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Algebraic partitioning problems from Luca Pacioli's Perugia manuscript (Vat. Lat. 3129)

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I Introduction

I.1 The manuscript

Given the importance attributed to Luca Pacioli's *Summa* for over five centuries, it can only come as a surprise that so little attention has been paid to his earlier book on arithmetic and algebra, the until recently unpublished manuscript Vat.Lat. 3129. The book bears no title although several have been attributed. van Egmond [51] names it *Opera di mathematica*, others have used the title *Tractatus mathematicus ad discipulos perusinos* [10] [9] [11], or *Trattato d'arithmetica* [31]. The first description of this work dates from 1879 by Baldassare Boncompagni in his *Bulletino* [7]. He included a transcription of the first three folio's of the manuscript. Emmett Taylor wrote a comprehensive biography of Pacioli based on what Jayawardene has described as "secondary sources and, where these were found wanting, on the author's imagination" ([31], 22). The biography includes a reliable description of the context in which the manuscript was produced ([47], 127-129). From the text itself it can be established that the manuscript was commenced on 12 December 1477 and completed on 29 April 1478 ($f.2^r$). The writing took place at the beginning of a three-year period where Pacioli was teaching at the University of Perugia (1477-1480). The book is dedicated to his students at this location. Taylor includes a reproduction and transcription of one page.[1] A detailed paleographic description and provenance is included in Warren van Egmond's *Catalogue* ([51], 219). With the fifth centennial of the publication of Pacioli's *Summa*, a more detailed description has become available from a conference in Sansepolcro in 1994. Giovanna Derenzini [15] adds to van Egmond's description with a complete foliation, a list of missing pages, the watermarks, a rough table of contents and some comments on its contents including the comparison of one problem with a corresponding one in the *Summa*. In 1996, Giuseppe Calzoni and Gianfranco Cavazzoni published a complete transcription of the manuscript [9], a considerable feat which went unnoticed by most scholars

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[1]Taken by Taylor from Boncompagni's article without acknowledgment, insert between pp. 132-3.

in the field.[2] The chapter titled *Tariffa mercantesca* has received some attention recently. Johanna Postma established that this section is identical with text on the bill of exchange written in 1475 by Kotruljević [40]. This may shed more light on the history of treatises on double-entry bookkeeping as the one in the *Summa*.

However, more than five centuries after the completion of this book, we are still mostly ignorant about its mathematical contents. A few scholars have referred to some passages, Jens Høyrup on the symbolism in a book review [26] and an article on abbaco algebra [27]. Jacques Sesiano included some of Pacioli's algebraic problems in his overview of negative solutions in Medieval algebra [42]. As we shall see below, he was not exhaustive with regards to this text. I know of no other studies on its mathematical contents beyond the fragments mentioned. Why have scholars shied away from a study of this book, while its importance has been recognized? Jayawardene compiled a list of eight priorities on the study of Pacioli and his writings and puts in the second place "publication of the *Trattato d'arithmetica* in the Vatican Library" and in third place "comparison of Vat. Lat. 3129 with the *Summa*" [31].

The lack of studies may be partly explained by the absence of this long awaited transcription. Pacioli's characteristic autograph made a study of its contents quite challenging. It has been described by van Egmond as "single hand, a very irregular, difficult, personalized cursive".[3] Figure 1 gives an overview of the hand with some typical abbreviations and ligatures. Recently we have a useful guideline for interpretation through a major discovery. It has been known some time that both Pacioli and Cardano wrote a book on chess, and both works were considered to be lost ([34], 417n). However, in 2006 the manuscript by Pacioli's hand was discovered in the private library of the Fondazione Palazzo Coronini Cronberg of Gorizia. It has since been published [36]. The editor Bartoli pays a lot of attention to Pacioli's autograph and includes useful guidelines for its reading [5]. Through the availability of a full transcription it can be hoped that more studies on this interesting work will come to light.

In this paper we present a transcription, an English translation and a mathematical commentary on one chapter of the book for which Calzoni gave the title *Divisioni e partimenti de numeri*. It runs from $f.229^r$ to $f.256^v$ which amounts to one fifteenth part of the whole text. The chapter is one of eight sections in the book in which algebra is used to solve problems. Other algebra sections deal with

[2]Before I started a study of Pacioli's text I asked some scholars in the field about ongoing work on this subject. None seemed to know about the transcription by Calzoni. I became aware of the published transcription only after my own transcription and in the process of concluding this article.

[3]Derenzini believes there is a second hand at work, suggested by a passage on $f.2^r$ which says that some *carta* at the end are written by "a certain of our disciples". However, for the chapter which is the subject of our study, everything has been written by Pacioli himself, including marginal comments.

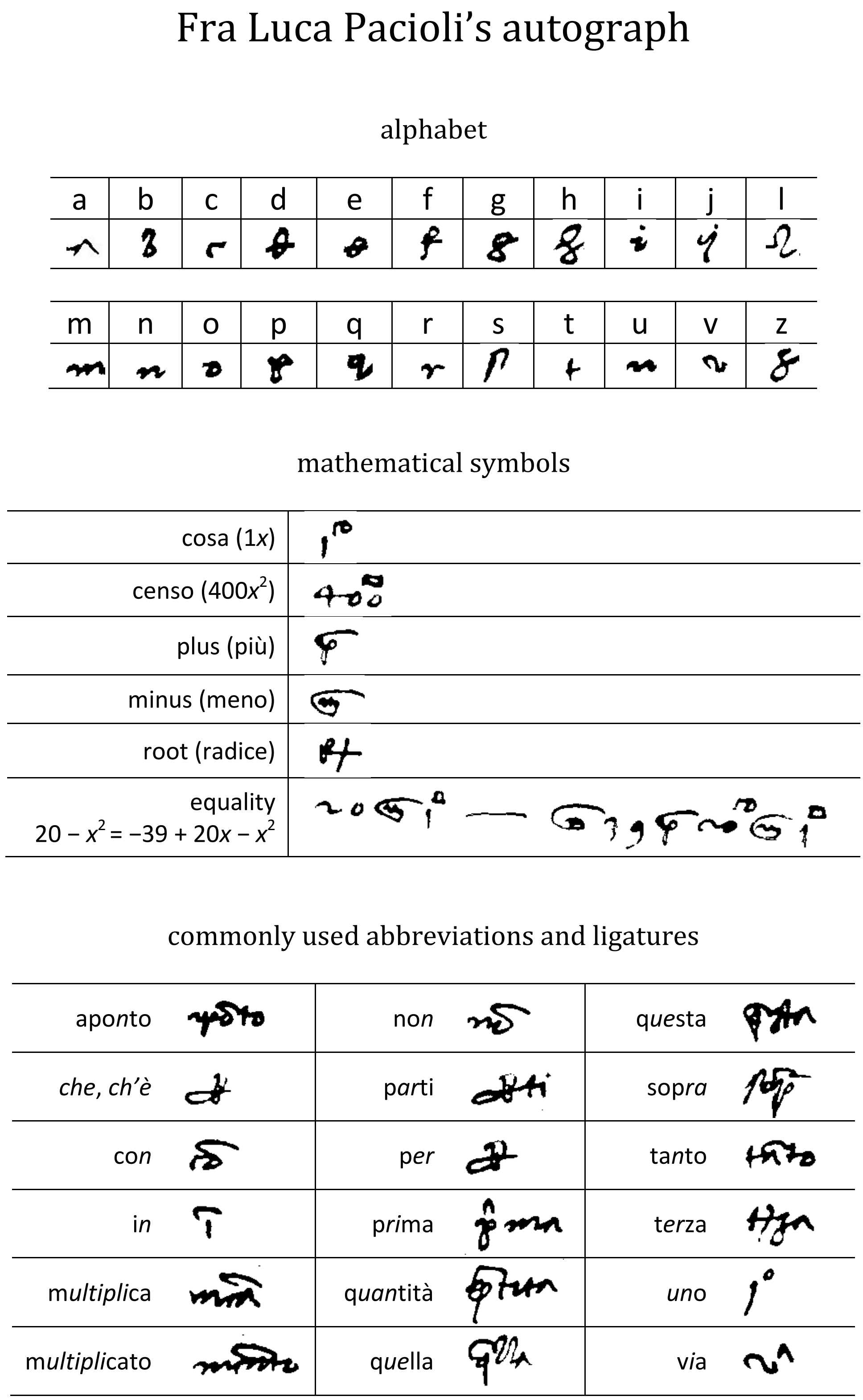

Figure 1: Pacioli's handwriting (digitized at 400 dpi from the Vat.Lat.3129 microfilm, with shadows manually cleaned and processed to 2-level b/w, all taken from f.236v-237r)

linear problems, finding a number, and progressions. Algebra is occasionally used in the sections on bartering, exchange and interest. The formal treatment of algebra, which is announced in the introduction, is unfortunately missing (25 pages). What is remaining in the chapter on algebra is a mixture of algebraic problems.

I.2 Pacioli's symbolism

Pacioli uses a consistent system of symbolism throughout the text (see Figure 1). He adopts the system for formal fractions as introduced in fourteenth-century abbaco treatises from Gherardi and Dardi. For the unknown and the square of the unknown he uses the superscript notation as first found in the anonymous *Alchune ragione* of 1424 (Vat. Lat. 10488) as in '100 e 1^{co} m̄ē $20^{\square}$' for $100+1x-20x^2$. Jens Høyrup sees a connection of this with the Maghreb notation as it developed around the same time [29]. Pacioli uses ligatures based on p̄ and m̄ for plus and minus and a long horizontal line for an equation sign. Thus with his notation for $20-x^2=-39+20x-x^2$ as in Figure 1, he comes closest to a symbolic equation than any other abbaco text before him. We still cannot interpret this as a modern equation and the equation sign here should be understood as the equality of two polynomials.

