficient of the square term even if this amounts to dividing by one.
6. *numerical test*: often, but not always, the validity of the solution is checked by a numerical test using the root of the equation. This test is always performed on the problem enunciation and not on the equation.

The lack of symbolism in abbaco algebra is compensated by the rigid rhetorical structure. Each problem is dealt with in the same way. Every rule is reformulated and applied as it were for the first time. Repetition, cadence and structure facilitates the understanding and memorization of the problem solving procedure. Only in very rare cases are problems and solutions generalized or is there a transfer of results from one problem to another.

III Our source text

The text we will be discussing is from an anonymous manuscript of the first half of the fifteenth century with the title *Ragioni appartenenti all'arismetricha* (Calculations which belong to arithmetic). Except for the description by van Egmond [27] it has never been described, discussed or published. As typical for an abbaco treatise it is a comprehensive text of about 230 folio's (depending on the copy) dealing with subjects ranging from hand reckoning to calender calculation and including arithmetic, geometry and algebra. We will further discuss only the chapters which deal with algebra or use algebra to solve arithmetical problems. However, one copy of the manuscript includes a coin list (see below) which makes it somewhat of an exception within abbaco treatises. Coin lists did appear frequently in *mercatura* treatises as documented by a recent study of Lucia Travaini [24], but they are less common in abbaco treatises. Only three are known from published texts and these are the earliest abbaco treatises such as the *Columbia Algorism* (c. 1290, [30]), the *Liber habaci*

(c. 1310, [4]) and Jacopo da Firenze's treatise of 1307, recently published and translated by Høyrup [11]. We know of two more fifteenth-century lists in unpublished manuscripts: Vat. lat. 10488 of 1424 and Luca Pacioli's Perugia manuscript (Vat. lat. 3129) of 1478. Our text is thus the sixth known abbaco treatise including a coin list.

Algebra is discussed in chapters 30 and 31 and the problems which are solved algebraically are spread in three sections: chapter 33, 41 and 48. Some geometrical problems of chapter 34 also use algebra but these are not included in our transcription. Within the algebra chapters we also find some features which makes the text an exception within the tradition. Firstly, and most importantly, it is the first abbaco text, and as far as we know, the first algebra treatise in Europe which makes explicit reference to two kinds of problem solving, a rhetorical one (*per scritura*) and a symbolic one (*figuratamente*). While the 'symbols' used in the symbolic part are not unique – some of them also appear in previous manuscripts – the explicit description and use of two methods is very unique and makes this text crucial in the history of symbolic algebra. In chapter 33 the author solves several problems, first symbolically and then repeats the solution in sentences. Possibly the practice of solving a problem *figuratamente* existed before but in any case it was not found suitable to be written down in a treatise. Here however, our author believes the symbolic notation contributes to a better understanding of the solution as he writes "I showed this symbolically as you can understand from the above, not to make things harder but rather for you to understand it better. I intend to give it to you by means of writing as you will see soon".[2]

A second feature of the text which makes it stand out of other treatises dealing with algebra is the systematic enterprize to generalize solutions of problems into rules. The solutions to most of the problems of chapter 33 are augmented with a paragraph which states "And you can do this by a rule" hence followed by a generalization of the problem solution. Generalized solutions to algebraic problems are found in European algebra only at the end of the sixteenth and the beginning of the seventeenth century, notably in Viète [29], Clavius [6] who generalizes the problems of Stifel [22] and Jacques de Billy [5] who abandons the terms 'problem' and *quaestio* altogether and instead talks about *propositio*.

A third feature of interest is that we have six extant copies of the manuscript. This is not exceptional – the *Trattato d'abacho* by Bendetto da Firenze is extant in eighteen copies – but it is the anonymous manuscript dealing with algebra with the most copies. Concerning the use of symbolism, this is particularly useful because we can compare the 'symbols' or *figure* between the copies and it informs us about

[2] $f.059^r$: "Ora io telo mostrata figurativamente come puoi comprendere di sopra bene ch'è lla ti sia malagievole ma per che tu lla intenda meglio. Io intende di dartela a intendere per scrittura come apresso vedrai".

the way scribes adopted or rejected particular ligatures and symbols. However, this aspect will be discussed elsewhere. We will now describe the six manuscripts and discuss further the contents of chapter 44.

III.1 The extant manuscripts

III.2 Sigla

A (c.1437) Florence, Biblioteca Nazionale Centrale, Magl. Cl. XI. 119, ff. 5r-213v, Inc: "Concio sia cosa che sono nove figure nell'abacho per le quali chi conoscie quelle agievolmente conosciera poi l'altre .." (neat *corsiva cancellaresca formata*)

B (c.1440) Florence, Biblioteca Mediceo-Laurenziana, Ash. 608, ff. 3r-173r, Inc: "Concio sia chosa che sono nove fighure nel abacho per le quali chi chonoscie quele agievolmente chonosciera poi l'altre .." (rapid *corsiva mercantesca*)

C (c.1440) London, British Library, Add. 10363, Inc: "Concio sie cosa che son 9 fighure nell'abbaco per le quali chi chonoscie quelle agevolmente conosciera poi l'altre . . . ", (very neat humanistic bookhand)

D (c.1440) Paris, Bibliothèque Nationale, It. 463, ff. 3r-66r, (irregular *corsiva mercantesca*)

E (1442) London, British Library, Add. 8784, Inc: "E choncio sia chosa che sono 9 fighure nell'abacho per le quale chi conosce quelle agievolmente conoscera poi l'altre...", (fairly neat Italian Gothic bookhand, by Agostino di Bartolo)

F (c.1444) Florence, Biblioteca Mediceo-Laurenziana, Ash. 343, ff. 3r-96v, Inc: "Chonciosia chosa che sono 9 fighure nell'abacho per le quale ci chonoscie quelle agievolmente chonoscera poi l'altre .." (neat *corsiva mercantesca*)

S (c.1463) Siena, Biblioteca Comunale degli Intronati, L. V. 46, ff. 1-17, Inc: "Conciosia cosa che sono 9 fighure nella arithmetricha, cioe nell'abacho per le quale che cognitione di quello facilmente ha poi di tucte l'autra ..", (*mercantesca*).

III.3 Sigla of other manuscripts

L (c.1330) Lucca, Biblioteca Statale, 2408, ff. 1r-87r, *Libro Di Molte Ragioni d'Abaco*, (transcription by Arrighi 1973)

R (c.1485) Florence, Biblioteca Riccardiana, Ms. 2408, ff. 1r-202v, *Trattato d'arismetricha*, (transcription by Arrighi 1973)

V (1307) Vatican Library, Vat. Lat. 4826. Jacobo da Firenze, *Tractatus algorismi.*

III.4 Description of the six copies

III.4.1 Manuscript A

All seven copies are described with their paleographic properties in the *Catalogue* of van Egmond [27] from which we also use the dating in the sigla. We used **A** as the basis for the transcription because it is dated by van Egmond as the oldest and together with **C** and **E** it is also the most complete. The manuscript has been in the library of Carlo di Tommaso Strozzi (1587-1670), catalogued under number 546 in 1670. Van Egmond dates the manuscript on basis of the watermark roughly similar to Briquet's 11702 from 1437 but this does not give us a definite date. The watermark is also similar to Briquet's 11726 from Florence dated 1432-3. We believe this to be closer to the actual date of the copy. There are passages which appear in all other five copies and which are omitted in **A**. Therefore we can establish that this is a copy of an earlier lost manuscript, which we call α. The lunar tables in chapter 59 run from 26 January 1418 to 1444, from which we assume that α was written in the year 1417 or 1418. The scribe was knowledgeable about the material he was copying. He rarely introduces any mistakes and often repairs errors or completes omissions from his source. The marginal corrections in the same hand suggest a second reading of his source and in general **A** can be considered as the most reliable extant copy.

III.4.2 Manuscript B

The first difference between **B** and **A** that attracted our attention is that the lunar tables run from 14 January 1438 to 1459. At some point, presumably in 1437 someone found it necessary to update superseded data from the original copy which has lead to a new family of manuscripts. However, this copy is not the first one within this family as **B** is dated by van Egmond at 1440 from internal evidence. A second peculiarity is that **B** includes a coin list, as discussed above (ff. $168^v - 170^r$). From chapter 57 the chapter numbering differs from the other copies. This copy omits several problems from all three sections dealing with algebra.

III.4.3 Manuscript C

This is the most legible handwriting from a professional hand taking great care in drawing the figures and the leading characters. The lunar tables run from 26 January 1418 to 1444 which differentiates it from the family including manuscript **B**. Chapter numbering is done by the scribe himself while most other copies have numbering done by a later hand.

III.4.4 Manuscript D

This text is an incomplete copy kept at the national library in Paris. Van Egmond dates the manuscript on basis of the watermark at c. 1440. However, three different kinds of paper are used with watermarks ranging from 1429 to 1457, which should situate the copy at the upper end of the range. Unfortunately, the lunar tables, which immediately would place it within one of the two families, are not included in the copy. However, from textual analysis, as will be discussed below, we can determine that this copy belongs to the **B** family. The text contains the table of contents but the chapter headings are not always included in the text and the chapter numbering is omitted.

III.4.5 Manuscript E

This is the only manuscript for which we not only have a definite date but also a name. As we read on $f.212^v$:"I finished this book of computations amen. Written in a Florentine prison on the day of the fifteenth of March 1442 by me, Agostino di Bartolo".[3] But given the earlier copies of the same text, di Bartolo cannot be the author and should be considered the name of a scribe. The scribe was not very knowledgeable about the subject. While this copy includes all the problems we also find in **A** and **C**, this is the most mutilated copy. Almost every problem solution has one or more lines missing to a degree that the solution text becomes incomprehensible. The lunar tables run from 26 January 1418 to 1444, so it does not belong to the **B** family.

III.4.6 Manuscript F

Written by a professional scribe in very neat but small *corsiva mercantesca*. Also this scribe was not familiar with the material. It is therefore very unlikely that he attempted to correct any mistakes in the original related to its contents. This copy has most chapter titles missing, and some chapter have been omitted or abbreviated to a few lines. All of the geometry problems come without figures and the scribe also left out many of the symbolic parts of the algebra sections. The lunar tables run from 26 January 1418 to 1444 so it belongs to the family of **A**, **C** and **E**.

III.4.7 Manuscript S

Van Egmond pointed out that the first 18 folio's of this Siena manuscript are from the same family of manuscripts. As this only deals with the multiplication tables and the basic operations of arithmetic, we discarded this copy for further research.

[3] "Finito detto libro di ragione amen. Scritto nelle stinche a di xv di Marzo 1442 per me Agostino di Bartolo" (Queen's Anna's new world of words, 1611: "stinche: the name of a prison in Florence").

III.5 Dependencies

We have already determined one archetype α from 1417 and an oldest extant copy **A** from around 1433. We cannot be sure that **A** is a direct copy from α but given the limited number of omissions and scriber errors in comparison with other copies, it probably is. However, we can establish for certain that none of the other copies depends on **A**. For example in RAA301E the text of **A** has "Truovami uno numero chel la metà d'esso numero..", while all other copies (which list the problem) use instead "che lla metà di quello numero". This is one of several instances where the reading of **A** differs from all other copies. Manuscript **B** has many of its alternative readings in common with **D** but **D** has chapter 48 missing. Therefore **B** cannot depend on **D**. On the other hand, there are several problems omitted in **B** which do appear in **D**. Therefore they cannot be considered copies of each other in either direction. This is further confirmed by the many different readings between both. We therefore must assume the existence of another lost copy which we call β from which **B** and **D** are copies. As both **B** and **D** are dated at c. 1440 and the lunar tables were updated from 1437 we can assume that β was created around that date. This leaves us with the **CEF** family. Copies **C** and **E** share all the problems with **A** but **F** does not. Therefore **F** must be a truncated copy of either **C** or **E** or from an intermediate source. The text shows us that **C** and **E** are very close. They share text additions such as "a mezza cose" in RAA301S, and "Perciò che se io multipricero una cosa via una cosa sie uno cienso." in RAA307S, text omissions such as the word *da'* in RAA315E and alternative readings especially in the concluding phrases of a problems. However they differ also in too many instances to consider them copies in either direction. In RAA305, **C** has several copy errors in the numbers, $\frac{10}{169}$ instead of $6\frac{10}{169}$ and $6\frac{7}{13}$ instead of $7\frac{6}{13}$ which are not in **E**. Given the many other mistakes in **E** it is unlikely that the errors in **C** were corrected by the scribe of **E**. On the other hand, there are so many serious omissions in **E** to even consider **C** as a copy of **E**. We therefore must assume a common source copy which we call γ and which fits between α and **CEF**. This leaves us with **F** as a copy of either **C** or **E**. The many omissions in **E** allow us to determine easily that **F** depends on **E**. This leads us to a chain of dependencies as shown in Figure 1. Having determined all this we are still left with some anomalies which apply to chapter 48 specifically. Leaving out the ending phrases like "esta bene, ed è fatta, e così fa l'altre" which differ the most between the copies, there are four instances in which **F** seems to deviate from **E** in a way which cannot be explained by corrections. I have pointed out these explicitly in the critical apparatus. For example in RAA303T, both **C** and **F** omit the line "agiugni sopra 5 sono 8. E poi di' radicie di 9 sie 3" but we find it in **E**! It is therefore possible that the network of dependencies for the **CEF** family is still more complex than pictured here. Another possible explanation, which we favor, is that the scribe of **F** had at its disposal another copy of either chapter 48 or an earlier version of