Why then did Pacioli abandon this consistent symbolism for the *Summa*? When algebra is introduced in distinction 8 of the *Summa*, *cosa*, *censo*, *cubo* and the higher order powers are written in full words or they are abbreviated to *co*, *ce* and *cu*. We can only assume that algebraic scratchpad calculation using symbolism were not very well understood outside the circle of abbaco masters and was not found appropriate for publication. We see the same happening with Regiomontanus's *De triangulis omnimodis* published by Petreus in 1533 where all of the symbolic scratchpad calculations are removed from the original manuscript (Moscou MS. 541).

I.3 Pacioli's sources

I.3.1 Determining sources in abbaco treatises

Before discussing possible sources of Pacioli's problems it might be useful to explicate our methodology. While traditional methods of critical text comparison naturally do apply to abbaco texts, this tradition features some special characteristics. Firstly, abbaco masters generally never cite their sources, unless they are explicitly compiling texts in a compendium as did Maestro Benedetto.[4] Secondly, it was customary to borrow problems and its solutions from each others manuscripts without any acknowledgment. The text could be copied literally or the values of the problems could be changed and the solution text paraphrased. It is precisely this interdependence

[4]In his *Trattato di practicha d'arismetrica* Maestro Benedetto explicitly mentions the authors Antonio de' Mazzinghi [2], Biagio [39], Giovanni di Bartolo [37] and Fibonacci [1], from which he uses considerable fragments.

of texts which makes of abbaco mathematics such a coherent tradition. As we have shown, one typical family of manuscripts contains problems which depend on the earliest extant abbaco texts and are copied in Cannaci's treatise of the end of the fifteenth century, thus covering two full centuries [24].

The most convincing argument for a dependence of problem texts is of course evidence of a literal copy. Enrico Giusti mostly depends on such evidence for his review of sources for the algebra part of Piero della Francesca's *Trattato d'abaco* (Ash 359*, c. 1480) [20]. He shows how several fragments in Piero's text literally appear in Maestro Gilio's *Questioni d'algebra* (operations on surds, c. 1378), Mariotto di Giovanni Guiducci's *Libro d'Arismetricha* (c. 1465), Matteo di Nicolò Cerretani's *Libro d'Abbaco* (1461). Giusti also accounts for contextual elements, such as abbaco masters operating within the same city but rarely ventures beyond that.

If we limit comparisons to literal text fragments only we will not get very far finding the sources of Pacioli's algebra problems. While it is established that Pacioli literally copied text for his *Summa*, he does not seem to have done so for the algebra problems. He did borrow problems and solution methods from earlier sources but formulated the solution text in his own wordings. In these cases we cannot depend on text comparisons only. The connecting elements are the algebra problems, their structure, their values and their practical context. But even then, abbaco masters sometimes conceal their source of problems by subtle changes in the structure, the values and the context of the problem. In a (unpublished) comprehensive study of problems on numbers in geometric progression (*GP* or continuous proportion) from 1380 to 1600, we clearly discern the mechanisms of this process. The factors that allow us to point at specific influences are the systematic correspondences of problems with the same values and same structures. Especially when these values or structures are different from the mainstream problems does their correspondence becomes significant. For the problems on *GP* a typical example is to find three numbers given their sum and an additional condition, in modern symbolism:

$$\begin{aligned} &\tfrac{x}{y} = \tfrac{y}{z} \\ &x + y + z = a \end{aligned}$$

Many additional conditions which make the problem determinate can be considered: the sum of squares, products of three or two, .. However, some of these conditions are so far fetched that an influence can be assumed if they appear in two treatises. For example the condition $x(y+z)+y(x+z)+z(x+y)=b$ was introduced by Maestro Antonio ([2], problem 1). We also find it in Pacioli's *Summa* with different values. If this was an isolated problem we could still be in doubt, but Pacioli systematically uses Antonio's problems in this way and therefore it is safe to assume an influence. Also, some values of a problem are so rare that the coincidence that two people devise these same values becomes very unlikely.

Apart from literal copying and problem borrowing there are other indicators which may reveal an influence between abbaco texts. Jens Høyrup pioneered the method of close analysis of linguistic and syntactic features in the study of mathematical sources for Old-Babylonian tablets. This allowed him to formulate a radical new geometrical interpretation of Babylonian algebra [25]. An additional benefit of his approach was the possibility to group and classify problem texts in chronological or geographical dimensions, which provided new information for many unprovenanced tablets. General arithmetical operations such as addition, subtraction, multiplication and division were subdivided by Høyrup in distinctive syllabic and logographic versions, each covering a different shade of meaning of the arithmetical operation in modern sense. The subtraction operation for example is subdivided into eight different linguistic classes.[5]. Some of them refer to concrete operations such as 'cutting off', being part of the 'cut-and-paste' geometrical model, others refer to the comparison of concrete magnitudes. In some cases the terminological peculiarities of these operations could be mapped on groups of tablets from different periods or different areas in Mesopotamia. In a recent publication [28] Høyrup explicitly draws the analogy between terminological subtilities of Old-Babylonian and abbaco mathematics. The distinction in 'partire in' and 'partire per' or 'divide in' and 'divide by' in early abbaco algebra is such an example and points at "traces of historical diffusion". The difference between the two lived on in writings from the Iberian peninsula as late as the end of fifteenth century, which may point to possible influences from that region. Linguistic peculiarities have now become part of the instrumentarium to trace influences between problem texts.

I.3.2 Dividing 10 into two parts

Dividing a number into two parts is the most common problem in abbaco algebra. It is the prototypical problem to illustrate the quadratic equation. It appears as such in the earliest Arabic treatise on algebra. The first problem that al-Khwārizmī treats is to divide 10 into two parts with their product being 21. Pacioli has the same problem with the product being 20 [PPM1202]. The history of these problems goes back much further. Although a continuity cannot be demonstrated, the resemblance with 'rectangle problems' or *igūm-igibūm* problems' from Babylonian algebra is remarkable ([25] 55). More complex forms [such as PPM1232], in which the sum of the division of the larger by the smaller and the smaller by the larger is given, already appear as al-Khwārizmī's fourth problem. The difference lies in the more systematic treatment of variations on the problem and the occurrence of radical solutions.

Given the vast number of abbaco treatises dealing with these kind of problems a precise source for Pacioli's treatment would be difficult to point out. However, in this

[5] For a good overview and a discussion of its methodological consequences, see his recent publication [28].

case there are two treatises on algebra which bear close resemblance with the chapter of Pacioli's manuscript: the *Questioni d'algebra* by Maestro Gilio, (L.IX.28 of the Biblioteca Communale in Siena, transcription by Franci [16]) and the anonymous Fond. prin. II.V.152 of the Biblioteca Nazionale Centrale in Florence [18]. Both these sources have strong links with Antonio de' Mazzinghi, as we shall demonstrate.

The problems section of Gilio's text contains only six partitioning problems of the type we are discussing (problems 1, 2, 9, 10, 11 and 12) but all of them appear in Pacioli's chapter (problems 1, 2 38, 26, 5 respectively) and three of them have the same value. Problem 12 by Gilio asks to divide 10 into two parts (say a and b) such that $\frac{a}{b}(a-b)=40$. This one has no direct equivalent but Pacioli does include several variations on this theme. We assume that Pacioli had to his disposal a text by Maestro Antonio also known to Gilio and which has since been lost. According to Franci ([16], iv) it is quasi certain that Maestro Gilio frequented Antonio's abbaco school in Florence and was familiar with his writings. The second claim can be easily established from a comparison of problems on numbers in geometric progression in his *Questioni* with the *Fioretti* by Antonio. Also Pacioli used this text by Antonio for the *Summa* (see below).

The Florence manuscript Fond. prin. II.V.152 also corresponds well with the partitioning problems in this chapter. The watermarks are of the last decade of the fourteenth century and several problems refer to dates from 1390 to 1393. The text is anonymous but there are strong indications that at least the section with problems is from Maestro Antonio de' Mazzinghi:

1. The text is written in Florence where Antonio lived and where his abbaco school was located.
2. Antonio died around 1390 [48] and could have written this shortly before his death. It is typical to refer to the current and following years in abbaco problems.
3. The text uses some typical expressions which are identical with the *Fioretti* such as "noi faremo posizione che l'una parte fosse una chosa", "sarà di bisogno di raguagliare le parti", etc.
4. Both texts frequently refer to the *quadrati* of numbers and unknowns. Such Latinism is otherwise seldom used in abbaco texts.
5. Both texts use fractions of polynomials within the text, eg. $\frac{5+x}{5-x}$. We know of only one other text who uses the same format, *Chasi exemplari alla regola dell'algibra* by Maestro Biagio [39] which has otherwise less affinity with either of these two texts.
6. Both texts have problems solved by the so-called *regula aequalis positionis*[6]

[6]This name is coined by Cardano in the *Ars magna* ([12] 1968, 192).

in which the two parts are expressed as $5 - x$ and $5 + x$.[7] This is a rather unorthodox way to solve partitioning problems in abbaco algebra.