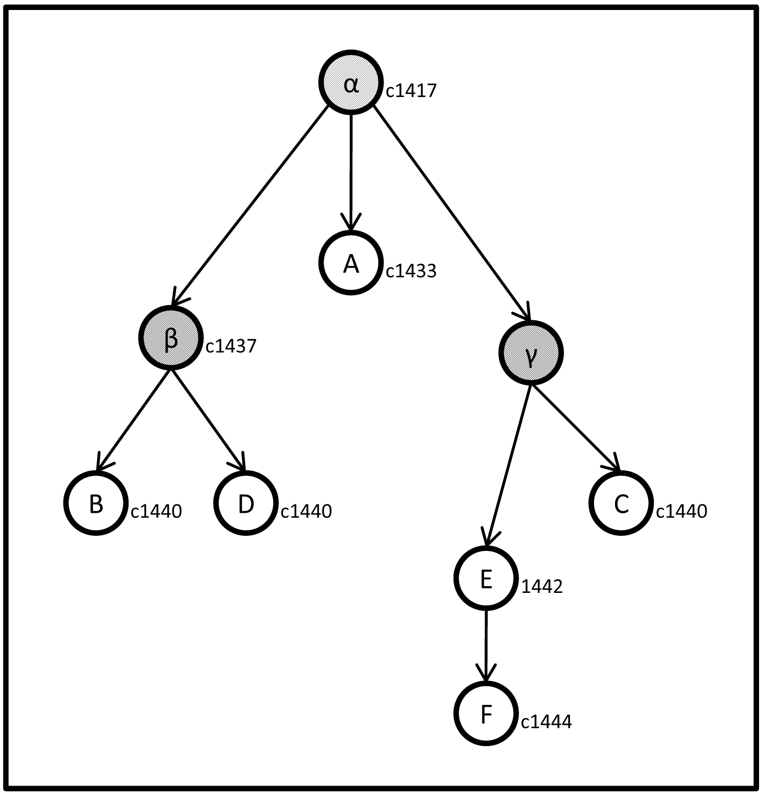

Figure 1: Dependencies between the manuscript copies

the text. As we shall see below, chapter 48 depends on a fourteenth-century text, so this last supposition makes some sense. In any case, the four exceptions provide us with insufficient data to revise our scheme.

III.5.1 The structure of the text

The algebra sections are spread in several parts over the text. Chapter 30 lists a total nineteen rules of algebra ('equations'), though some copies omit one or more of the rules. The rules are given without illustrating examples which is not uncommon in other abbaco treatises. Chapter 31 is titled "on the rules of algebra and their roots and other things" but is limited to an explanation of what square and cube roots are. Chapter 33 is a first section of 39 problems and their solutions which we will refer to as the first part. It does not contain any further rules or explanation of rules. The symbolic solutions to problems only appear in this part. The second section is chapter 41 with the cryptical title "To know, when one leaves money [to others] in a circle, how much each is left" which refers to the typical recreational problem of two or more men having a sum of money and finding a purse. The reference to the circle probably stems from one cyclic variant of the problem with the following structure in modern symbolism:

$$\begin{aligned} x + a_1 &= b_1(y + z - a_1) + c_1 \\ y + a_2 &= b_2(x + z - a_2) + c_2 \\ z + a_3 &= b_3(x + y - a_3) + c_3 \end{aligned} \tag{1}$$

However, the chapter deals with a range of problems not limited to this specific type. The fifteen problems include legacy problems, compound interest problems, exchange problems, a meeting problem and also the traditional Arabic problems. Also in the second section there are no additional rules of algebra. The third section is chapter 48 which refers to algebra in the title: "More diverse computations which are done by algebra" for which we here provide a critical edition and translation. It includes 21 problems of which 10 are solved by applying a rule rather than by algebra. The third section includes five rules of algebra each illustrated by a problem.

It is evident from even a brief comparison that the three sections have been compiled from three different sources. We can deduce that from the style, vocabulary and internal references.

Concerning the style the most salient feature of section one is the presence of symbolic solutions. None of the two other sections use similar solutions, similar symbols or even reference to solving a problem *figuratamente*. Nevertheless, very similar problems, solutions and operations appear in all three sections. If we assume that the author of α introduced the method of symbolic solutions in section one of this treatise it appears that he was not the author but rather a compiler of the two other parts. Also specific for section one is that it repeats the complete text of the rule which is being applied in solving the problem. In part three the solutions do not refer to a specific rule but five problems are preceded by the description of the rule. In section two there are occasional references to a rule, while the type of rule is being described.

The curious repetition of the whole rule within the solution text is not very common. So it may provide us some clues about the possible sources of the author. The most common formulation is “dicie la regola della cosa che quando il ciensi ..”, which is used 14 times within the first part. Fourteenth-century reatises in which each rule is followed by an example do not repeat the rule within the solution. They usually refer to it by the expression “observing the rule given above” (“oservando la regola di sopra data”) or “proceed according the rule said above” (“prociede siconda la regula sopra ditta”). This includes Jacopo da Firenze (1307) [11], the Lucca manuscript [3], the anonymous Florence text Fond.prin.II V.152 (c.1390) [8], Maestro Dardi and Maestro Benedetto. Others first list all the rules and in the solution text refer to the rule by number such as “this is the fifth rule” (“che è la quinta regola”), or “following the first rule” (“sechondo la prima reghola”). Examples are Maestro Biaggo and the anonymous treatise Ricc. 2263 (c. 1365) [21]. Maestro Gilio of Siena (c.1380) [7] is somewhat of an exception as in his algebra he does not refer to any rules at all. The only algebra treatise where we do find a similar formulation is the pseudo-Paolo dell’Abaco [1]. However, rules, problems and solutions differ too widely to assume any relationship between this treatise and the current one.

We also discern a use of vocabulary which is specific to each of the three sections. The first section is the only one to use the expression “and so much is worth every cosa” (“e tanto vale ogni cosa”) both in the description of the rules of algebra as in seven of the problems.[4] The second section is the only one to use the expression *seguita*, in the meaning of “follow this rule”. The third section is the only one to use the term *dividere* (in RAA305 and RAA317)[5] and ‘ecco’ and occasionally uses “né più, né meno”.

Lastly, there are some idiosyncrasies, specific to one of the sections. The third uses the expression “to divide by half”, actually meaning to halve or to divide by two. We do not find this used in the other parts.

This leads us to the conclusion that the author of this text did not write all three parts by himself. As no other abbaco texts are know which adopt the distinction between ’symbolic’ and ’rhetorical’ solutions we assume that the first part is from the hand of our author. If we look for previous sources for the problems in this treatise, they are likely to be found for the second and third part.

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[4]RAA312 in the third part is an exception.

[5]It is noteworthy that in these two problems the use of *dividere* is slightly different. The first is used in the sense of sectioning into two parts, in the second problem we have a division of two numbers. The more common term for that is *partire per*.

III.6 Its place within the abbaco tradition: sources and influences

In the introduction we have characterized abbaco texts on arithmetical problem solving as a very coherent tradition. We discern only minor variations in terminology and problem solving methods within two centuries of text production. The family of texts we are discussing here provides an instructive illustration of this and allow us to link the earliest extant abbaco treatise on algebra (1307) with one of the last ones (1495) within this tradition. It is precisely this interdependence of texts which causes the consistency in the treated subjects and methods.

III.6.1 A direct source for the third part

While the dates for the manuscript copies determined by van Egmond all point toward the 1440's, several factors lead us to think that the source of the text might be a lot earlier, before the date of 1418 which we conjectured for the archetype copy. Firstly, the type of problems are typical for the early abbaco period. Secondly, the terminology and spelling is often closer to fourteenth then fifteenth-century texts. Looking for similar problems in earlier treatises we found a direct source for the third section in *Libro Di Molte Ragioni d'Abaco*, the Lucca manuscript 1754 from c. 1330 (henceforth **L**). This text, transcribed by Gino Arrighi in 1973 [3], is an abbaco treatise of medium length including two short parts on algebra. One part runs from $f.50^r$ to $f.52^r$ and the second $f.81^r - f.81^v$. It appears that a consecutive section of eight problems from our family of texts correspond with the first algebra section of **L**, as follows:

Chapter 48	Lucca 1754
RAA308	50r, first problem
RAA309	50r, third
RAA310	50r, second
RAA311	50v, first
RAA312	50v, second
RAA313	50v, third
RAA314	51r, second
RAA315	51r, third
RAA316	51v, second

These eight problems have been literally copied from the Lucca manuscript or from a common source. We can demonstrate that by the following fragment of a problem about a man going two business trips and making the same proportional profit on both trips.

Libro d'abaco (c. 1330)	Problem RAA312
Uno huomo è che va in due viaggi, al primo viaggio guadagna fiorini 6 d'oro e al seghondo viaggio guadagna a quella medesma ragione, e da sezzo si truova 27 fiorini d'oro. Adomandotj con quanti danari si mosse. In prima de' fare chosìe. Pognamo che ssi movesse chon una chosa, al primo viaggio al primo viaggio guadagnerebbe 6 fiorini d'oro, eccho una chosa e 6 fiorini d'oro. Or di' chosì: d'una chosa fo io una chosa e 6 fiorini d'oro, che farce d'una chosa e 6 acciò che ne vegna 27? Or di' chosì: multipricha una chosa e 6 per se medesmo, fanno uno censo e 12 chose e 36. Ora dèj partire uno censo e 12 e 36 per una chosa, che nne dìe venire 27 senpre. Dèi multiprichare il partitore contra quella che nne de' venire, e sserà eguale a quello che ssi dìe partire. Dièi multiprichare una chosa contra 27, fae 27 chose; abiamo che 27 chose sono egualj a uno censo e 12 e 36. Ora trai 12 di chatuna parte, abbiamo che 15 chose sono egualj a uno censo e 36. Ora dimezza le chose, ch'è 7 e 1/2, dèlo multiprichare per se medesmo che fano 56 e 1/4; dène chavare 36 rimane 20 e 1/4. E abbiamo che ssi mosse con 7 e 1/2 meno radice di 20 e 1/4, e voglio dire con 20 e 1/4 meno radice di 7 e 1/2 che ssono a punto o voglio con 12 o voglio con 3.	Eglie uno huomo che va in due viaggi, al primo viaggio guadagnia 6 fio. d. e al secondo viaggio guadagnia a quella medesimo ragione et da sezzo si truova 27 fio. d. Dimmi con quanti d. simosse. Farai così: pogniamo che ssi movesse chununa cosa. Al primo viaggio guadagnia 6 fio. d. Eccho una cosa e 6 fio. Ora di' una cosa fa io una cosa e 6 fio. d'oro che faro io al secondo viaggio d'una cosa e 6 acciò che nne venga 27. Debbi multipricare una cosa e 6 fio. d. per lo medesimo fanno uno cienso e 12 cose e 36 fio. d'oro. Ora debbi partire uno cienso e 12 cose e 36 fio. d'oro per una cosa dene venire 27 fio. d'oro. Sempre dèi lo partitore multipricare e quello che nne viene farà iguali a quello che ssi de partire. Sicché debbi multipricare una cosa via 27 fio. d'oro fanno 27 cose. Abbiamo che 27 cose sono iguali a uno cienso e 12 cose e 36 per numero. Ora trai 12 cose di ciascuna parte. Abbiamo che 15 cose sono iguali a uno cienso e 36 per numero. Ora dimezza le cose, sono 7 $^{1}/_{2}$. Multipricare per se medesimo fanno 56 $^{1}/_{4}$. Trane 36 resta 20 $^{1}/_{4}$ e con 7 $^{1}/_{2}$ e radicie di 20 $^{1}/_{4}$ si mosse o vuoi dire con 12 o vuoi con 3 per numero.