7. The *Fioretti* is the first abbaco treatise to use the second unknown. The II.V.152 is the first one to use the second unknown for a linear problem ($f.177^r$). Both innovations seem to stem from Antonio.
8. Some uncommon problems in the *Fioretti* also appear in the II.V.152 with same values and the same solution method, though the text is not a copy (e.g. problem numbered 31 in both texts).

We know that Pacioli used other material from Maestro Antonio for the *Summa* (see below) and that he held a high esteem for this abbaco master. It therefore makes sense that he used a text by Antonio with the same material as in the II.V.152. Pacioli did not copy the problems and their solutions literally. He takes some problems and creates his own variations on the theme. Neither does he employs the *regula aequalis positionis* for any of the problems. Twelve of the partitioning problems correspond directly with the II.V.152, three of them use the same values. The most conspicuous of correspondences is the problem by Maestro Antonio which ask to divide 10 into two parts (a and b) such that $\sqrt{97 - a^2} + \sqrt{100 - b^2} = 17$ (problem 31 in the *Fioretti* and the II.V.52). I know of no other treatise with this problem. Pacioli uses the condition $\sqrt{100 - a^2} = 2 + \sqrt{120 - b^2}$. It would be rather unlikely that Pacioli came up with structurally the same problem by himself.

I.3.3 Dividing 100 by a number

The other main section of the manuscript is shorter and contains fourteen problems asking to divide a given number by an unknown quantity, given certain conditions. In most of these problems the number to be divided is 100. A typical example is [PPM851]:

$$\frac{100}{a} + \frac{100}{a+3} = 20 \tag{1}$$

Such problems are not so common in abbaco algebra and thus can facilitate the identification of possible sources of Pacioli. The problem most probably descended from early Arabic sources. Abū Kāmil's *Algebra* contains several such problems set in a context in which a sum of money is divided amongst a number of men ([43], 370; [32], 104):

> One says that 50 was divided among a certain number of men. Increase the number of men by 3 and divide 50 among all of them. For every one of the last ones, it comes out to less than what would come out for each of the first ones by $3\frac{3}{4}$.

[7] In the *Fioretti* problems 6, 8, 27, 29a, 30 and 31 in the Arrighi numbering and in the II.V.152 problems 6 to 10, 17, 19, 20, 27, 29 to 31 and 33.

The structure of the problem and the context of a sum of money being divided already appeared in al-Khwārizmī's *Algebra*:[8]

> Instance: You divide one dirhem amongst a certain number of men, which number is thing. Now you add one man more to them, and divide again one dirham amongst them, the quota of each is then one-sixth of a dirhem less than at the first time.

$$\frac{1}{1+x} = \frac{1}{x} - \frac{1}{6} \tag{2}$$

So it appears as if Abū Kāmil expanded on this problem and created several variations of it. Five of the problems from his *Algebra* also appear in the *Liber abbaci* by Fibonacci and in the same order. The problem cited above has different values ([8], 413; [44], 561):

> I divided 60 by a number of men, and each had an amount, and I added two more men, and I divided the 60 by all of them, and there resulted for each $\frac{1}{2}2$ denari less than that which resulted first.

For the number being divided Fibonacci uses the values of 60, 20 and 10 but never 100. In the rare instances where we see 60 being divided in abbaco treatises it concerns an adaption or translation from Fibonacci. The unpublished Lincei Cors. 1875 (c.1340, f. 81v), the unpublished Palat.573 (c.1460, $f.438^v$), Maestro Bendetto's *Trattato di praticha d'arismetrica* (Siena, L.IV.21 f.409v) as well as the unpublished Ottob.lat.3307 (c.1465, $f.312^v$) contains a literal Italian translation of this problem together with the geometrical demonstration. The anonymous *Trattato dell'Alcibra amuchabili* (Ricc.2263)[9] contains three problems of 60 denari being divided by a number of men ([46] 58-60).

The first abbaco text to divide 100 appears in the *Libro di ragioni* by Paolo Gherardi from a transcription by Arrighi (Magl. XI. 87, [4] 101):

> Io porto 100 in una quantità e tengho a mente quello che ne viene prima con quello che ne viene poi e fa 20. Adomando in che fu parto prima.

Clearly, this is a corrupted version of an earlier formulation of the problem. Warren van Egmond reconstructed the problem on basis of a version in a different manuscript

[8]From Rosen's English translation: [41], 63. The Latin text from Cremona's translation is: "Quod si dixerit tibi: 'Divisi dragmam per homines, et provenit eis res. Deinde addidi eis hominem. Et postea divisi dragmam per eos, et provenit eis minus quam ex divisione prima secundum quantitatem sexte dragme unius", [30], 255.

[9]This manuscript is discussed below.

(Ricc. 2252, $f.161^v$, [50] 168):

> Io p[a]rto 100 in una quantità e tengho a mente quello che ne viene [et poi parto in 5 più che la prima volta e poi agiungo insieme quello che ne viene] prima con quello che ne venne poi, e fa 20. Adomando in che fu parto prima?

In modern symbolism this is the type of problem Pacioli deals with:

$$\frac{100}{a} + \frac{100}{a+5} = 20 \tag{3}$$

We find this problem in a very similar wording, as an illustration of the rule to solve the equation $ax^2 = bx + c$, in a series of manuscripts. In all of these occurrences the same expression is used: "to keep in mind" (tieni a mente) the result of the first division. These texts form a thread of copying and appropriation spanning over two centuries. We provide here a chronological overview. The unpublished manuscript Cl.XI.120, $f.11^v$, attributed to Maestro de' Mazzinghi by Rafaella Franci, contains a single problem of this type (originally written around 1380):

> Parti 100 in una quantità e tenni a mente e poi lo parti in 4 più e giunsi insieme con la somma di prima e fece 20. Adomando in quanto lo parti in prima e poi.

$$\frac{100}{a} + \frac{100}{a+4} = 20 \tag{4}$$

Another unpublished manuscript rephrases Gherardi's text (c. 1395, Conv.Soppr. G7.1137, $f.143^r$):

> Io parto 100 in una quantità e tengnio a mente quello che ne vene e poi il parto in più 5 che prima e agiungo quello che vene in prima con quello che ne vene poi e facie 20. Adomando in che e il parto in prima.

We find the problem again in the unpublished Palermo manuscript 2Qq.E.13 (1398, $f.41^r$):

> Io parto 100 in una quantità e tengo a mente quello che ne viene e poi lo parto in più 5 che la prima e poi agiungo quello in prima e questo insieme e fa 20. Adomando in che lo parto prima e poi.

In the *Trattato d'abbaco* by Piero della Francesca we find it twice again ([3] 125, $f.53^v$, 135, $f.60^v$):

> Partime 10 per una quantità et quello che ne vene tieni a mente, poi 10 parti per 5 più che prima; e quello che ne vene, gionto con quello che ne venne prima, faccia 20.

A recent article [45] discusses a sixteenth-century Portugese manuscript in which we find the problem in the same formulation of Paolo Gherardi (Bento Fernandes, 1555, $ff.85^v - 86^r$):

> I divide 100 by a quantity and I memorize what results; and then I divide again by 5 more than the first time, and I add together the result of the first time with the result of the second time, and it makes 20. I ask: by what did I divide, the first time and the second time?

The author expresses her surprise that this problem did not originate in the *Summa*. This shows that Italian abbaco treatises spread to the Iberian peninsula already during the fifteenth century ([45], 190):

> At a time when Pacioli's Summa, the first printed text that includes algebraic methods, was already so diffuse, it is surprising that it turns out not to have been the source of the algebraic material of Bento Fernandes. The comparative study I have carried out between the *Tratado da arte de arismetica* and a number of abacus books from the 14th and the 15th centuries shows that Bento Fernandes's algebra had its origin in abacus manuscripts antedating the *Summa*.

Then we come to a text in which the problem does not appear as an illustration of one of the rules of algebra but as a selection of similar problems around the same theme. The Florence manuscript Ricc.2263 of c.1365 has nine such problems [46]. As this is the only abbaco text that we know about which deals with these problems in a systematic way it is a likely source for Pacioli. Below is a selection of the problems which correspond with the Perugia manuscript.