Our author followed the text of **L** quite faithfully with only some minor reformulations. So he prefers 'restare' over 'rimanere'. The only technical term changed in this and the other problems is 'cavare' by 'trane' as also used in **L**. Remarkable is that while the rule describes two solutions, one by adding the root and a second by subtracting it, the two calculation in the manuscripts give only one. The Lucca manuscript only gives the subtractive one and the present text gives the additive calculation. In the end both texts list the two solutions. This seems to indicate a common source for these two texts. The Lucca manuscript does not include verifications of the solution. Our author adds the test for problem RAA316 as he does for the other problems, not included in **L**. Two other problems which are solved by algebra (RAA305 and RAA306) are not included in **L**.

In 1978 van Egmond wrote an article on what he then considered "The Earliest Vernacular Treatment of Algebra: The *Libro di Ragioni* of Paolo Gherardi" [26]. It has since been established that Gherardi's algebra, written at Montpelier in 1328, is preceded by that of Maestro Jacopo da Firenze. Jacopo wrote the text, also at Montpelier, in 1307. It now appears that **L** depends on this text. Høyrup (2006, 21) has called the two algebra sections of **L** as "two free abridgments of Jacopo da Firenze's algebra" (1307) (henceforth **V**). Indeed, let us look at the text of problem RAA314 in these two earlier manuscripts.

Jacopo, $f.39^v$	Lucca 1754, $f.51^r$
Fammi de 10 dui parti, che multipricata la magiore contra la minore faccia 20. Adimando quanto serà ciascheuna parte. Fa chosì, poni la minore parte fosse una chosa. Dunqua la magiore serà rimanente infino in 10, che serà 10 meno una chosa. Appresso si vole multiprichare la minore, che è una cosa, via la magiore, che è 10 meno una cosa. Et diciamo che vole fare 20. Et però multipricha una cosa via 10 meno una cosa. Fa 10 cose meno uno censo, la quale multiprichatione è oguale a 20. Ristora ciascheuna parte, cioè de aggiongere uno censo a ciascheuna parte, et arai che 10 cose sonno oguali a uno censo et 20 numeri. Arrecha a uno censo, et poi dimezza le cose, ve ne vienne 5. Multipricha per se medesimo, fa 25. Cavane el numero, che è 20, remane 5, del quale piglia la sua radice, la quale è manifesta che non l'à apponto. Adunqua vale la cosa 5, cioè el dimezzamento meno radice de 5. Et noi ponemo che la parte, cioè la minore, fosse una chosa. Adunqua è 5 meno radice de 5. Et la seconda è rimanente infino in 10, che è 5 et più radice de 5. Et sta bene.	[F]ammi, di 10, due parti che multiprichata l'una parte per l'altra faccia 20. De' fare chosì. Poni che ll' una parti sia una chosa e l'altra è rimanente infine in 10 ch'è 10 meno una chosa. Dèi multiprichare una chosa via 10 meno una chosa, fanno 10 chose meno uno censo. Abbiamo che 10 chose meno uno cienso sono egualj a 20. Or di': da' uno censo a chatuna, abiamo che 10 chose sono eguali a 20 e uno censo; or dimena le chose che sono 5 chose, or lo multipricha per se medesimo fanno 25, ora ne trae 20 rimane 5. Abiamo che ll'una parte è 5 e radicie di 5, e l'altra è 5 meno radicie di 5. Ed è fatta.

It is not established that the Lucca text is derived from Jacopo's treatise. According to Høyrup, Jacopo must have used still earlier vernacular texts. Therefore, **L** could have been derived from these lost sources. However, considering the concordance of the use of 'partire per' and 'partire in' Høyrup considers the relation between **V** and **L** very close [10](footnote 33).

This problem of dividing a number into two parts given their product is one of the most common problems within the abbaco tradition. It may ultimately go back to Babylonian algebra as it is very similar to rectangle problems where the sum of the sides is known. However it does not appear in Babylonian algebra with values leading to irrational solutions. Dividing 10 into two parts is also very common in Arabic algebra. However, as far as we know not the problem of dividing 10 into two parts with 20 as their product. The text of **L**, which is literally copied by our author, can be considered as an abbreviated version of **V**. In many cases however, abbaco masters change the values of the problems while copying problems from earlier sources and adapt the solutions accordingly. We can witness that from some other problems taken from **V**. Problem RAA308 ($f.50^r$ in **L**) asks to divide 10 into two parts while dividing the larger by the smaller the result is 5. The original problem in **V** has instead 100 for the quotient (problem 16.2, $f.36^v$ in **V**). Problem RAA311 is a compound interest problem with a starting capital of 400 £ which amounts to 480 £ after two years ($f.50^v$ in **L**). The original problem starts from a capital of 100 £ which grows to 150 £ after two years (problem 16.9, $f.38^r - 38^v$ in **V**). Changing the values of problems may have been a way to conceal the source, this technique has also been used to increase the complexity of the solution. What we often see is that a problem with a simple integral solution is tweaked to arrive at complex irrational solutions. We give one example from Maestro Antonio on three numbers in geometric progression with their sum and product given:

$$\begin{aligned} &\tfrac{x}{y} = \tfrac{y}{z} \\ &x + y + z = a \\ &xyz = b \end{aligned} \tag{2}$$

The method of solution stands as a model for many other problems of this kind. Its first occurrence in extant manuscripts is to be found in Antonio de Mazzinghi's *Trattato di fioretti* written before 1383, [ATF023] with 14 as the sum and 64 as the product. As often with abbaco problems their solution depends on a clever choice of a single unknown. In this case the best choice is the middle term in order to exploit the identity $y^2 = xz$. His approach is to reduce the problem to the sub problem of finding the two extremes, given their sum. Taking the mean term as the unknown, the sum of the two extremes is thus $14 - x$, which has to be divided into two parts such that their product is equal to x^2, the square of the middle term. His procedure to do this in modern symbolism corresponds with:

$$\frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 - ab} \tag{3}$$

Applied to the sum of the extremes $14 - x$, this results in:

$$7 - \frac{1}{2}x \pm \sqrt{49 - \left(\frac{3}{4}\right)^2 - 7x} \tag{4}$$

So, now that we have the three terms expressed in one unknown, Antonio observes that multiplying the three terms together is the same as taking the third power of the middle term because $y^2 = xz$. The middle term is therefore the cube root of the product of the three, or 4. Substituting 4 in (4) leads to the solution (2, 4, 8) for the three numbers. The very same problem solved with exactly the same procedure is to be found in Pacioli's *Summa* [16] [PSA609]. The abacus master Dionigi Gori from Siena treats the exact same problem in an unpublished manuscript of 1544 (Siena L.IV.22, $f.83^v$; [17], 32). He sets the problem in a practical context of three types of wool and adds a version about the money of three men in geometric progression, with sum 10 and product 30 leading to the irrational solution:

$$\left(3\frac{1}{2} - \sqrt{2\frac{1}{4}}, 3, 3\frac{1}{2} + \sqrt{2\frac{1}{4}}\right) \tag{5}$$

The irrational solution does not contribute anything new. At the contrary, it obfuscates the reading of the solution method, which is identical to the one of Maestro Antonio. It is also very peculiar to take a 'theoretical' problem and set it within a context of merchant practice but end up with a solution of $3\frac{1}{2} - \sqrt{2\frac{1}{4}}$ lira which has no practical use whatsoever. We assume that this practice of replacing simple integral solutions with complex irrational ones was a way of demonstrating the skill of abbaco masters.

III.6.2 Later reproduction of the first two parts

We previously expressed our surprise of finding generalized rules for solving problems in an abbaco treatise by Rafaello di Giovani Canacci [12]. For a recent study on Canacci as an abbacist see [25]. Given that our manuscript also generalizes problem solutions we were curious to see if there is any relation between the two treatises. It appears that Canacci was not the first one to do so and that he found his inspiration from our family of manuscripts. Canacci wrote several texts on arithmetic and algebra:

- *Trattato d'arismetricha* (c.1485)
- *Fioretto dell'abacho* (1490)

- *Vilume del argibra* (c.1490)
- *La regola dell'argibra* (c.1495)
- *Alchuna ragione* (1496)

Only the fourth treatise has been published under the title *Ragionamenti d'algebra*, the first forty folios by Angiolo Processi [18][19] who later also published the second part [20]. The other texts have not been published but we know that the *Vilume del argibra* is quite similar to the fourth one and that the last one does not contain algebra. We have only been able to compare our manuscript with the second part of the *La regola dell'argibra* and this shows a clear dependence as can be seen from one problem shown below.

RAA135	Cannaci f. 63v
E puossi fare per regola. Cioè multipricare <e> metti per partitore sempre quello che vuoi che rimanga, cioè 3 3/4 . Ora viponi sempre su 2 ài 5 3⁄3. Ora multiprica 10 via 5 3⁄3 fa 57 1⁄2. Ora lo parti per 3 3⁄3 vienne 15 1/3. Ora multiprica 10 per sé fa 100. Parti pure per 3 3⁄3 vienne 26 2/3. Ora dimezza 15 1/3 ch'è 7 2/3. Ora lo multiprica per se medesimo fa 58 7/9. Ora netrai 26 2/3 resta 32 1/9 e dirai che una parte sia 7 2/3. Cioè quello dimezzamento che faciesti meno la radicie di 32 1/9. E l'altra dirai che sia 5 2/3.	Puossi fare per reghola. Cioè multipricha e metti per partire per sempre quello vuoi che rimangha, cioè 3 3⁄3. Cioè sempre imponi suso 2 fa 5 3⁄3. Ora mulipricha 10 che vuoi dividere vie 5 3⁄3 fa 57 1⁄2 e questo parti per 3 3⁄3 vienne 15 1/3. Ora multipricha 10 per sé fa 100. Parti pure per 3 3⁄3 vienne 26 2/3. Ora dimezza 15 1/3 che è 7 2/3 e questo multipricha per sé fa 58 7/9. Ora netrai 26 2/3 resta 32 1/9 e dirai che l'una parte sia 7 2/3. Cioè quello dimezzamente che facesti meno la radice di 32 1/9. El l'altra dirai sia e resto per insino in 10.

This demonstrates that Canacci not only adopted the idea of generalizing problem solution from our manuscript family but also literally reproduced parts of the the text. We can even determine from which specific manuscript Canacci copied his problems an solutions from. For example in problem RAA129 the **ABD** manuscripts write "e tanto farà andato el primo" while **CEF** and Canacci use "e tanto sia ito il primo". There are important omissions which are shared by the **CEF** copies and Canacci. For example problem RAA132 has the sentence "Dunque e più uno di e vedi che nel primo sisono recari a 3 di e 7 cose di di del primi e del secondo sisono recari a 8 di e 4 cose". **CEF** and Canacci have instead "Dunque e più uno di e vedi che nel primo sisono recari a 8 di e 4 cose", which by omitting a line, becomes a false statement. Some omissions specific to **C**, which are not easily corrected, are not found in Canacci's version, so this excludes **C**. Omissions which are specific to **E** and **F** such as "e ponemmo che uno di del secondo sia per una cosa del primo" in the paragraph following previous quotation, are not found in Canacci. Therefore Canacci must have used the mother copy of **CEF** which we conjectured as γ.

RAA	Problem	Canacci
101	Truovami uno numero che multipricato .. 300	
102	Truovami due numeri .. tre più che l'altro	
103	Truovami due numeri .. quanto multipricato	
104	Fammi di 10 due parti .. faccia 24	
105	Fammi di 10 due parti .. facciano 60	45
106	Fammi di 10 due parti .. ne venga 24	47
107	Fammi di 10 due parti .. ne venga 50	48
108	Fammi di 10 due parti .. rimanga 40	46
109	Truovami tre numeri .. faccia 60	
110	Eglie uno huomo che à .. nogli rimase lira	
111	E sono tre che ànno denari .. 8 per uno	
112	E sono tre che ssi puosono a giuocho	
113	E sono tre che ànno denari .. tre cotanti dite	
114	E sono tre huomini .. quatro cotanti di vuoi	
115	Tre huomini ànno d. .. io ò 5	
116	E sono due huomini che truovano una borsa	
117	Tre huomini truovano una borsa	
118	E sono due che vogliono comperare	
119	Tre huomini vogliono comperare	70
120	Uno padre à alquanti figliuoli	
121	Eglie uno che à una somma di mele	
122	Di qui a Roma sono 200 miglia	
123	Eglie uno che va ogni di 25 miglia	
124	Uno maestro toglie a ffare una lavorio in 10 di	
125	Eglie una botte che à 3 canelle	
126	Tre huomini ànno lira .. due cotanti di lui	71
127	E sono tre che ànno lira .. 3 e più 5 di voi	72
128	Eglie uno che à bisogno di lira .. lira 80	73
129	Eglie uno ch'è a Firenze per andare a Roma	74
130	Eglie uno che ffa alquanti viaggi	75
131	E sono 3 che tolgone ascharicare .. in 2 di	
132	E sono 3 huomini che toglono .. in 2 di	76

133	Fammi di 10 due parti .. ne venga 16	
134	Fammi di 10 due parti .. faccia 20	
135	Fammi di 10 due parte .. rimanga 3 $^3/_3$	
136	Fammi di 10 tali due parte .. facciano 20	51*
137	Truovami 2 numeri .. e 10 più	
138	Truovami 2 numeri .. posto insulsecondo	
139	Uno presto a un altro d. 100	
201	E sono due huomini che ànno d. .. 5 più dite	78
202	Uno merchantante à suoi d. .. spendesse nulla	79
203	Uno huomo manda uno suo famiglio .. b. 3	
204	Eglie uno huomo che presta .. in prima	
205	Uno huomo presto a uno altro fior. 40 b.	
206	Eglie uno huomo che presta .. fior. 5 chonessi	
207	Uno huomo a uno sua fiorino .. b.20 divolterrani	
208	Uno huomo à 20 fior. d'oro .. secionio 100	88
209	Uno à 10 bologni .. 24 pisani	
210	Uno maestro tolse a cavare uno pozzo	
211	Fammi di 10 due parti .. faccia 10	
212	Fammi di 10 due parte .. faccia 4	
213	Fammi di 10 due parte .. faccia 20	52
214	Fammi di 10 2 parte .. faccia 3 $^1/_2$	53
215	Eglie uno corriere che va .. ciesciendo uno miglio	90

III.7 Conclusion

We have discussed the algebra sections of one anonymous treatise from the abbaco tradition. While this text has some specific characteristics which are of historical interest, primarily its explicit acceptance of 'symbolic' solutions, from a mathematical point it contains no real surprises. However, the text and its relation to other treatises provides us with an informed insight in the way abbaco text are produced, copied and appropriated. The fact that we have here a family of at least ten copies of a single manuscript demonstrates that abbaco treatises were considered valuable properties which were collected by merchants, bankers and nobility in Renaissance Italy. Professional scribes were paid to copy abbaco texts with little knowledge about their contents. For the production of such texts abbaco masters relied heavily on their tradition. They copied literally from previous manuscripts, abbreviated solutions, in the process changing the order of the problems and the problem values to conceal their sources. Often problem values or parameters were tweaked to increase the complexity of the problem solutions. In the process of almost three hundred years of text production the practice of algebraic problem solving within the abbaco tradition paved the road to the emergence of symbolic algebra during the first half of the sixteenth century. The expansion of the number concept, in particular the full acceptance of irrational numbers, and occasional occurrences of negative solutions contributed to this process. The preparation of a symbolic algebra was realized, not only through the acceptance of 'symbolic' solutions such as in this treatise, but also through the epistemic validity attributed to the operations of algebra. The validity of the rules of signs is derived from correctly expanding the binomial product $(a-b)(a-b)$, which we consider as a form of symbolic reasoning. The close interdependency of abbaco treatises and the relation between problems and solution types makes this tradition a very coherent one.

IV Transcription

IV.1 Conventions

I have numbered all rules and problems. Problems are broken down in (E) enunciation, (S) solution, (T) test and (R) rule generalization for easy reference. Line breaks are added for distinguishing the problem enunciation, the problem solution, the numerical test and the formulation of a general rules, if present. Abbreviations in the text have been expanded except for the 'symbolic' parts, which literally follow **A**. Accents have been added to make the distinction in meaning between words such as *ch'è* and *che*, *ànno* and *anno*, *ài* and *ai*, *di'* and *di*, *à* and *a*, *è* and *e*. Punctuation and word separations have been normalized where modern Italian uses separate words. Spelling has been normalized to the dominant spelling of manuscript **A** (e.g. "multipricato" as most commonly spelled in **A**). The distinction between *i* and *j* has been dropped in favour of *i*.

The critical apparatus has been used only with respect to differences in contents, not spelling, minor word omissions like 'ora', minor word transpositions or the use of words for numbers and vice versa. As common, I used < > to complete omissions and { } to indicate passages that are included by error. Single brackets [ ] are used to add my comments and pointers, such as problem numbers. Double brackets [[ ]] enclose letters or words deleted by the scribe himself, and only apply to **A**. If no *suppl.* is added, completions were done by me, otherwise the sigla of the source is given. Footnotes are inserted at the point where the reading differs from the variants and conjectures. If only one word or number differs, the single bracket is omitted.

IV.2 Chapter 48: Di più ragioni mischiate lequale si fanno per l'algibra

[*f*.178^v]

IV.2.1 [RAA301]

(E) Truovami uno numero che lla metà d'esso[6] numero multipricato per se medesimo faccia quattro[7] cotanti che quello numero.[8]

(S) Farai così: pogniamo che quello numero fosse una cosa. La metà d'una cosa sie $1/2$ cosa, multiprica[9] in sé fa $1/4$ cienso. Dunque è una cosa iguale a $1/4$ cienso e noi vogliamo che ffa 4 cotanti. Multiprica 4 via $1/4$ fa 16 e parti per una cosa

[6] **BCE** d'esso] di quello

[7] **E** *om.* quattro

[8] **C** *om.* che quello numero, RAA301 is proceeded by one problem which does not use algebra. **F** *om.* RAA301

[9] **CE** *add.* una mezza cosa

ne viene 16. E diremo che quello numero sia 16. Ed è fatta.[10]

(T) E se ttu la volessi provare piglia el mezzo di 16 che è 8 e multipricа 8 via 8 fa 64. Dunque 64 venne è 4 cotanti di 16. E per questa regola si truovano di più forti <ragioni>.

IV.2.2 [RAA302]

(E) Truovami uno numero che'l 1/4 el 1/5 di quello numero giunti insieme et multipricati per loro medisimi sia 2 cotanti che quello numero medisimo.[11]

(S) Farai così: pogniamo che quello numero fosse una cosa. Ora di 1/4 e 1/5 di una cosa sie 9/20 di cosa. Ora multiprica 9/20 di cosa in sé[12] fa 81/400 di cienso. Adunque 81/400 di cienso[13] sono iguali a una cosa. E tu vuoi due cotanti recha assano per 400 e dirai[14] 400 via 81</400>[15] fano <81>[16] e 400 via 2 fa 800. Parti 800 per 81 ne viene 9 71/81 e tanto è quello numero. Cioè 9 71/81. Ed è fatta.[17]

(T) E se ttu la voglia provare prendi el 1/4 di 9 71/81 el quale è 2 38/81 e poi prendi il 1/5 simigliantemente di 9 71/81 el quale è 1 79/81. Ora ragiugni insieme questi 2 numeri. Cioè 2 38/81 e 1 79/81 che ssono 4 36/81. Ora multiprica 4 36/81 in se medesimo. Cioè 4 4/9 fanno fanno 19 61/81 [18] e sono due cotanti chel numero che truovasti, cioè 9 71/81. Ed è fatta e provata.[19] [$f.179^r$]

IV.2.3 [RAA303]

(E) Fammi di 10 tali 2 parte che multipricata l'una contro all'altra faccia 16.

(S) Noi sappiamo che è 2 e 8 ma facciamo questo leggieri[20] per intendere le più forti. Farai così: pogniamo che quello numero fosse una cosa. Trai una cosa di 10 resta 10 meno una cosa. Ora multiprica una cosa via una cosa fa una cienso. Dunque 10 meno uno cienso sono iguali a una cosa. E sempre quando ài a dimezzare uno numero si cienci dall'una parte che venga tondo per potere

[10] **C** *add.* esta bene

[11] **F** *om.* RAA302

[12] **BCE** in sé] via 9/20 cose

[13] **E** *om.* Adunque 81/400 di cienso

[14] **B** *om.* 400 e dirai

[15] **E** *suppl.*

[16] **CE** *suppl.*

[17] **C** *add.* esta molto bene, **E** *add.* esta bene

[18] **C** 16 61/81

[19] **CE** Ed è fatta e provata] bene sono 2 cotanti, esta bene e provata

[20] **CE** facciamo questo leggieni] diciamo questo leggiere, F direma questo leggiere

dimezzare.[21] E per conosciere dove il cienso ella cosa per seguita la regola. Dunque giugni uno cienso sopra 10 meno uno cienso e ai 10 cose agiugni uno cienso sopra al numero. Cioè sopra a 16 e ài[22] 16 e uno cienso.[23] Ora dicie che 16 e uno cienso sono iguali a 10 cose. Dimezza le cose. Cioè 10 sono 5 e multiprica 5 in sé fa 25 cose. E di' da' 16 insino a 25 [24] sia 9. Dunque di' che ll'una parte sia 5 più radicie di 9 e l'altra 5 meno radicie di 9. Ed é fatta.

(T) Esse la vuoi provare dirai radicie di 9 sie 3 agiugni sopra 5 sono 8. E poi di' radicie di 9 sie 3 [25] tralo di 5 resta 2 e 2 è l'altra parte. E multiprica 2 via 8 fa 16 apunto e ragiugni 8 e 2 fanno 10. Esta bene.

(R) In questa regola potremmo mostrare più leggieremento ma non farebbe regola della cosa e fa così: dirai il 1/2 di 10 sie 5. Multiprica 5 in sé fa 25. Trai 16 di 25 resta 9. E rispondi e di' l'uno è 5 [26] più radicie di 9 e l'altro e 5 meno radicie di 9.[27]

IV.2.4 [RAA304]

(E) Ancora diremo fammi di 10 tali 2 parte che multiplichata l'una per l'altra faccia 17 23/37.

(S) Farai così: parti 10 per 1/2 ne viene 5. Multiprica 5 via 5 fa 25. E poi di' da' 17 23/37 per insino in 25 sia 7 14/13.[28] Dunque rispondo che ll'una parte farà 5 meno radicie di 7 14/37 e l'altra parte farà 5 più radicie di 7 14/37.[29] Ed è fatta.[30] [*f*.179^v]

IV.2.5 [RAA305]

(E) Fammi di 10 tali 2 parte che multipricata l'una contro all'altra faccia 18 159/169 ni più ni meno.[31]

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[21]**CEF** *om.* uno numero si cienci dall'una parte che venga tondo per potere dimezzare.

[22]**CEF** *om.* 10 meno uno cienso e ài 10 cose agiugni uno cienso sopra al numero. Cioè sopra a 16 e ài

[23]**B** *om.* e ài 16 e uno cienso

[24]**CEF** *add.* cose

[25]**CF** *om.* agiugni sopra 5 sono 8. E poi di' radicie di 9 sie 3 (not in **E**!)

[26]**F** *om.* 5

[27]**C** *add.* Ed è fata et provata esta bene, **E** *add.* Ed è fatta, esta bene, e così fa tutte l'altre, **F** *add.* Ed è fatta, esta bene

[28]sic all copies

[29]**C** *om.* e l'altra parte farà 5 più radicie di 7 14/37

[30]**CE** *add.* esta bene, **F** Ed è fatta] Esta bene.

[31]**F** *om.* RAA305

(S) Farai così: dividi 10 per mezzo che è 5. Multiprica 5 via 5 fa 25. Ora trai di 25 il numero che domandi. Cioè 18 159/169 [32] resta 16 10/169.[33] E diremo che ll'uno[34] numero sia 5 meno radicie di 6 10/169.[35] <E l'altra 5 più dette radicie>.[36] Ed è fatta.

(T) E se lla vuoi provare sappi che radicie di 6 10/169 el quale è 2 6/13. Tralo di 5 resta 2 7/13. Ora poni 2 7/13 [37] sopra 5 e ài 7 6/13 [38] e tanto é l'altro numero. Ora multiprica 2 7/13 via 7 6/13 fanno 18 159/169. Dunque esta bene, cioè 18 159/169. Ed è fatta e provata.[39]

IV.2.6 [RAA306]

(E) Truovami uno numero chel mezzo di quello numero multipricato per se medisimo faccia 2 cotanti chel numero e 15 più.[40]

(S) Farai così: poni che quello numero fosse una cosa. Prendi il $1/2$ d'una cosa che è $1/2$ cosa. Multiprica $1/2$ cosa via $1/2$ cosa fa $1/4$ cienso. Adunque una cosa è iguali a $1/4$ di cienso e 15 più. Fa sempre de ciensi rotti interi. Die percio che $1/4$ di cienso, die 4 vie uno e anche fa 4 via una cosa fa 4 cose e 4 via 15 fanno 60. Ora agiugni 4 sopra a 60 fa 64 e dimezza le cose sono 2. E diremo che quello numero sia radicie di 64 e più 2. Cioè il dimezzamento delle cose e questo numero sie 10.