1. Uno partì 100 in una quantità e poi partì 100 in più 5 che prima e, giunti questi due avenimenti insieme, fecie 20. Vo' sapere in che 100 si partì in prima ed in che si part poscia [ABR013]. Pacioli has four variations of this problem (51, 54, 60 and 63).[10]

$$\frac{100}{a} + \frac{100}{a+5} = 20 \tag{5}$$

2. Uno partì 100 in una quantità e poi partì 100 in più 5 che prima e cioè che nne

[10] In the correspondence of Regiomontanus ([13] 237, 256) we find another variation: "Ego divisi 100 per certum numerum, deinde divisi 100 per eundem divisorem sibi addito 8, et numeros cotientes prime et secunde divisionis aggregavi, et fuerunt in summa 40: queritur quantitas primi divisoris".

venne di chatuno partimento, tratto l'uno dell'altro rimase 10. Vo' sapere in che si partì 100 in prima e in che si partì poscia [ABR014]. Pacioli's problems 50 and 52 are structurally the same but use different values.

$$\frac{100}{a} - \frac{100}{a+5} = 10 \tag{6}$$

3. Uno partì 100 in una quantità e poi partì 100 in più 5 che prima e multipricha l'uno avenimento contro all'altro, fecie 50. Vo' sapere in che 100 si partì in prima e 'n che si partì poscia [ABR016]. This problem is basically the same as Pacioli's number 56.

$$\left(\frac{100}{a}\right)\left(\frac{100}{a+5}\right) = 50 \tag{7}$$

4. Uno partì 100 in una quantità e, quello che nne venne, posto sopra '1 partitore, fecie 30. Vo' sapere in che 100 si partì [ABR017]. This problem corresponds with problem 62 by Pacioli.

$$\frac{100}{a} + a = 30 \tag{8}$$

5. Uno partì 100 in una quantità e poi part 100 in più 5 che prima e poi lo partì in due cotantj che prima e, giunti insieme questi 3 avenimentj, fecie 20. Vo' sapere in che 100 si partì chatuna volta [ABR020]. This problem is very similar to Pacioli's problem 49.

$$\frac{100}{a} + \frac{100}{a+5} + \frac{100}{2a} = 20 \tag{9}$$

I.4 Negative solutions

Jacques Sesiano is one of the few who discussed the mathematical contents of the manuscript in his comprehensive study on negative solutions in Medieval mathematics [42]. He discusses six examples from abbaco treatises, five of which are by Pacioli, three in the Perugia mansucript and two in the *Summa*. Although we now know that there are more such occurrences in abbaco algebra, they are quite rare and are generally not accepted as negative quantities *in se*. The practical context of problems with negative solutions is usually chosen in such a way that they can be interpreted as a debt. However, by considering them as a debt the quantities are forced to become positive and we cannot speak anymore about the acceptance of negative solutions. In our opinion, a real acceptance of negative solutions is offered by Cardano in his *Ars Magna* [21]. Chapter 37 of this book is titled *de regula falsum ponendis*, or the "Rule of Postulating a Negative". Cardano introduces the subject by giving three reasons why it may be useful to postulate a negative.

1. "One can assume a negative". The term "assuming" here refers to the rhetorical structure of algebraic problem solving. Every solution in abacus texts

commences with the assumption that the rhetorical unknown equals some unknown quantity in the problem formulation. Assuming a negative thus means that one can use a negative *cosa*, or $-x$ instead of x, for the unknown quantity. This application is illustrated with three problems.

2. "One seeks the square root of a negative". After a failed geometrical demonstration Cardano solves two problems using imaginary numbers, for the first time in the history of mathematics.
3. "One can seek what is not". With this cryptic formulation, Cardano refers to a negative which is neither of the first category nor of the second. However, his result stems from a wrong manipulation of imaginary numbers.

The first application of hypothesizing a negative provides a path to the acceptance of negative solutions embedded within the rhetoric of abbaco problem solving. By postulating a negative *cosa*, a negative solution should not come as an anomaly. It is a confirmation of a hypothesis and thus should be accepted within the standard practice of algebraic problem solving.

In Pacioli's writings we do not find this rhetorical trick, but Pacioli is quite liberal in his acceptance of negative solutions. In problems 9, 28 and 29 discussed by Sesiano, Pacioli does not resort to an interpretation as debt but literally writes "thus say that the cosa values minus 5" ($f.230^v$). The section on partitioning problems contains four more instances of negative solutions not discussed by Sesiano: problems 26, 30, 48 and 61. The first two are interesting as they both contain a calculation error by Pacioli which are otherwise quite rare. Problem 48 is also listed in the anonymous manuscript Magl. Cl.XI.120 from the the Biblioteca Nazionale Centrale in Florence ($f.11^r$). Franci and Rigatelli have pointed out that this text contains extracts from Antonio de' Mazzinghi's treatises and therefore links again Pacioli with Antonio [17]. However, the problem is solved incorrectly in this source. The condition for the two parts that $3a = \sqrt{8}b$ leads to the equation $3x = \sqrt{8}(10 - x)$. In the text the right hand side is squared but not the left hand side. It seems that solution to problems with negative solutions are more prone to errors than regular problems. Pacioli and Gilio arrive at the correct solution $\sqrt{7200} - 80$ and $90 - \sqrt{7200}$ without mentioning an oddity. Given the link with abbaco masters Antonio and Gilio we can thus say that negative solutions appeared already in fourteenth century treatises without much apprehension. However, remark that they are represented as binomials and not as isolated negative quantities. Also Pacioli presents the negative solutions in problems 28, 29, 30 and 61 as $(10 - 20)$, $(10 - 40)$, $10 - \sqrt{200}$ and $\frac{4}{5} - 3\frac{1}{5}$ respectively.

I.5 Comparison with the Summa

I.5.1 From problem solving to algebraic theory

I have argued elsewhere that the Pacioli's treatment of algebra in the *Summa* is a major departure from the abbaco tradition [22]. Where abbaco treatises are concerned with problem solving only, Pacioli moves towards the development of algebraic theory. He presents algebraic solutions to well-known problems as theorem of algebra. Generalized solutions, or keys as he calls them, become theorems with a general validity as the propositions of Euclid's *Elements*.

One example concerns numbers in geometric progression (*GP*). Pacioli discusses thirty such problems (from the 35) in distinction 6, treaty 6, article 14. They are presented before he treats algebra itself (in distinction 8). Most of these problems correspond with problems from Maestro Antonio's *Trattato di Fioretti* of c.1380 [2], often using the same values. More importantly, the original problem solving methods are followed closely by Pacioli, including one rare instance using two unknowns and one which Maestro Antonio calls "without *cosa*". The theoretical principles on three numbers in *GP* are discussed in two introductory sections preceding the problems. In the section called *De tribus quantitatibus continue proportionalium* (distinction 6, treaty 6, article 12, *f*.88^v).[11] Another section on keys, lists theoretical principles on four numbers in *GP* under the heading *De clavibus seu evidentiis quantitatum continue proportionalium*, (distinction 6, treaty 6, article 11, *f*.88^r). Pacioli does not explain where these principle are derived from. He only gives some numerical examples. However, a close comparison with the *Trattato di Fioretti* shows that several are extracted from Maestro Antonio's solution. Let us look at one example involving three numbers in *GP* with their sum given and an additional condition.

Pacioli's *Summa*	Maestro Antonio's *Trattato*
Make me three parts of 13 in continuous proportion so that the first multiplied with the sum of the other two, the second [multiplied] with the [sum of the] other two, the third [multiplied] with the [the sum of the] other two, and these multiplications added together makes 78.	Make me three parts of 19 in continuous proportion so that the first multiplied with [the sum of] the other two, the second part multiplied with the [sum of the] other two, the third part multiplied with the [the sum of the] other two, and these sums added together makes 228. Asked is what are the parts.

In modern symbolism, using multiple unknowns, the general structure of the

[11] I have used the 1523 edition but the numbering of pages and sections is practically identical with the original.

problem is as follows:

$$\begin{aligned}&\tfrac{x}{y} = \tfrac{y}{z}\\ &x + y + z = a\\ &x(y + z) + y(x + z) + z(x + y) = b\end{aligned}$$

The *Trattato di Fioretti* is the first extant source we could find in which this type problem is treated and Maestro Antonio poses the problem with values $a = 19$ and $b = 228$. Expanding the products and summing the terms gives

$$2xy + 2xz + 2yz = 228,$$

but as $y^2 = xz$ we can write this expression also as

$$2xy + 2y^2 + 2yz = 228 \quad or \quad 2y(x + y + z) = 228$$

Given that the sum of the three terms is 19, dividing 228 by 19 thus gives us the double of the middle part. Therefore the middle part is 6. Antonio then proceeds to find the other terms with the procedure of dividing a number into two extremes such that their product is equal to the square of the middle term. The problem thus reduces to dividing $19 - 6 = 13$ into two parts so that 6 is the middle term. Thus the product of the two parts is the square of the middle term or 36. Given the product and the sum of two numbers their values can be found as the roots of the quadratic equation

$$x^2 + 36 = 13x.$$

Maestro Antonio uses the rule for the two positive roots of the quadratic equation to find the two parts as 4 and 9. [12]

Pacioli poses the problem with values $a = 13$ and $b = 78$ and solves it in essentially the same way. However, the rhetorical structure is quite different. Maestro Antonio performs an algebraic derivation on a particular case. Instead, Pacioli justifies the first part of the solution as an application of a more general principle, defined by a general key:[13]

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[12] The recognition of two positive roots for this type of equations was known from Arabic and even Babylonian algebra, where the two roots correspond with the sides in some rectangle problems. However, during the abbacus tradition it gradually disappears. In [23] we argued that this evolution is invoked by the specific rhetoric of abbacus problem solving. The rhetorical unknown stands for one specific unknown quantity of the problem and can therefore be only that single quantity.

[13] Pacioli 1494, *f*.91^r: "Questa solverai p*er* la 14^a chiave. La q*uella* dice che stu p*ar*tirai la su*m*ma de ditte multiplicatio*n*i, cioè 78 per lo doppio de 13. E q*uella* 13 sira la summa de ditte q*uan*tità ne virra la 2^a parte. Donca parti 78 in 26 ne ve*ne* 3 p. la 2^a parte".

> This can be solved using the fourteenth key. Which says that you have to divide the sum of these multiplications, thus 78, by the double of 13. And this 13 is the sum of these quantities, which will give you the second part. Thus divide 78 by 26 gives 3 for the second part.