(T) Percio se lla vuoi provare dirai il $1/2$ di 10 sie 5 e multiprica 5 via 5 fa 25, bene. E 25 [41] due cotanti di 10 e più 15. E a questo esempro farai le più forti.[42]

IV.2.7 [RAA307]

(E) Fami di 10 tale 2 parte che'l quadrato dell'una parte multipricato per 2 7/9 faccia 10. Il quadrato vuole dire il quadrato di 3 sie 9, il quadrato di 5 sie 25.[43]

(S) Farai così: poni che quello numero sia una cosa. Il quadrato d'una cosa sie uno

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[32] **B** *om.* cioè 18 159/169

[33] **E** 6 10/169

[34] **C** *om.* cioè 18 159/1 resta 16 10/169. E diremo che ll'uno

[35] **C** 6 10/169] 10/169

[36] **B** *suppl.*

[37] **BE** 2 6/13

[38] **C** 6 7/13

[39] **BC** *om.* ed è fatta e provata, **B** *add.* ragungnigli insieme fanno 10

[40] **B** *om.* RAA306

[41] **C** 15

[42] **C** add. ragione esta bene, **E** *add.* ragione è fatta e così fa tutte l'altre, **F** *add.* ragione è fatta

[43] **BF** *om.* RAA307

cienso. <Perciò che se io multipricero una cosa via una cosa sie uno cienso.>[44] Segui quello che ài detto di sopra nella proposta. Multiprica cienso via 2 7/9 fa 2 7/9. Ora è iguali 2 7/9 a una cosa. E noi vogliamo 10. Parti 10 per 2 7/9. Recha assano [f.180^r] e ài 25 e 90. Parti 90 per 25 ne viene 3 3/5. E diremo che'l quadrato di 3 3/5 sia l'una parte. Cioè il quadrato della radicie di 3 3/5 sia l'una parte. El quadrato del compimento per insino in 10 sia l'altra parte, cioè 6 2/5. Ma 3 3/5 è gli numero che noi domandiamo.

(T) E se ttu la vuoi provare multiprica 3 3/5 via 2 7/9 [45] fanno 10. Ed è bene fatta.

(R) E nota che nnoi potremmo rispondere sanza dire e lle dicie e de tutto uno[46] ma dovremo rispondere il quadrato di 3 3/5 della sua radicie in percio che la radicie di 3 3/5 multiprica in sé rende il suo quadrato 3 3/5. Sicché l'una parte è il quadrato di 3 3/5 e l'altra parte è la radicie di 6 2/5.[47]

IV.2.8 [RAA308]

(E) Ancora diro fammi di 10 tali 2 parte che partito l'una per l'altra ne venga 5.[48]

(S) Fari così: poni che ll'una parte sia una cosa e l'altra sia 10 meno una cosa. Ora diremo partire 10 meno una cosa per una cosa e debene venire 5. Dobbiamo multipricare 5 via una cosa fa 5 cosa. Abbiamo che 5 cose sono iguali a 10 meno una cosa. Ora da' una cosa a ciascuna parte e abbiamo che 6 cose sono iguali a 10 per numero. Diremo partire 10 per 6 nne viene 1 2/3 e tanto vale la cosa. Di' che noi ponemmo che ll'una parte fosse una cosa. Adunque farà 1 2/3 l'una parte e l'altra parte sia insino in 10. Cioè per 8 1/3 sicché a partire 8 1/3 per 1 2/3 ne vienne 5.

IV.2.9 [R02]

Quando i ciensi sono iguali al numero dobbiamo partire ne' ciensi e quello che nne vienne farà radicie e tanto varrà la cosa.[49]

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[44] **CE** *suppl.*

[45] **CE** 27/9

[46] **E** *add.* modo

[47] **CE** add. Ed è fatta

[48] **B** *om.* RAA308

[49] **E** *om.* sono iguali al numero dobbiamo partire ne ciensi e quello che nne vienne farà radicie e tanto varrà la cosa

IV.2.10 [RAA309]

(E) Truovami uno numero che trattone el 1/4 el 1/5 el rimanente multipricato per se medesimo faccia quello medesimo numero.[50]

(S) Farai così: poni [$f.180^v$] che quello numero fosse una cosa. Piglia il 1/4 el 1/5 d'una cosa sono 9/20. Tralo d'una cosa resta 11/20 di cosa. Ora multipricata 11/20 di cose fa per se medesimo <fa>[51] 121/400 di cienso. E così sono iguali a una cosa. Dobbiamo partire una cosa per 121/400 di cienso che nne viene 3 37/400. E quello numero fa 3 37/400.

IV.2.11 [R04]

Quando e ciensi e lle cose sono iguali al numero. Dobbiamo partire ne' ciensi e poi dimezare lo cose e multipricare per se medesimo e porre sopra al numero e farra radicie di quello e meno el dimezzamento delle cose e tanto varrà la cosa.[52]

IV.2.12 [RAA310]

(E) Truovami uno numero che trattone il 1/3 el 1/4 lo rimanente multipricato per se medesimo faccia 20.[53]

(S) Farai così: pogniamo che quello numero fosse una cosa. Trane il 1/3 el 1/4 resta 5/12 di cosa. Multiprica per se medesimo fa 25/144 di cienso. Abbiamo che 25/144 di cienso sono[54] iguali a 20 per numero. Dobbiamo partire 20 per 25/144 di cienso che nne viene 115 1/5. Abbiamo che radicie di 115 1/5 vale le cosa. E noi ponemmo che quello numero fosse una cosa. Sicché farà quello numero[55] radicie di 115 1/5.

IV.2.13 [R03]

Quando <gli>[56] ciensi sono iguali alle cose. Dobbiamo partire le cose per gli ciensi e quello che nne vienne sia numero e tanto varrà la cosa.[57]

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[50] **B** faccia quello medesimo numero] faccia quello medesimo numero faccia quello medesimo numero

[51] **CF** *suppl.* (not in **E**!)

[52] **C** *add.* esta bene, **EF** *add.* esta molto bene

[53] **F** *om.* RAA310

[54] **CEF** *om.* Abbiamo che 25/144 di cienso sono

[55] **B** *om.* fosse una cosa sicché farà quello numero

[56] **CBE** *suppl.*

[57] **C** *add.* Ed è fatta, esta bene

IV.2.14 [RAA311]

(E) Uno presta a uno altro £ 400.[58] E quando vienne in capo di 2 anni gliene rende £ 480 di mi. Aquanti ₰ gli venne prestata la lira il mese a ffare a capo d'anno.[59]

(S) Fari così: poni che lla £ fosse prestata el mese a una cosa, che nne viene l'anno una lira 12 cose. Ora per le 12 cosa piglia lo 1/20 di 400 che ssono 20 cose. Abbiamo 400 £. e 20 cose che ssono 20 cose[60] e uno cienso. Agiugnilo sopra a 400 e 20 cose fanno 400 £ e 40 cose e uno cienso che ssono $[f.181^r]$ iguali a 480 £. Ora trai 400 di 480 resta 80 £. Abbiamo che 40 cose sono iguali a 80 £.[61] Dimezza le cose sono 20. Multiprica per se medesimo fanno 400. Poni sopra 80 fanno 480. Abbiamo che ffu prestata la £ el mese a radicie di 480 meno 20 per numero. E così farai le simiglianti.[62]

IV.2.15 [R05]

Quando i ciensi e <gli>[63] numeri sono iguali alle cose. Dobbiamo partire ne' ciensi e poi doveremo dimezzare le cose e poi multipricare per se medesimo e cavare lo numero e quello che rimane sie radicie di quello e più lo dimezzamente delle cose vale la cosa. O vero lo dimezzamento delle cose e meno lo dimezzamento del numero.[64]

IV.2.16 [RAA312]

(E) Eglie uno huomo che va in due viaggi, al primo viaggio guadagnia 6 fio. d. e al secondo viaggio guadagnia a quella medesimo ragione, et da sezzo si truova 27 fio. d. Dimmi con quanti ₰ simosse.[65]

(S) Farai così: pogniamo che ssi movesse chun una cosa. Al primo viaggio guadagnia 6 fio. d. Eccho una cosa e 6 fio. Ora di': una cosa fa io una cosa e 6 fio. d'oro che faro io al secondo viaggio d'una cosa e 6 acciò che nne venga 27? Debbi multipricare una cosa e 6 fio. d. per lo medesimo fanno uno cienso e 12 cose e 36 fio. d'oro. Ora debbi partire uno cienso e 12 cose e 36 fio. d'oro per una cosa dene venire 27 fio. d'oro. Sempre dèi lo partitore multipricare e quello

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[58]**C** *om.* 400 quando vienne in capo di 2 anni gliene rende.

[59]**F** *om.* RAA311

[60]**E** *om.* Abbiamo 400 £ e 20 cose che ssono 20 cose

[61]**C** *add.* innanzi

[62]**C** e così farai le simiglianti] ed è fatta, esta bene

[63]**BCE** *suppl.*

[64]**L** meno lo dimezzamento del numero] meno la radice di quelle, The mistake stems from α as it is repeated in all copies.

[65]**BF** *om.* RAA312, **CE** no solution given

che nne viene farà iguali a quello che ssi de partire. Sicché debbi multipricare una cosa via 27 fio. d'oro fanno 27 cose. Abbiamo che 27 cose sono iguali a uno cienso e 12 cose e 36 per numero. Ora trai 12 cose di ciascuna parte. Abbiamo che 15 cose sono iguali a uno [$f.181^v$] cienso e 36 per numero. Ora dimezza le cose, sono 7 1/2. Multipricare per se medesimo fanno 56 1/4. Trane 36 resta 20 1/4 e con 7 1/2 e radicie di 20 1/4 si mosse o vuoi dire con 12 o vuoi[66] con 3 per numero.

IV.2.17 [R06]

Quando le cose e 'l numero sono iguali a' ciensi. Dobbiamo partire ne' ciensi e poi dimezzare le cose e multipricare per se medesimo e porre sopra al numero e farà radicie di quello e più lo dimezzamento delle cose e quello vale ogni cosa.[67]

IV.2.18 [RAA313]

(E) Ancora diro truovami uno numero che postrovi suso 30 faccia altrettanto quanto multipricato per se medesimo.[68]

(S) Farai così: pogniamo che quello numero fosse una cosa. Multiprica in se medesimo. Fa uno cienso. Ora poni 30 [69] sopra una cosa e ài 30 e una cosa. E abbiamo che una cosa e 30 sono iguali a uno cienso. Dimezza le cose che sono 1/2 cosa. Multiprica per se medesimo fa 1/4. Ponlo sopra a 30 fa 30 e 1/4. Abbiamo che quello numero fu 1/2 e radicie di 30 1/4, cioè 6. Ed è fatta.

IV.2.19 [RAA314]

(E) Fammi di 10 tali 2 parte che multipricata l'una per l'altra faccia 20.[70]

(S) Farai così: pogniamo che ll'una parte fosse una cosa e l'altra insino in 10 sia 10 meno una cosa. Ora multiprica una cosa via 10 meno una cosa, fanno 10 cose meno uno cienso. Abbiamo che 10 cose meno uno cienso[71] sono iguali a 20 per numero. Ora da'[72] uno cienso da ciascuna parte e abbiamo che 10 cose sono iguali a uno cienso e a 20. Dimezza le cose sono 5. Multiprica per se medesimo fa 25. Trane 20 resta 5. Abbiamo che ll'una parte sia 5 radicie di 5 e l'altra parte e per insino in 10, cioè 5 meno radicie di 5.