The fourteenth key he is referring to, is formulated as follows, some pages earlier: [14]

> On three quantities in continuous proportion, when multiplying each with the sum of the other two and adding these products together. Then divide this by double the sum of these three quantities and this always gives the second quantity.

This particular key is one of several variations on the algebraic derivation of Maestro Antonio, each presented as a general principle. In modern notation:

$$y = \frac{x(y+z) + y(x+z) + z(x+y)}{2(x+y+z)} \quad (Key14)$$

Having determined the value for the middle part, Pacioli continues to solve Antonio's problem in a different way, by means of algebra. The problem reduces to one of dividing 10 into two parts so that 3 is the middle term in *GP*. Using the *cosa* for the smaller term and $(10 - x)$ for the larger, the product of the two is 9, the square of the middle term. He arrives at the quadratic equation with 1 and 9 as its roots. Elsewhere Pacioli writes that this sub problem, finding the two extremes of three numbers in *GP* with the middle term given, can be solved either by algebra or following a theorem of Euclid.[15] Drawing on Euclid, he provides legitimation for the procedure which needs no further explanation or proof.

I.5.2 Evolution in Pacioli's writings

We have shown how Pacioli's mines Antonio's treatise for general principles such as the one we have discussed. He has chosen to present some typical derivations as general rules which are later applied to solve problems in a clear and concise way. Even with the body of evidence against him, we should be careful in accusing Pacioli of plagiarism as has been done so often.[16] At best, we observe here an appropriation of problems and methods. The restructuring of material and the shift in rhetoric is

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[14] Pacioli 1494, $f.89^v$: "De 3 quantità c*on*tinue proportionali che multiplicata ciascuna in l'altre doi e quelli multiplicationi gionti insiemi. E poi questo p*ar*tito nel doppio de la summa de ditte 3 qua*n*tita e sempre l'avenimento sera la 2^a quantità".

[15] Pacioli 1494, $f.89^r$. Pacioli only refers to "the second of Euclid". On other occasions he uses "the fourth of the second of Euclid". This leads us to book 2, prop. 4.

[16] Ever since Giorgio Vasari's encyclopedic biography of painters, sculptors and architects it was suspected that Pacioli based his published work on several manuscripts from the abbacus tradition

in itself an important aspect in the development of sixteenth-century textbooks on algebra. Pacioli raised the testimonies of algebraic problem solving from the abbacus masters to the next level of scientific discourse, the textbook. When composing the *Summa*, Pacioli had almost twenty years of experience in teaching mathematics at universities all over Italy. His restructuring of abbacus problem solving methods is undoubtedly inspired by this teaching experience.[17] By reformulating algebraic derivations of abbacus masters as theorems of algebra, and using Euclid's theorems for algebraic quantities, Pacioli introduces a new style of argumentative reasoning which was absent from abbacus algebra.

Contrasting the Perugia manuscript with the *Summa* we can now point out this evolution within Pacioli's own writings. Most of the partitioning problems within the chapter under discussion appear in some form or another in the last part of distinction 7 of the *Summa*. Again, these are presented before his exposition on algebra and not as problems but as *conclusiones seu evidentie* which have a general validity. It is clear that this list of seventy propositions ($f.106^r$ to $f.111^r$) has been compiled by going through partitioning problems as the ones presented in the Perugia manuscript. However, the emphasis is not on the solution to these problems but on properties which are common to all such partitioning problems. Each proposition is followed by an example in which we recognize many of the problems from our chapter. Interestingly, there are no examples with irrational solutions. Most of the partitioning problems result in the partitioning of 10 into 8 and 2, or 12 into 8 and 4, numerical examples which only have the purpose of demonstrating the validity of the proposition. Problems with radical solutions are

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[49]. These claims have partially been substantiated in relation to the *Geometry*. Gino Loria was the first to show that the *Libellus* by Pacioli is a literal translation of *De corporibus regularibus* of Piero della Francesa. Margaret Daly Davis [14] demonstrated that 27 of the problems on regular bodies in the geometry part of Pacioli's *Summa* are reproduced almost literally from Pierro della Francesco's *Trattato d'abaco*. Ettore Picutti [38] cites the historian Girolamo Mancini [33] who discovered that treatise XI of distinction 9 of the *Summa*, entitled *De scripturis*, is a transcription of a manuscript by Giorgi Charini. Picutti himself has shown that "all the 'geometria' of the *Summa*, from the beginning to page 59^v (119 folios), is the transcription of the first 241 folios of the Codex Palatino 577". He includes a reproduction of one part of $f.51^r$ of the geometry part and the corresponding text from the manuscript to prove his claim. In relation to the algebra contents Franci and Rigatelli [17] further claim that a detailed study of the sources of the *Summa* would yield many surprises. Yet, for the part dealing with algebra, no hard evidence for plagiarism has been given.

[17] Although Pacioli was trained in abbacus mathematics and taught in an abbaco school in Venice, he is often wrongly considered an abbacist (e.g. [6]. In fact, he enjoyed the social status of a well-paid university professor. Between 1477 and 1514, he taught mathematics at the universities of Perugia, Zadar (Croatia), Florence, Pisa, Naples and Rome. Taylor refers to some archival document which list quite substantial fees [47].

often used in abbaco algebra to show off the calculating skills of the abbaco masters. A notable example is problem PPM1260 in which the number of men receiving a sum of money is $9 + \sqrt{101}$ and $11 + \sqrt{101}$. The *Summa* on the other hand is a textbook intended to instruct its audience in more general principles. For that purpose, simple numerical examples suffice.

We could collate the problems from the Perugia manuscript with the propositions of the *Summa* but many of these are quite trivial. Instead we will show the transformation of presentation at work by one example, problem PPM1233. This problems asks to divide ten into two parts such that the sum of ten divided by each part is equal to ten again.

$$\begin{cases} a + b = 10 \\ \frac{10}{a} + \frac{10}{b} = 10 \end{cases} \tag{10}$$

Pacioli solves the problem in the classic way by using x and $(10 - x)$ for the two parts and arrives at a quadratic equation with solution: $5 + \sqrt{15}$ and $5 - \sqrt{15}$. In the *Summa* we find the problem under proposition 23 ($f.108^r$):

> E se una q*ua*ntità sia divisa i*n* doi p*ar*ti quomodocu*mque* chi partira la ditta q*ua*ntità per ciascuna d*i* ditte p*ar*ti: semp*re* li ditti doi avaneime*n*ti sara*n*no ta*n*to agio*n*ti q*ua*nto m*ultipli*cati uno i*n* l'altro. E q*ue*sto se verifica i*n* tutte. Exemplu*m* sia 10 diviso i*n* 2 e 8, poi parti 10 per 2 ne ve*ne* 5 e p*ar*tito per 8 ne ve*ne* $1\frac{1}{4}$ che gio*n*ti insiemi fanno $6\frac{1}{4}$. Dico ch*e* similimente ancora fa*r*ra tanto multiplicati uno in l'altro, cioè 5 via $1\frac{1}{4}$ che aponto fa $6\frac{1}{4}$.

Rather than solving the partitioning problem with some arbitrary values Pacioli now makes an observation which is valid *quomodocumque* (in whatever way) the partitioning is being made. If the number being divided say a, is divided by the parts x and y, the sum of these divisions is equal to the product. In modern symbolism this amounts to stating that if $x + y = a$ then

$$\frac{a}{x} + \frac{a}{y} = \left(\frac{a}{x}\right)\left(\frac{a}{y}\right) \tag{11}$$

This can be easily demonstrated by bringing the equation to the same denominators, but Pacioli does not do the algebra. In distinction 7 he only lists general propositions to which he comes back in his chapter on algebra.

The fifteen years separating the Perugia manuscript from the *Summa* lead to an evolution in Pacioli's approach to writing and teaching about mathematics. His treatment of proportions and partitioning problems is a major departure from the abbaco tradition in which mathematics consist of no more than problems solving. Pacioli can be considered as one of the first executors of the humanist program initiated by Regiomontanus, to provide new foundations to mathematics and in particular to algebra. He was the first to use the solution to algebraic problems as theorems with general validity. He was most likely inspired to do so by the writings of Maestro Antonio de' Mazzinghi, which he gradually discovered from the time of the production of the Perugia manuscript and before completing the *Summa*.

II The transcription

II.1 The Calzoni and Cavazzoni transcription

In retrospect, my transcription may be considered superfluous given the availability of the publication by Calzoni and Cavazzoni [9]. However, it appears that our reading of the manuscript differs in several respects. I did not include the differences in the transcription listed below, but I would like to illustrate differences in interpretation with one example (problem 71):

Current transcription	Calzoni and Cavazzoni (1996)
Fame de 10 2 parti che multiplichata la prima per sé e trato de 100 e la Rx di questo che resta serba, poi multiplicha l'altra in sé e trato de 101 e preso la Rx di ciascuni questi remanimenti gionte insiemi facino 14 a ponto. Dimando le parti. [f.255v]	Fame de 10 doi parti che multiplichata la prima per se stessa e trato de 100, e la Rx di quanto che resta serba, e poi multiplichata l'altra in sé e trata de 101, e presa la Rx di ciascuno di questi avenimenti gionti insemi facino 14 aponto, dimando le parti. [p.419]

This calls for some general observations on the published transcription:

1. Numerals are arbitrarily expanded to words as '2' has here been changed into 'doi'.
2. Sometimes words are introduced which do not appear in the manuscript, as here 'stessa'.
3. Words are sometimes interpreted wrongly, as here 'quanto' instead of 'questo' and 'avenimente' instead of 'remaninmente'
4. Spelling sometimes is silently changed, in 'trata' instead of 'trato', 'insemi' instead of 'insiemi'

Given that there are differences in reading, I did not find any differences which would influence the correct interpretation of the problems. While below my own transcription will be used, the work by Calzoni and Cavazzoni remains an excellent basis for the further study of Pacioli's text.

II.2 Conventions

The numbering of the problems follows the marginal numbering of the manuscript (omitted in [9]). Abbreviations and ligatures in the text have been expanded. Accents have been added to make the distinction in meaning between words such as *ch'è* and *che*. Punctuation and word separations have been normalized where modern Italian uses separate words. The spelling follows the manuscript. Expansions of abbreviations follow the dominant spelling of manuscript (e.g. "multiplichato"). The distinction between i and j has been dropped in favor of i.

II.3 Partitioning problems

1. Fame de 10 2 parti che partito la magiore per la menore ne venga 7. Dimando le parti. [$f.229^r$]
2. Fame de 10 2 parti che multiplichato l'una contra l'altra facia 20. [$f.229^r$]
3. Fame di 10 2 parti che l'una sia 3 tanto più de l'altra. Dimando le parti. [$f.229^r$]
4. Fame de 10 2 parti che multiplichato zaschuna in sé e quelle 2 multiplicationi gionte insiemi facino 60 aponto. [$f.230^v$]
5. Fame de 10 2 parti che partito la magiore per la menore e l'avenimento gionto sopra a la magiori facia 7 1/2. Dimando le parti. [$f.230^v$]
6. Fame de 10 2 parti che partito la magiore per la minore quello ne vene gionto sopra la minore fa 5 aponto. [$f.230^v$]
7. Fame de 10 2 parti che partito la menore per la magiore e quello ne vene gionto sopra a la menore facia 5 aponto. [$f.231^r$]
8. Fame de 10 2 parti che multiplichato zaschuna in sé e trato l'una multiplichatione de l'altra resti 49. Dimando che sia le parti. [$f.230^r$]
9. Fame de 10 2 parti che multiplichato zaschuna in sé e quelle multiplichationi facia 200 più l'una che l'altra. Dimando le parti. [$f.230^v$]
10. Fame de 10 2 parti che multiplichato l'una contra l'altra e a la multiplichatione giontoci la diferentia che è da l'una parte e l'altra facia 22 aponto. Dimando le parti. [$f.230^v$]
11. Fame de 10 2 parti che multiplichato zaschuno in sé e quelle doi multiplichationi gionti insiemi e la summa giontoci la diferentia che è da una parte e l'altra quest'ultima summa fa 54 aponto. Dimando le parti. [$f.231^r$]
12. Fame de 10 2 parti che multiplichato zaschuna in sé e quelle doi multiplichationi gionte insiemi facia tanto quanto la diferentia ch'è da una parte e l'altra gionta con 54. Dimando le parti. [$f.231^r$]
13. Fame de 10 2 parti che multiplichato l'una contra l'altra e quel' che fa partito per la diferentia che è da l'una parte e l'altra ne venga 18. Dimando le parti. [$f.231^v$]
14. Fame de 10 2 parti che multiplichato l'una contra l'altra e quel' che fa partito

per la diferentia ch'è da l'una parte e l'altra ne venga la differentia che è da l'una parte e l'altra. Dimando che siranno le dicte parti aponto. [$f.231^v$]

15. Fame de 10 2 parti che partito el quadrato de la magior parte in la multiplichatione de l'una parte in l'altra ne venga 6. Dimando le parti. [$f.232^r$]
16. Fame de 10 2 parti che partito 10 per ciascuna de quelle e quelli avenimenti gionti insiemi facino 5. Dimando le parti. [$f.232^r$]
17. Fame de 10 2 parti che multiplichata zaschuna in sé e quel' che fa gionto insiemi e giontoci la diferentia che è da l'una e l'altra parte facia 55. Dimando le parti. [$f.232^v$]
18. Fame de 10 2 parti che multiplichato l'una contra l'altra facia tanto quanto la diferentia multiplichata per 4. [$f.232^v$].
19. Fame de 10 2 parti che multiplichato l'una contra l'altra facia tanto quanto la diferentia in se medema. [$f.233^r$]
20. Fame de 10 2 parti che $3/4$ de l'una facia tanto tanto quanto li $4/5$ de l'altra. [$f.233^r$]
21. Fame de 10 2 parti che multiplichato l'una contro l'altra facia 20 e ℞ 20 [$f.233^r$]
22. Fame de 10 2 parti che multiplichato l'uno per ℞ 7 facia tanto quanto l'altra multiplichato per ℞ 6. [$f.233^v$]
23. Fame de 10 3 parti che multiplichata la magiore per la menore facia tanto quanto la seconda multiplichata per se medema. [$f.233^v$].
24. Fame de 10 2 parti che multiplichato l'una per se medesima e trato la de 100 e l'altra multiplichata per se medesima tratola de 120 presa la ℞ de ciascuna facia 2 più l'una de l'altra. Dimando che sia le parti. [$f.233^v$]
25. Fame de 10 3 parti che multiplichato l'una per 3 e l'altra per 4 e l'altra per 5 facia tanto l'una quanto l'altra. Dimando le parti. [$f.237^r$]
26. Fame de 10 2 parti che multiplichato ziascheduna in sé e gionti insiemi poi partita quella summa per la diferentia ch'è da l'una parte e l'altra quando siamo multiplichate in sé che ne venga 20. Dimando le parti. [$f.237^v$]
27. Fame de 10 2 parti che multiplichato zaschuna in sé e gionto 10 sopra a la multiplichatione de la menore facia tanto l'una quanta l'altra. [$f.238^r$]
28. Fame de 10 2 parti che multiplichato l'una per 2 e l'altra per 3 e quelle doi multiplichatione gionti insiemi facino 10. [$f.238^r$]
29. Fame de 10 2 parti che partito l'una per 2 e l'altra per 3 e quelli avenimenti gionti insiemi facia 10. [$f.238^r$]
30. Fame de 10 2 parti che partito el quadrato de la magiore per lo quadrato de la menor ne venga 2. Dimando le parti. [$f.238^r$]
31. Fame de 10 2 parti che multiplichato la magior per sé facia tanto quanto la menor per 10. [$f.238^v$]
32. Fame de 10 2 parti che partita la magiore per la menore e poi la menor per la magiore e poi li avenimenti gionti insiemi facino 9. [$f.238^v$]
33. Fame de 10 2 parti che partito 10 per ciascuna d'esse e li avenimenti gionti

insiemi facino 10 aponto. [$f.239^r$]

34. Fame de 10 3 parti che multiplichato la prima contra la terza facia tanto quanto la secondo in sé. [$f.239^r$]
35. Fame de 10 3 parti che multiplichato la prima via la seconda e la summa che fa multiplichata in la terza facia 6. Domandi le parti. [$f.239^r$]
36. Fame de 10 2 parti che multiplichato l'una contra l'altra e partito per diferentia che è da l'una parte e l'altra ne venga R 18. Dimando le parti. [$f.239^v$]
37. Fame de 10 doi parti che preso la R de cadauna gionte insemi facia 3 $^1/_2$. Dimando le parti, etc. Sapi che questa se vol pigliare al contrario, e tanto vol dire questa ragione commo a dire: Fame de 3$^1/_2$ 2 parti che multiplichato zaschuna in sé e quelle multiplichationi gionte insiemi facino 10. Dimando le parti, etc. [$f.240^r$]
38. Fame de 10 2 parti che preso la de ciasche d'una e gionte insiemi facino 3$^1/_2$. [$f.241^r$]
39. Fame de 10 2 parti che partito la magiore per la menor ne venga la diferentia. [$f.242^v$]
40. Fame de 10 2 parti che la menore sia R de la magiore. Dimando le parti. [$f.243^r$]
41. Fame de 10 2 parti che partito la magiore per la minore e la minore per la magiore e quelli avenimenti {gionti} tracti l'uno del'altra el rimamente sia 10. Dimando le parti. [$f.243^v$]
42. Fame de 10 2 parti che preso el $^1/_3$ de la R de l'una e preso el $^1/_4$ de la R de l'altra gionti insiemi facia 3 $^1/_2$. Dimando le parti. [$f.244^r$]
43. Fame de 10 2 parti che li $^3/_4$ de R de l'una gionti con li $^2/_3$ de la R de l'altra facino 3 $^1/_2$. Dimando le parti. [$f.244^v$]
44. Fame de 10 2 parti che tanto sia li $^3/_4$ de la R de l'una quanto li $^2/_3$ de la R de l'altra. Dimando le parti. [$f.245^r$]
45. Fame de 10 2 parti che preso la R di ziascuna gionta insiemi facino la differentia che è da l'una parte e l'altra. Dimando le parti. [$f.245^v$]
46. Fame de 10 2 parti che partito la magiore per la minore e quel' che ne vene multiplichato in sé facia 2 meno che la magiore. O voli ancho più però che a ogni modo virà a doi termeni. Dimando le parti. [$f.245^v$]
47. Fame de 10 5 parti che multiplichato zasche d'una de le 5 in sé per 10 e quelle 5 multiplichatione gionte insiemi facino 90 aponto. Dimando le parti. [$f.246^r$]
48. Fame de 10 2 parti che tanto facia l'una multiplichato per R 8 quanto l'altra multiplichato per 3. Dimando le parti. [$f.246^v$]
49. Partime 100 per una quantità e quel' che ne vene salva e poi lo parti per 5 più che non partisti prima e quello che ne vene salva e poi lo parti per 6 meno che non partisti la seconda volta e quello che ne vene salva e poi agiogni insiemi questi 3 avenimente e fa che facino 20 aponto. Dimando in che quantità fo' partito la prima volta e in che la seconda e in che la terza. [$f.247^r$]