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[66]**C** *add.* dire

[67]**C** *add.* esta bene

[68]**B** *om.* RAA313

[69]**F** *om.* 30

[70]**F** *om.* RAA314

[71]**C** *om.* Abbiamo che 10 cose meno uno cienso

[72]**CE** *om.* da'

IV.2.20 [RAA315]

(E) Fammi di 10 tali 2 parte che multipricata [$f.182^r$] l'una per l'altra e agiunto insieme faccia 60 né più né meno.[73]

(S) Farai così: poni che ll'una parte sia una cosa e l'altra conviene che ssia 10 meno una cosa. Multiprica una cosa per sé fa uno cienso. E di' 10 meno una cosa multipricata per se medesimo fa 100 meno uno cienso. E 20 cose e uno cienso ch'era per la prima parte. Ecco 100 meno 2 ciensi meno 20 cose che ssono iguali a 60.[74] Ora da' 20 cose a ciascuna parte e trai 60 di ciascuna parte e abbiamo che 40 e 2 cienso sono iguali a 20 cose. Parti ogni cosa per 2 che sono gli ciensi. E abbiamo che 20 e uno cienso sono iguali a 20 cose.[75] Ora dimezza 10 che è 5. Multiprica per se medesimo che ffa 25. Trane 20 resta 5. Abbiamo che ll'una parte è 5 meno radicie di 5 [76] e l'altra è 5 più radicie di 5. Ed è fatta.[77]

IV.2.21 [RAA316]

(E) Ancora dirò fammi di 10 tali 2 parte che multipricata ciascuna per se medesimo faccia 6 più l'una parte che l'altra.[78]

(S) Farai così: pogniamo che ll'una parte fosse una cosa e l'altra parte sia per insino in 10, cioè 10 meno una cosa. Multiprica una cosa in sé fa uno cienso e multiprica 10 meno una cosa in se medesimo. Fa 100 e uno cienso meno 20 cose. Trane uno cienso resta 100 e 20 cose che ssono iguali a 6.[79] Ora da' 20 cose a ciascuna parte e trai 6 [80] di ciascuna parte. E ài che 20 cose sono iguali a 94.[81] <Parti 94 per 20 che nne viene 4 7/10. Abbiamo che ll'una parte è 4 7/10 e l'altra insino in 10, cioè 5 3/10. Ed è fatta.>[82]

(T) Ora se lla vuoi provare, multiprica ciascuna parte per sé come dicie la domandata ragione e se fa 6, l'una più che l'altra. {Istara bene. Tu di' che truovi

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[73]**B** *om.* RAA315

[74]**L** Ecco 100 meno 2 ciensi meno 20 cose che ssono iguali a 60] Eccho 100 e 2 ciensi meno 20 cose che ssono iguali a 60

[75]**L** E abbiamo che 20 e uno cienso sono iguali a 20 cose] abiamo che 20 e uno censo sono iguali a 10

[76]**CF** *om.* meno radicie di 5 (not in **E**!)

[77]**C** ed è fatta] esta bene, **E** *om.* Ed è fatta

[78]**B** *om.* RAA316

[79]**CEF** Trane uno cienso resta 100 e 20 cose che ssono iguali a 6] Trane uno cienso meno 20 cose che ssono iguali a 6

[80]**C** *add.* cose

[81]**C** 49

[82]**CEF** *suppl.*, **EF** Parti 94 per 20 che] perciò

che l'una parte è 5 7/10 e l'altra insino in 10 [f.182^v] cioè 10 meno una cosa. Multiprica una cosa in sé fa uno cienso e multiprica 10 meno una cosa in se medesimo fa 100 e uno cienso meno 20 cose. Trane uno cienso resta 100 e 20 cose che ssono iguali a 6. Ora da' 20 cose a ciascuna parte e trai 6 di ciascuna parte. E ai che 20 cose sono iguali a 95. Parti 95 per 20 ne viene 4 7/10. Abbiamo che ll'una parte è 4 7/10 e l'altra insino in 10, cioè 5 3/10. Ed è fatta.}[83]

{Ora se lla vuoi provare, multiprica 4 7/10 in se medesimo come dicie la domandata ragione e se fa 6, l'una più che l'altra.} Istara bene. Tu di' che truovi che l'una parte è 4 7/10 e l'altra 5 3/10.[84] Ora multiprica 4 7/10 in se medesimo fa 22 9/100 e multiprica 5 3/10 [85] in se medesimo fa 28 9/100.[86] Ed è fatta che ella fa 6 più che ll'altra come tu adomandi.[87]

IV.2.22 [RAA317]

(E) Fammi di 10, 2 parti che partito la maggiore per la minore ne venga 7.

(S) Farai così: agiugni sempre uno sopra 7 fa 8. Dividi 10 per 8 ne vienne 1 $1/4$. E diremo che ll'una parte sia 1 $1/4$ e l'altra parte l'avanzo per insino in 10 cioè 8 $3/3$. Ed è fatta.[88]

(T) E se ttu la vuoi provare parti 8 $3/3$ per 1 $1/4$ che nne viene 7. Ed è provata. Esta bene.

IV.2.23 [RAA318]

(E) Fammi di 12 due parti che partito la maggiore per la minore ne venga 59.[89]

(S) Farai così: agiugni uno sopra a 59 fa 60. Parti 12 per 60 ne viene 1/5.[90] E diremo che ll'una sia 1/5 e l'altra insino a 12, cioè 11 4/5. Ed è fatta.

(T) E se la vuoi provare parti 11 4/5 per 1/5 che nne viene 59 come domandi. Esta bene.[91]

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[83] The confusion seems to have originated in α as we have here two different readings. This part is missing in **CEF** and repeats much of the problem solution. At several instances 4 7/10 and 5 7/10 are mixed up. L has no test.

[84] **F** 4 3/10

[85] **F** 4 3/10

[86] **F** 28 9/10

[87] **CEF** *om.* come tu adomandi

[88] **B** fatta] provata, **C** *add.* esta bene

[89] **F** *om.* RAA318

[90] **C** 2/5

[91] **E** *add.* Ed è provata.

IV.2.24 [RAA319]

(E) Ancora diremo fammi di 10 due parti che partita la maggiore per la minore e la minore per la maggiore <e partimenti agiunti insieme>[92] ne venga 4 1/4.[93]

(S) Questo fo per mostrare la regola delle più forti ma bene. Sappiamo che ll'una è 8 e l'altra è 2. [*f*.183^r]
Farai così: poni 2 sopra a 4 1/4 fa 6 1/4 e multiprica 10 via 10 fa 100. E parti per 6 1/4 che nne viene 16. E multiprica il 1/2 di 10 che è 5 in se medesimo, fa 25. Ed di' da' 16 per insino a 25 sia 9. Rispondi che ll'una parte sia 5 meno radicie di 9 [94] e l'altra sia 5 più radicie di 9. Ed è fatta. E così fa di tutti.[95]

(T) E sse la vuoi provare farai così: tu di' che l'una parte è 5 meno radicie di 9. Dunque è l'una parte 2 appunto[96] in percio che lla radicie di 9 <sie 3 e tu di' che 5 meno radicie di 9>.[97] Dunque viene a esse 5 meno 3 che resta 2 e 2 è l'una parte. Ora per l'altra parte tu di' che ll[98] è 5 più radicie di 9. Dunque viene a esse e 8 in percio che lla radicie di 9 sie 3 agiunta a 5 fa 8.[99] Ora tu di' che partita l'una per l'altra e agiunti insieme apartimenti faccino 4 1/4. Parti 8 che l'una parte ella maggiore per 2 ne viene 4. Ora serba 4 e parti 2 per 8 che è la maggiore che nne viene 1/4 agiugni al 4 che serbasti e ài 4 1/4 come adomandi.[100] <Esta bene.>[101]

IV.2.25 [RAA320]

(E) Ancora diremo fammi di 10 tali 2 parti che partita la maggiore nella minore ella minore nella magiore mi venga 7.[102]

(S) Farai così: agiugni 2 sopra a 7 sono 9. Multiprica 10 in sé fa 100. Parti 100

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[92] **A** *marg.* in different hand

[93] **E** 4 1/4] 1/4

[94] **F** *add.* e a esse 5 meno

[95] **CE** e così fa di tutti] esta bene

[96] **B** *om.* in percio che lla radicie di 9. Dunque viene a esse 5 meno 3 che resta 2 e 2 è l'una parte. Ora per l'altra parte tu di' che lle 5 più radicie di 9. Dunque viene a esse e 8 in percio che lla radicie di 9 sie 3 agiunta a 5 fa 8, *add.* l'altra parte è 5 piu radicie di 9 ch'è 8.

[97] **CEF** *suppl.*

[98] **E** che ll] che l'una parte cioè l'altra (not in **F**!)

[99] **B** Dunque viene a esse 5 meno 3 che resta 2 e 2 è l'una parte. Ora per l'altra parte tu di che lle 5 più radicie di 9. Dunque viene a esse e 8 in percio che lla radicie di 9 sie 3 agiunta a 5 fa 8] ch'è 8

[100] **C** *om.* come adomandi

[101] **BCF** *suppl.*

[102] **C** has a blank, **F** *om.* RAA320

per 9 ne vienne 11 1/9.[103] E multiprica il ½ di 10 in sé fa 25. E poi di' da' 11 1/9 insino a 25 sia 13 8/9 e diremo che ll'una parte sia 5 e radicie di 13 8/9 e l'altra parte sia 5 meno radicie di 13 8/9.

IV.2.26 [RAA321]

(E) Fammi di 10, 2 parte che partito l'una nell'altra ne venga 10 né più né meno.[104]

(S) Fa così: poni <uno>[105] sopra a 10. E ài 11 parti 10 per 11 [106] ne viene 10/11. Trai 10/11 di 10, resta 9 1/11.[107] E l'una parte sie 9 e 1/11 e l'altra parte è 10/11.

(T) E se la vuoi provare parti l'una nell'altra e se ttu vuoi che faccia 10. Esta bene e così fa le simili.[108] [f.183^v]

[103] **B** *add.* ~~per insino di 25~~

[104] **F** *om.* RAA321

[105] **CE** *suppl.*

[106] **C** *om.* parti 10 per 11

[107] **C** 8 1/11

[108] **B** e così fa le simili] e così fa le tutti l'altri, **C** esta molto bene, **E** e così fa tutte, esta bene

V English translation

V.1 About the translation

There is no tradition of translating abbaco texts. In fact, the first translation of a complete abbaco treatise is published only very recently by Høyrup [11]. Even translations of partial texts and fragments are quite rare. We have some by Warren van Egmond [26] [28] and a review article by Franci which includes some translated problems. Hence we are in a situation where there is no agreement on how to translate abbaco algebra. Van Egmond "opted for a literal rendition of the text, in order to retain as much of the original style as possible" [26] (165). However, several modern terms sneak into his translation, 'a fare capo d'anno' becomes 'at compound interest', 'agiugnere' and 'porre sopra' are translated as 'to add' and 'trane' as 'subtract'. Høyrup is more consequent in his very literal translation and we follow him in that. We study abbaco texts to learn about the concepts and operations specific to the practice of fourteenth and fifteenth-century algebra. We should therefore be very careful in the terms we use for rendering this practice in modern English. For example, though we are dealing with algebra the word 'equation' does not appear in any of the algebra chapters. With the exception of Maestro Dardi's *adequatione*, the term was hardly used at all within abbaco texts. The reason for this is simple: abbaco masters did not operate on equations but on polynomials. Therefore, the concept of an equation, as it emerged in Europe by the end of the sixteenth century, is absent from abbaco algebra. We previously coined the term 'co-equal polynomials' as a suitable substitute for the modern term 'equation' [13]. Less obvious may be the terms for abbaco operations which we too easily take for granted as modern-day operations: adding, dividing, subtracting, etc. Høyrup notices the subtle distinction between 'partire per' and 'partire in' in the formulation of the rules of algebra in early abbaco treatises [10]. Not only is this a valuable pointer to determine dependencies between the earliest texts its etymological origin "points to a source in time or space where the distinction was still semantically alive" [10] (note 31). It is only by making these subtle distinctions that we are able to see the operations as different from our modern conceptions and that we can determine their evolution in time. For this reason do we follow Høyrup's very literal translations. For example '*resto per insino in 10*' is not translated as "the rest subtracted from 10" but literally as "the rest until 10". For operations on polynomials, we avoided the use of 'add' and 'subtract' in favor of 'join' and 'detract'. However, we made one concession to the modern reader in order to facilitate the understanding of problems. We opted to use the modern symbols x and x^2 for *cosa* and *cienso* and their plurals *cose* and *ciense*. Consider them, together with the formulas as meta-descriptors for the words they designate. When *cosa* is used in the sense of "rules of the *cosa*" we translated this is as "the rules of algebra".

V.2 Chapter 48: More diverse computations which are done by algebra

[f.130^v]

V.2.1 [RAA301]

Find me a number so that half of that number multiplied in itself makes four times as much as this number.

$$\left(\frac{x}{2}\right)^2 = 4x \tag{6}$$

Do as such: let us posit that this number was one x. The half of one x is $\frac{1}{2}x$, multiply it in itself makes $\frac{1}{4}x^2$. Hence is one x equal to $\frac{1}{4}x^2$ and we want it makes 4 times as many. Multiply 4 times $\frac{1}{4}$ it makes 16 and divide by one x and 16 results. And we shall say that this number is 16. And it is done.

And in case you want to verify this. Seize half of 16 which is 8 and multiply 8 times 8 it makes 64. Hence 64 results [which] is 4 times as many as 16. And by this rule you can find more difficult computations.