50. Partime 100 per una quantità e quello che ne vene poni da chanto e poi lo parti per 5 più che prima e fa' che ne venga uno meno che non vene prima. Dimando per che lo partirai prima e che poi. Alia petita quod idem est, et nichil aliud importat.[$f.247^r$]
51. Partime 100 per una quantità e poi lo parti per una quantità più 3 e questi 2 avenimenti gionti insiemi facino 20. Dimando che lo partirai prima e in che poi. [$f.247^v$]
52. Partime 100 per una quantità e poi partime 120 per 3 più che non partisti 100 e chava questo secondo avenimente del primo e fa che che resti 5. Dimando in che partisti 100 e in che 120. Idem est. [$f.248^r$]
53. Multiplichame 10 per uno quantità e quel' che fa samvalo et poi lo multiplicha per 2 più che prima e quel' che fa salvalo e poi lo multiplicha per 6 meno che la seconda volta e poi piglia queste 3 multiplichationi e giognile insiemi e fa' che sieno 10. Dimando per che lo multiplicharai la prima volta e per che la seconda e per che la terza. [$f.248^v$]
54. Partime 10 per una quantità e quel' che ne vene salva e poi parti 12 in un'altra quantità e quelli doi avenimenti gionti insiemi facino 4. Dimando che sia el partitore. [$f.249^r$]
55. Partime 100 per una quantità e quel' che ne vene chavato de 100 e poi parti quel' remanente in uno più che non partisti prima e fa' che ne venga meno 4 che non ne vene prima. Dimando che numero fo partitore. [$f.249^v$]
56. Partime 100 per una quantità e quel' che ne vene serba e poi partime 130 per quella quantità più 3 e quello che ne vene multiplichato contra a quello che ne vene prima e fa' che facia 101. Dimando in che fo partito 100 e 130. [$f.250^r$]
57. Partime 100 per una quantità e quel' che ne vene serbalo e poi multiplicha questa quantità in che partisti più 6 e la somma che farà chavala de 100 e poi lo resto tornalo e partir in la quantità che partisti prima e poi parti la muliplichatione de la quantità via 6 in 3 e li avenimenti de l'uno partimento e de l'altro sieno equali. Dimando che sia partitore. [$f.250^v$]
58. Uno parte 100 fl. in alquanti homini e quello che thochò per uno chadauno ne dà via 2 a un altro homo e tanto se trovar uno quanto l'altro. Dimando quanti fo' li primi homini e che tochò per uno. [$f.251^v$]
59. Fame de 10 2 parti che multiplichato la differentia che è da l'una parte e l'altra in sé fa 20 $1/4$. Dimando le parti. [$f.252^r$]
60. Partime 100 fl. in una quantità d'omini e quello che ne vene salva e poi li parti in 2 homini più che non partisti prima e quello che ne vene de questo sechondo partimento giogni con quello che te vene del primo e fa' che facia 10. Dimando quanti con ne vene esser li primi homini. [$f.252^r$]
61. Partime un numero per un più e fa che ne venga un meno. Dimando che sirà el numero. [$f.252^v$]
62. Parti 100 fl. in alquanti homini e quello che tocha per uno posto sopra a li

homini facia 25 per numeri. [f.252^v]

63. Partime 20 per una quantità; quel' che ne vene salva, poi lo parti per quella quantità più 2 e questi doi avenimente gionti insiemi fa' che facaino 9. Dimando che sia partitore [f.252^v]
64. Fame de 10 doi parti che partito la magiore per la minore ne venga 7 e R 7. Dimando le parti. [f.253^r]
65. Fame de 10 3 parti che sia tal parte: la prima della seconda commo la seconda della terza e multiplichata la prima per 3 e la sechonda per 4 el la terza per 5 e gionti insiemi questi multiplichamenti facino 35 aponto. Dimando le parti. [f.253^r]
66. Fame de 10 4 parti che la prima sia tal parte che la seconda commo la seconda de la terza e la terza de la quarta e multiplichata la prima per 8 e l'altra per 4 e l'altra per 3 e l'altra per 1 e quelle multiplichatione gionte insiemi facino 16 aponto. Dimando le parti. [f.254^r]
67. Fame de 10 2 parti che partito la magior per la minore e la minor per la magiore e questi doi avenimenti gionti insiemi facino R 16 aponto. Dimando le parti. [f.254^r]
68. Fame de 10 2 parti che multiplichato l'una contra l'altra facia 5 tanto più che partito la magior per la minore. Dimando le parti. [f.254^v]
69. Fame de 12 doi parte che multiplichata la prima in sé e giontoci 13 e multiplichata l'altra in sé e tratone 11 e preso la radice de l'una somma e de l'altra gionte insiemi facino 12. Dimando le parti. [f.254^v]
70. Fame de 12 doi parti che multiplichata zascuna in sé e de la multiplichatione de la prima tratone 7 e la multiplichatione de l'altra postovi sopra 8, presa la radice di ciaschuna parte e gionta insiemi facia 12 aponto. Dimando le parti. [f.255^v]
71. Fame de 10 2 parti che multiplichata la prima per sé e trato de 100 e la R di questo che resta serba, poi multiplicha l'altra in sé e trato de 101 e preso la R di ciascuni questi remanimenti gionte insiemi facino 14 a ponto. Dimando le parti. [f.255^v]
72. Fame de 10 doi parti che partito la magior per la minore e quella che ne vene multiplichato via la differentia che è da l'una parte e l'altra facia tanto quanto ragionta la differentia con 10. Dimando le parti. [f.256^r]
73. Fame de 10 {doi parte} tre tal parti che sia tal parte la prima de la seconda commo la seconda de la terza e multiplichato ciaschuna in sé e quelle multiplichatione gionte insiemi facino 40 aponto. Dimando le parti. [f.256^v]
74. Fame de 10 doi parti che multiplichato l'una contra l'altra e partita la multiplichatione per la R de la diferentia ne venga R della diferentia. Dimando le parti. [f.256^v]

III The translation

III.1 Translation principles

I adopted a very literal style of translation as argued for in an earlier critical edition of an abbaco treatise on algebra [24].

III.2 Partitioning problems

III.2.1 [PPM1201]

Make me 2 parts of 10 such that dividing the larger by the smaller 7 results from it. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{a}{b}=7 \end{cases} \tag{12}$$

(solution: $8\frac{3}{4}$ and $1\frac{1}{4}$)

III.2.2 [PPM1202]

Make me 2 parts of 10 such that multiplying the one against the other it makes 20. I ask for the parts.

$$\begin{cases} a+b=10 \\ ab=20 \end{cases} \tag{13}$$

(solution: $5+\sqrt{5}$ and $5-\sqrt{5}$)

III.2.3 [PPM1203]

Make me 2 parts of 10 such that the one is three times [and three][18] more than the other. I ask for the parts.

$$\begin{cases} a+b=10 \\ a=3b+3 \end{cases} \tag{14}$$

(solution: $8\frac{1}{4}$ and $1\frac{3}{4}$)

III.2.4 [PPM1204]

Make me 2 parts of 10 such that multiplying each by itself and these two multiplications joined together makes 60 exactly.

$$\begin{cases} a+b=10 \\ a^2+b^2=60 \end{cases} \tag{15}$$

(solution: $5+\sqrt{5}$ and $5-\sqrt{5}$)

[18]Without this addition the correct solution would be $7\frac{1}{2}$ and $2\frac{1}{2}$. From the solution text it becomes apparent that Pacioli had this problem in mind.

III.2.5 [PPM1205]

Make me 2 parts of 10 such that dividing the larger by the smaller, and the outcome joined above the larger make $7\frac{1}{2}$. I ask for the parts.

$$\begin{cases} a + b = 10 \\ \frac{a}{b} + a = 7\frac{1}{2} \end{cases} \tag{16}$$

(solution: $9\frac{1}{4} + \sqrt{10\frac{9}{16}}$ and $\frac{3}{4} + \sqrt{10\frac{9}{16}}$)[19]

III.2.6 [PPM1206]

Make me 2 parts of 10 such that dividing the larger by the smaller, and that what comes out of it, joined above the smaller makes 5 exactly. I ask for the parts.

$$\begin{cases} a + b = 10 \\ \frac{a}{b} + b = 5 \end{cases} \tag{17}$$

(solution: impossible [no real solutions])

[Alternative problem]
Make me 2 parts of 12 such that dividing the larger by the smaller, and that what comes out of it, joined with the smaller makes 6. (Solution: 8 and 6)

III.2.7 [PPM1207]

Make me 2 parts of 10 such that dividing the smaller by the larger, and that what comes out of it, joined with the smaller makes 5 exactly.

$$\begin{cases} a + b = 10 \\ \frac{b}{a} + b = 5 \end{cases} \tag{18}$$

(solution: $8 - \sqrt{14}$ and $2 + \sqrt{14}$)

III.2.8 [PPM1208]

Make me 2 parts of 10 such that each part multiplied by itself, and the one multiplication detracted from the other leaves 49. I ask what are the parts.

$$\begin{cases} a + b = 10 \\ a^2 - b^2 = 49 \end{cases} \tag{19}$$

(solution: $7\frac{19}{20}$ and $2\frac{11}{20}$)

[19] The correct solution is 6 and 4. Pacioli interprets the problem as $\frac{a}{b} + b = 7\frac{1}{2}$ which has no real solutions.