V.2.2 [RAA302]

Find me a number so that 1/4 and 1/5 of this number joined together and multiplied by themselves is 2 as much as this same number.

$$\left(\frac{1}{4}x + \frac{1}{5}x\right)^2 = 2x \tag{7}$$

Do as such: let us posit that this number was one x. Now say: $\frac{1}{4}$ and $\frac{1}{5}$ of one x is $\frac{9}{20}$ of one x. Now multiply $\frac{9}{20}$ of one x in itself it makes $\frac{81}{400}$ of x^2. Therefore $\frac{81}{400}$ of x^2 are equal to one x. And you want twice as many, bring it together by 400 and you will say 400 times 81 [109] it makes 81 together and 400 times 2 it makes 800. Divide 800 by 81 and $9\frac{71}{81}$ results and so much is this number. That is $9\frac{71}{81}$. And it is done.

And if you want to verify take one $\frac{1}{4}$ of $9\frac{71}{81}$ which one is $2\frac{38}{81}$ and next take the $\frac{1}{5}$ in the same way of $9\frac{71}{81}$ which one is $1\frac{79}{81}$. Now join together those 2 numbers. That is $2\frac{38}{81}$ and $1\frac{79}{81}$ which is $4\frac{36}{81}$. Now multiply $4\frac{36}{81}$ in itself. That is $4\frac{4}{9}$ makes $19\frac{61}{81}$ and is twice as much as the number that you found, that is $9\frac{71}{81}$. And it is done and verified. [f.179^r]

V.2.3 [RAA303]

Make me 2 parts of 10 so that multiplying the one against the other makes 16.

$$\begin{cases} a + b = 10 \\ ab = 16 \end{cases} \tag{8}$$

[109] should be $\frac{81}{400}$

We know that it is 2 and 8 but we do this swiftly so that it may be understood better. Do as such: let us posit that this number was one x. Detract one x from 10 is left 10 less one x. Now multiply one x times one x it makes one x^2. Hence 10 less one x^2 are equal to one x.[110] And always when you want to halve a number of x^2 make one part even so that what results from it you will be able to halve. And to know what is the x^2 of that x follow the rule. Hence join one x^2 above 10 [111] less one x^2 and you get $10x$ joined one x^2 above the number. That is above a 16 and you get 16 and one x^2. Now say that 16 and one x^2 are equal to $10x$. Halve the x. That is 10, is 5, and multiply 5 in itself it makes $25x$. And say from 16 until 25 is 9. Hence say that one part is 5 more the root of 9 and the other 5 less the root of 9. And it is done.

And if you want to verify this, you will say the root of 9 is 3 joined to it 5, it makes 8. And next say the root of 9 is 3 detract it from 5 is left 2 and 2 is the other part. And multiply 2 times 8 it makes 16 exactly and joined 8 and 2 it makes 10. And it goes well.

We should show this rule in a more simple way but it would not make a rule of algebra, and do as such: you will say half of 10 is 5. Multiply 5 in itself it makes 25. Detract 16 of 25 is left 9. And respond that the one is 5 more than the root of 9 and the other and 5 less the root of 9.

V.2.4 [RAA304]

Again we shall say make me 2 parts of 10 so that multiplying the one by the other it makes $17\frac{23}{37}$.

$$\begin{cases} a+b=10 \\ ab=17\frac{23}{37} \end{cases} \tag{9}$$

Do as such: divide 10 by half and 5 results. Multiply 5 times 5 it makes 25. And next say from $17\frac{23}{37}$ until 25 is $7\frac{14}{13}$. Hence I respond that one part makes 5 less the root of $7\frac{14}{13}$ and the other part makes 5 more the root of $7\frac{14}{13}$. And it is done. [$f.179^v$]

V.2.5 [RAA305]

Make me 2 parts of 10 so that multiplying the one against the other it makes $18\frac{159}{169}$, neither more nor less.

$$\begin{cases} a+b=10 \\ ab=18\frac{159}{169} \end{cases} \tag{10}$$

Do as such: divide 10 by half that is 5. Multiply 5 times 5 it makes 25. Now detract of 25 the number that is asked. That is $18\frac{159}{169}$ is left $6\frac{10}{169}$. And we shall say that one number is 5 less the root of $6\frac{10}{169}$. And the other 5 more this root. And it is done.

[110] Should be: "Hence 10 less one x^2 are equal to 16".

[111] Should be $10x$.

And if you want to verify this know what the root is of $6\frac{10}{169}$, which is $2\frac{6}{13}$. Detract it from 5 is left $2\frac{7}{13}$. Now posit $2\frac{6}{13}$ above 5 and you get $7\frac{6}{13}$ and so much is the other number. Now multiply $2\frac{7}{13}$ times $7\frac{6}{13}$ which makes $18\frac{159}{169}$. Hence it goes well, that is $18\frac{159}{169}$. And it is done and verified.

V.2.6 [RAA306]

Find me a number so that half this number multiplied in itself makes 2 as many as that number and 15 more.

$$\left(\frac{x}{2}\right)^2 = 2x + 15 \tag{11}$$

Do as such: posit that this number was one x. Take $1/2$ of one x that is $1/2\ x$. Multiply $1/2\ x$ times $1/2\ x$ it makes $1/4\ x^2$. Therefore one x is equal to $1/4$ of x^2 and 15 more. Always make of the broken x^2 integer. Say: therefore that $1/4$ of x^2, say 4 times one, and also it makes 4 times one x it makes 4 x and 4 times 15 make 60. Now join 4 above 60 it makes 64 and halve the x is 2. And we shall say that this number is the root of 64 and 2 more. That is the halving of the x and this number is 10.

Therefore if you want to verify this you will say the $1/2$ of 10 is 5 and multiply 5 times 5 it makes 25, is good. And 25 is twice 10 and 15 more. And with this example you will do the more difficult.

V.2.7 [RAA307]

Make me 2 parts of 10 so that the square of one part multiplied by 2 7/9 makes 10. The square means that the square of 3 is 9, the square of 5 is 25.

$$\begin{cases} x + y = 10 \\ 2\frac{7}{9}x^2 = 10 \end{cases} \tag{12}$$

Do as such: posit what this number is one x. The square of one x is one x^2. Therefore if I multiply one x times one x this is one x^2. Follow that what was told above as in the proposal. Multiply x^2 times $2\frac{7}{9}$ it makes $2\frac{7}{9}[x^2]$. Now are equal $2\frac{7}{9}$ and one x. And we want 10. Divide 10 by $2\frac{7}{9}$. Bring it together [$f.180^r$] and you get 25 and 90. Divide 90 by 25 and $3\frac{3}{5}$ results. And we shall say that the square of $3\frac{3}{5}$ is the one part. That is the square of the root of $3\frac{3}{5}$ is the one part. The square of what is missing to reach 10 is the other part, that is $6\frac{2}{5}$. But $3\frac{3}{5}$ is the number that we asked.

And if you want to verify this multiply $3\frac{3}{5}$ times $2\frac{7}{9}$ it makes 10. And it is well done.

And note that we could respond without saying those things said, and of all but one be able to respond with the square of $3\frac{3}{5}$ of its root because the root of $3\frac{3}{5}$ multiplied in itself returns its own square $3\frac{3}{5}$. So that the one part is the square of $3\frac{3}{5}$ and the other part is the root of $6\frac{2}{5}$.

V.2.8 [RAA308]

Again I shall say make me 2 parts of 10 such that dividing the one by the other 5 results from it.

$$\begin{cases} a + b = 10 \\ \frac{a}{b} = 5 \end{cases} \tag{13}$$

Do as such: posit that one part is one x and the other is 10 less one x. Now we shall say divide 10 less one x by one x and 5 has to result from it. We shall multiply 5 times one x it makes $5x$. We have that $5x$ are equal to 10 less one x. Now give one x to each part and we have that $6x$ are equal to 10 by number. We shall say divide 10 by 6 and $1\frac{2}{3}$ results from it and so much is x worth. Say that we posited that one part was one x. Therefore it shall make $1\frac{2}{3}$ for the one part and the other part is the rest until 10. That is by $8\frac{1}{3}$ so that dividing $8\frac{2}{3}$ by $1\frac{2}{3}$ results in 5.

V.2.9 [Rule 2]

When the x^2 are equal to the number we shall divide the x^2 and from that which results we take the root, and so much be worth the x.[112]

$$\begin{aligned} ax^2 = dx \rightarrow x^2 &= bx \\ x &= \sqrt{b} \ (R02) \end{aligned}$$

V.2.10 [RAA309]

Find me a number when detracted the 1/4 and the 1/5 the remainder multiplied by itself makes that same number.

$$\left(x - \frac{1}{4}x - \frac{1}{5}x\right)^2 = x \tag{14}$$

Do as such: posit [$f.180^v$] what this number was one x. Seize the $\frac{1}{4}$ and the $\frac{1}{5}$ of one x which are $\frac{9}{20}$. Detract from one x is left $\frac{11}{20}$ of x. Now multiply $\frac{11}{20}$ of x by itself it makes $\frac{121}{400}$ of x^2. And such are equal to one x. We shall divide one x by $\frac{121}{400}$ of x^2 from which results $3\frac{37}{400}$. And this number becomes $\frac{37}{400}$. And it is done.

V.2.11 [Rule 4]

When the x^2 and the x are equal to the number. We shall divide the x^2 and then halve the x and multiply by itself and carry it above the number and it will be the root of this less the halving of the x, so much be worth the x.

$$\begin{aligned} ax^2 + dx = e \rightarrow \quad x^2 + bx &= c \\ x &= \sqrt{(\tfrac{b}{2})^2 + a} - \tfrac{b}{2} \ (R04) \end{aligned}$$

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[112]We follow the numbering of the rules of algebra as they appear in **L** and **V**. The first rule for $bx = c$ is omitted.

V.2.12 [RAA310]

Find me a number when detracted the 1/3 and the 1/4 the remainder multiplied by itself makes 20.

$$\left(x - \frac{1}{3}x - \frac{1}{4}x\right) = 20 \tag{15}$$

Do as such: let us posit that this number was one x. Detract $\frac{1}{3}$ and $\frac{1}{4}$ and is left $\frac{5}{12}$ of a x. Multiply by itself it makes $\frac{25}{144}$ of x^2. We have that $\frac{25}{144}$ of x^2 are equal to 20 by number. We shall divide 20 by $\frac{25}{144}$ of x^2 from which results $115\frac{1}{5}$. We have that root of $115\frac{1}{5}$ is worth the x. And we posited that this number was one x so that makes this number the root of $115\frac{1}{5}$.

V.2.13 [Rule 3]

When the x^2 are equal to the x. We shall divide the x by the x^2 and that which results from it is number and so much be worth the x.

$$ax^2 = bx \rightarrow x = \tfrac{b}{a} \ (R03)$$

V.2.14 [RAA311]

Someone lent to another 400 lire and when it came to the end of two years he gave back to him 480 lire. How much denaro he came to lend in lira per month to make [up at] the end of the year.

Do as such: posit that the lira lent per month are one x, from which results $12x$ lira per year. Now from the $12x$ seize 1/20 of 400 which are $20x$.[113] We have 400 lire and $20x$ which are $20x$ and one x^2. Join above 400 and $20x$ make 400 lire and $40x$ and one x^2 which are [$f.181^r$] equal to 480 lire. Now detract 400 of 480 is left 80 lire. We have that $40x$ are equal to 80 lira.[114] Halve the x which are 20. Multiply by itself makes 400. Posit 80 above makes 480. We have that the lira was lent per month the root of 480 less 20 by number. And so you shall do in similar [cases].

V.2.15 [Rule 5]

When the x^2 and the numbers are equal to the x. We shall divide the x^2 and next we shall halve the x and next multiply by itself and remove the number and what remains be the root of this and the halving of the x more is worth the x. Or indeed

[113] Omitted here is the specification of the ratio of lira to denaro which we find in **J** $f.38^v$:"$12x$ of denaro are the twentieth of one libra, so that the libra is worth the 1/20 of one libra".

[114] Missing a part in all the copies. Should be: "We have that $40x$ and one x^2 are equal to 80 lire".

the halving of the x less the halving the number.[115]

$$ax^2 + e = dx \rightarrow x^2 + c = bx \; (R05)$$

$$x = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

V.2.16 [RAA312]

There is a man that goes on two business trips. On the first trip he wins 6 gold *fiorini* and on the second trip he wins in the same rate and at last he found himself with 27 gold *fiorini*. Tell me with how much *fiorini* he set out.

$$\frac{(x+6)^2}{x} = 27 \tag{16}$$

Do as such: let us posit that he set out with one x. On the first trip he wins 6 fio. Here one x and 6 fio. Now say: from one x I make one x and 6 gold fio., what do I make on the second trip to that purpose of one x and 6 so that 27 results from it? You shall multiply one x and 6 fio. by itself to make one x^2 and $12x$ and 36 gold fio. Now you shall divide one x^2 and $12x$ and 36 gold fio. by one x so that 27 gold fio. results. You shall always multiply the divisor and that which results from it to become equal to that what you divide. So that you shall multiply one x times 27 gold fio. to make $27x$. We have that $27x$ are equal to one x^2 and $12x$ and 36 by number. Now detract $12x$ from each part. We have that $15x$ are equal to one [$f.181^v$] x^2 and 36 by number. Now halve the x, which are $7\frac{1}{2}$. Multiply that by itself makes $56\frac{1}{4}$. Detract 36, is left $20\frac{1}{4}$ and with $7\frac{1}{2}$ and the root of $20\frac{1}{4}$ he set out, or you want to say with 12, or you want with 3 by number.

V.2.17 [Rule 6]

When the x [and] the number are equal to x^2. We shall divide the x^2 and next halve the x and multiply by itself and join above the number and make the root of this and increased with the halving of the x and such is worth each x.

$$ax^2 = dx + e \rightarrow x^2 = bx + c \; (R06)$$

$$x = \sqrt{\left(\frac{b}{2}\right)^2 + c} + \frac{b}{2}$$

[115] **L** less the halving the number] less the root of which remains, The mistake stems from the archetype α as it is repeated in all copies.

V.2.18 [RAA313]

Again say: find me a number which carried above 30 makes as much as multiplied by itself.

$$x + 30 = x^2 \tag{17}$$

Do as such: let us posit that this number was one x. Multiply in itself, it makes one x^2. Now posit 30 above one x and you get 30 and one x. And we have that one x and 30 are equal a one x^2. Halve the x which are $\frac{1}{2}x$. Multiply by itself it makes $\frac{1}{4}$. Join above 30 it makes $30\frac{1}{4}$. We have that this number becomes $\frac{1}{2}$ and the root of $30\frac{1}{4}$, that is 6. And it is done.

V.2.19 [RAA314]

Make me of 2 parts of 10 such that the one multiplied by the other makes 20.

$$\begin{cases} a + b = 10 \\ ab = 20 \end{cases} \tag{18}$$

Do as such: let us posit that the one part was one x and the other [the rest] until 10 which is 10 less one x. Now multiply one x times 10 less one x makes $10x$ less one x^2. We have that $10x$ less one x^2 are equal to 20 numbers. Now give one x^2 to each part and we have that $10x$ are equal to one x^2 and to 20. Halve the x which are 5. Multiply by itself it makes 25. Detract 20 is left 5. We have that the one part is 5 and the root of 5 and the other part the rest until 10, that is 5 less the root of 5.

V.2.20 [RAA315]

Make me 2 parts of 10 such that multiplying [$f.182^r$] the one for the other and joined together it makes 60, neither more nor less.

$$\begin{cases} a + b = 10 \\ a^2 + b^2 = 60 \end{cases} \tag{19}$$

Do as such: posit that the one part is one x and it is appropriate that the other is 10 less one x. Multiply one x by itself it makes one x^2. And say 10 less one x multiplied by itself it makes 100 less one x^2 and $20x$ and one x^2 which is the first part. Here is 100 less $2x^2$ less $20x$ which are equal to 60. Now give $20x$ to each part and detract 60 from each part and we have that 40 and $2x^2$ are equal to $20x$. Divide each x by 2 which are the x^2 and we have that 20 and one x^2 are equal a $20x$. Now halve 10 which is 5. Multiply it by itself which makes 25. Detract 20 and 5 is left. We have that the one part is 5 less the root of 5 and the other is 5 more than the root of 5. And it is done.

V.2.21 [RAA316]

Again say: make me 2 parts of 10 so that each part multiplied by itself, the one makes 6 more than the other part.

$$\begin{cases} a+b=10 \\ a^2-b^2=6 \end{cases} \tag{20}$$

Do as such: let us posit that the one part was one x and the other part is [the rest] until 10, that is 10 less one x. Multiply one x in itself it makes one x^2 and multiply 10 less one x in itself. It makes 100 and one x^2 less 20 x. Detract one x^2 and what is left is 100 and $20x$ which are equal to 6. Now give $20x$ to each part and detract 6 from each part. And you get that $20x$ are equal a 94. Divide 94 per 20 from which results $4\frac{7}{10}$. We have that the one part is $4\frac{7}{10}$ and the other [the rest] until 10, that is $5\frac{3}{10}$. And it is done.

Now if you want to verify this, multiply each part by itself as the computation requires and it makes 6, the one more than the other. And it shall be good. You say that you found that the one part is $4\frac{7}{10}$ and the other $5\frac{3}{10}$. Now multiply $4\frac{7}{10}$ in itself it makes $22\frac{9}{100}$ and multiply $5\frac{3}{10}$ in itself it makes $28\frac{9}{100}$. And it is done that it makes 6 more than the other as you demand.

V.2.22 [RAA317]

Make me 2 parts of 10 so that dividing the larger by the smaller 7 results from it.

$$\begin{cases} a+b=12 \\ \frac{a}{b}=7 \end{cases} \tag{21}$$

Do as such: always join one above 7, it makes 8. Divide 10 by 8 and $1\frac{1}{4}$ results from it. And we shall say that one part is $1\frac{1}{4}$ and the other part what is left over until 10 that is $8\frac{3}{4}$.[116] And it is done.

$$\begin{cases} x+y=a \\ \frac{y}{x}=b \end{cases} \rightarrow x=\frac{a}{b+1}, y=a-\frac{a}{b+1} \ (R07)$$

And if you want to verify, divide $8\frac{3}{4}$ by $1\frac{1}{4}$ from which results 7. And it is verified. And it goes well.

V.2.23 [RAA318]

Make me two parts of 12 so that dividing the larger by the smaller 59 results from it.

$$\begin{cases} a+b=12 \\ \frac{a}{b}=59 \end{cases} \tag{22}$$

[116] All copies have here $8\frac{3}{3}$.

Do as such: join one above 59 and it makes 60. Divide 12 by 60 and $\frac{1}{5}$ results from it. And we shall say the one is $\frac{1}{5}$ and the other is [the rest] until 12, that is $11\frac{4}{5}$. And it is done.

$$\begin{cases} x + y = a \\ \frac{y}{x} = b \end{cases} \rightarrow x = \frac{a}{b+1}, y = a - \frac{a}{b+1} \; (R08)$$

And if you want to verify, divide $11\frac{4}{5}$ by $\frac{1}{5}$ from which results 59 as you ask. And it goes well.

V.2.24 [RAA319]

Again we shall say make me of 10 two parts so that dividing the larger by the smaller and the smaller by the larger and the partitions joined together $4\frac{1}{4}$ results from it.

$$\begin{cases} a + b = 10 \\ \frac{a}{b} + \frac{b}{a} = 4\frac{1}{4} \end{cases} \tag{23}$$

The following I do to show the stronger rule but in a good way. Let us know that one is 8 and the other is 2. [$f.183^r$]

Do as such: posit 2 above $4\frac{1}{4}$ which makes $6\frac{1}{4}$ and multiply 10 times 10 it makes 100. And divide by $6\frac{1}{4}$ from which results 16. And multiply half of 10, which is 5 by itself, it makes 25. And say from 16 until 25 is 9. Respond that one part is 5 less the root of 9 and the other is 5 more the root of 9. And it is done. And as such you can make all.

$$\begin{cases} x + y = a \\ \frac{x}{y} + \frac{y}{x} = b \end{cases} \rightarrow x, y = \frac{a}{2} \pm \sqrt{(\frac{a}{2})^2 - \frac{a^2}{b+2}} \; (R09)$$

And if you want to verify this, do as such: you say that the one part is 5 less the root of 9. Hence the one part is 2, right so because the root of 9 is 3 and you said 5 less the root of 9. Hence it becomes 5 less 3 and what is left is 2 and 2 is the one part. Now for the other part you say 5 more than the root of 9. Hence it becomes 8, because the root of 9 is 3 joined with 5 it makes 8. Now you say that the one divided by the other and the partitions joined together it makes $4\frac{1}{4}$. Divide 8, which is the one and the larger part, by 2 and 4 results. Now hold 4 and divide 2 by 8 which is the larger from which results $\frac{1}{4}$ joined with the 4 that you hold and you get $4\frac{1}{4}$ as was asked for. And it goes well.

V.2.25 [RAA320]

Again we shall say make me 2 parts of 10 so that dividing the larger in the smaller and the smaller in the larger 7 results to me.

$$\begin{cases} a + b = 10 \\ \frac{a}{b} + \frac{b}{a} = 7 \end{cases} \tag{24}$$

Do as such: put 2 above 7 it becomes 9. Multiply 10 in itself it makes 100. Divide 100 by 9 and $11\frac{1}{9}$ results from it. And multiply half of 10 in itself it makes 25. And next say: from 11 1/9 until 25 is $13\frac{8}{9}$, and we shall say that one part is 5 and the root of $13\frac{8}{9}$ and the other part is 5 less the root of $13\frac{8}{9}$.

$$\begin{cases} x+y=a \\ \frac{x}{y}+\frac{y}{x}=b \end{cases} \rightarrow x,y=\frac{a}{2}\pm\sqrt{(\frac{a}{2})^2-\frac{a^2}{b+2}}\ (R10)$$

V.2.26 [RAA321]

Make me 2 parts of 10, so that dividing the one in the other 10 results from it, neither more nor less.

$$\begin{cases} a+b=10 \\ \frac{a}{b}=10 \end{cases} \qquad (25)$$

Do as such: posit one above 10 and you get 11. Divide 10 by 11 and $\frac{10}{11}$ results from it. Detract $\frac{10}{11}$ from 10, is left $9\frac{1}{11}$. And the one part is 9 and $\frac{1}{11}$ and the other part is $\frac{10}{11}$.

$$\begin{cases} x+y=a \\ \frac{x}{y}=a \end{cases} \rightarrow x=a-\frac{a}{a+1}, y=\frac{a}{a+1}\ (R11)$$

And if you want to verify, divide the one in the other and as you want it makes 10. And it goes well and thus you shall do similar [problems]. [$f.183^v$]

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VI Appendix: Index of problems

Nr	Text	A	B	C	D	E	F
301	Truovami uno numero che lla metà	⊠	⊠	⊠	□	⊠	□
302	Truovami uno numero che'l $^1/_4$ el 1/5	⊠	⊠	⊠	□	⊠	□
303	Fammi di 10 tali 2 parte .. faccia 16	⊠	⊠	⊠	□	⊠	⊠
304	Fammi di 10 tali 2 parte .. faccia 17 23/37	⊠	⊠	⊠	□	⊠	⊠
305	Fammi di 10 tali 2 parte .. faccia 18 159/169	⊠	⊠	⊠	□	⊠	□
306	Truovami uno numero .. e 15 piu	⊠	□	⊠	□	⊠	⊠
307	Fammi di 10 tali 2 parte chel quadrato .. faccia 10	⊠	□	⊠	□	⊠	□
308	Fammi di 10 tali 2 parte .. ne venga 5	⊠	□	⊠	□	⊠	⊠
309	Truovami uno numero che trattone el $^1/_4$ el 1/5	⊠	⊠	⊠	□	⊠	⊠
310	Truovami uno numero che trattone il 1/3 el $^1/_4$	⊠	⊠	⊠	□	⊠	□
311	Uno presta a uno altro £400	⊠	⊠	⊠	□	⊠	□
312	Eglie uno huomo che va .. si truova 27 sid	⊠	□	⊠	□	⊠	□
313	Truovami uno numero che postrovi suso 30	⊠	□	⊠	□	⊠	⊠
314	Fammi di 10 tali 2 parte chel quadrato .. faccia 10	⊠	⊠	⊠	□	⊠	□
315	Fammi di 10 tali 2 parte .. faccia 60	⊠	□	⊠	□	⊠	⊠
316	Fammi di 10 tali 2 parte .. faccia 6	⊠	□	⊠	□	⊠	⊠
317	Fammi di 10 2 parte .. ne venga 7	⊠	⊠	⊠	□	⊠	⊠
318	Fammi di 12 due parti .. ne venga 59	⊠	⊠	⊠	□	⊠	□
319	Fammi di 10 due parti .. ne venga 4 $^1/_4$	⊠	⊠	⊠	□	⊠	⊠
320	Fammi di 10 2 parte .. mi venga 7	⊠	⊠	⊠	□	⊠	□
321	Fammi di 10 2 parte .. ne venga 10	⊠	⊠	⊠	□	⊠	□

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