III.2.9 [PPM1209]

Make me 2 parts of 10 such that each part multiplied by itself, and these multiplications make the one 200 more than the other. I ask for the parts.

$$\begin{cases} a+b=10 \\ a^2-b^2=200 \end{cases} \tag{20}$$

(solution: 15 and -5)[20]

III.2.10 [PPM1210]

Make me 2 parts of 10 such that the one multiplied by the other and to this multiplication join the difference there is between one part and the other, makes 22 exactly.

$$\begin{cases} a+b=10 \\ ab+(a-b)=22 \end{cases} \tag{21}$$

(solution: 8 and 2)[21]

III.2.11 [PPM1211]

Make me 2 parts of 10 such that each one multiplied by itself and these two multiplications joined together and the sum, joined with the difference there is between one part and the other, and this final sum, makes 54 exactly. I ask for the parts.

$$\begin{cases} a+b=10 \\ a^2+b^2+(a-b)=54 \end{cases} \tag{22}$$

(solution: 6 and 4)[22]

III.2.12 [PPM1212]

Make me 2 parts of 10 such that each multiplied by itself and these two multiplications joined together makes as much as the difference there is between one part and the other joined with 54. I ask for the parts.

$$\begin{cases} a+b=10 \\ a^2+b^2=(a-b)+54 \end{cases} \tag{23}$$

(solution: 7 and 3)[23]

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[20] Sesiano discusses this negative solution in [42] p. 144).

[21] Pacioli does not mention the alternative solution: 6 and 4.

[22] Pacioli does not mention the alternative solution: 7 and 3.

[23] Pacioli does not mention the alternative solution: 6 and 4.

III.2.13 [PPM1213]

Make me 2 parts of 10 such that the one multiplied by the other and that what comes out of it, divided by the difference there is between one part and the other, 18 results from it. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{ab}{a-b}=22 \end{cases} \tag{24}$$

(solution: $\sqrt{349}-13$ and $23-\sqrt{349}$)

[Alternative problem]
Make me 2 parts of 10 such that the one multiplied by the other and that what comes out of it, divided by the difference there is between one part and the other, 12 results from it. (Solution: 6 and 4)

III.2.14 [PPM1214]

Make me 2 parts of 10 such that the one multiplied by the other and that what comes out of it, divided by the difference there is between one part and the other, makes the difference there is between one part and the other. I ask what are these parts exactly.

$$\begin{cases} a+b=10 \\ \frac{ab}{a-b}=a-b \end{cases} \tag{25}$$

(solution: $5+\sqrt{5}$ and $5-\sqrt{5}$)

III.2.15 [PPM1215]

Make me 2 parts of 10 such that the square of the largest divided by the multiplication of the one part by the other, 6 results from it. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{a^2}{ab}=6 \end{cases} \tag{26}$$

(solution: $8\frac{4}{7}$ and $1\frac{3}{7}$)

III.2.16 [PPM1216]

Make me 2 parts of 10 such that 10 divided by each of them and these outcomes joined together makes 5. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{10}{a}+\frac{10}{b}=5 \end{cases} \tag{27}$$

(solution: $5+\sqrt{5}$ and $5-\sqrt{5}$)

III.2.17 [PPM1217]

Make me 2 parts of 10 such that each multiplied by itself and that what comes out of it, joined together and joined with the difference there is between one and the other part, makes 55. I ask for the parts.

$$\begin{cases} a+b=10 \\ a^2+b^2+(a-b)=55 \end{cases} \tag{28}$$

(solution: $5\frac{1}{2}-\sqrt{2\frac{3}{4}}$ and $4\frac{1}{2}+\sqrt{2\frac{3}{4}}$)

[similar problem]

Make me 2 parts of 10 such that each multiplied by itself and these multiplications joined together, and joined with the difference there is between one and the other part, 54 results from it. I ask for the parts. (solution: 6 and 4)

III.2.18 [PPM1218]

Make me 2 parts of 10 such that the one multiplied by the other makes as much as the difference multiplied by 4.

$$\begin{cases} a+b=10 \\ ab=4(a-b) \end{cases} \tag{29}$$

(solution: $9-\sqrt{41}$ and $1+\sqrt{41}$)

[similar problem]

Make me 2 parts of 10 such that the one multiplied by the other makes as much as the difference multiplied by 12. (solution: 4 and 6)

III.2.19 [PPM1219]

Make me 2 parts of 10 such that the one multiplied by the other makes as much as the difference multiplied by itself.

$$\begin{cases} a+b=10 \\ ab=(a-b)^2 \end{cases} \tag{30}$$

(solution: $5+\sqrt{5}$ and $5+\sqrt{5}$)

III.2.20 [PPM1220]

Make me 2 parts of 10 such that $\frac{3}{4}$ of the one makes as much as $\frac{4}{5}$ of the other.

$$\begin{cases} a+b=10 \\ \frac{3}{4}a=\frac{4}{5}b \end{cases} \tag{31}$$

(solution: $5\frac{5}{31}$ and $4\frac{26}{31}$)

III.2.21 [PPM1221]

Make me 2 parts of 10 such that the one multiplied by the other makes 20 and the root of 20.

$$\begin{cases} a+b=10 \\ ab=20+\sqrt{20} \end{cases} \tag{32}$$

(solution: $5+\sqrt{\sqrt{20}}$ and $5-\sqrt{\sqrt{20}}$)[24]

III.2.22 [PPM1222]

Make me 2 parts of 10 such that the one multiplied by the root of 7 makes as much as the other multiplied by the root of 6.

$$\begin{cases} a+b=10 \\ \sqrt{7}a=\sqrt{6}b \end{cases} \tag{33}$$

(solution: $\sqrt{4200}-\sqrt{3600}$ and $\sqrt{4900}-\sqrt{4200}$)

III.2.23 [PPM1223]

Make me 3 parts of 10 such that the larger multiplied by the smaller makes as much as the second multiplied by itself.

$$\begin{cases} a+b+c=10 \\ ac=b^2 \end{cases} \tag{34}$$

(solution: $1\frac{3}{7}$, $2\frac{6}{7}$ and $5\frac{5}{7}$)[25]

III.2.24 [PPM1224]

Make me 2 parts of 10 such that the one multiplied by itself and detracted from 100 and the other multiplied by itself and detracted from 120, taking the root of each of them, make two more the one than the other. Asked is what are the parts.

$$\begin{cases} a+b=10 \\ \sqrt{100-a^2}=2+\sqrt{120-b^2} \end{cases} \tag{35}$$

(solution: $4\frac{1}{26}+\sqrt{3\frac{31}{676}}$ and $5\frac{25}{26}-\sqrt{3\frac{131}{676}}$)

[24] Pacioli arrives at the correct equation $10x=20+\sqrt{20}+x^2$ but gives the wrong roots. The correct solution is $5+\sqrt{5-\sqrt{20}}$ and $5-\sqrt{5-\sqrt{20}}$.

[25] This is an indeterminate problem about three numbers in geometric progression. Pacioli solves it by posing 1 for the first and deriving 2 and 4 for the others. He then uses counterfactual arithmetic to divide 10 proportionally over the three ratios.

[similar problem]
Make me 2 parts of 10 such that the one multiplied by itself and detracted from 20 and the other multiplied by itself and detracted from 61, and take the root of the one and the root of the rest makes three more, the one against the other. I ask for the parts.

$$\begin{cases} a+b=10 \\ \sqrt{20-a^2}+3=\sqrt{61-b^2} \end{cases} \tag{36}$$

(solution: $6\frac{13}{109}+\sqrt{\frac{9216}{11881}}$ and $3\frac{96}{109}-\sqrt{\frac{9216}{11881}}$ or 6 and 4)

III.2.25 [PPM1225]

Make me 3 parts of 10 such that the one multiplied by 3 and the other by 4 and the other by 5 makes as much as the one as the other. I ask for the parts.

$$\begin{cases} a+b+c=10 \\ 3a=4b=5c \end{cases} \tag{37}$$

(solution: $2\frac{26}{47}$ [and $3\frac{9}{47}$, $4\frac{12}{47}$])

[marginal problem]
Find 3 numbers which make equal as told and joined together make 20.

[marginal problem]
Divide 10 by 5 and that what comes out of it is one part and next divide 20 by 4 and that what results is the other and next divide by 3 and that which results makes the third part. (no solution given)

III.2.26 [PPM1226]

Make me 2 parts of 10 such that each multiplied by itself and joined together, and then this sum divided by the difference there is between the one and the other, when it is multiplied by itself, 20 results from it. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{a^2+b^2}{a^2-b^2}=20 \end{cases} \tag{38}$$

(solution: not given)[26]

[marginal problem]
Find 3 numbers which make equal as told and joined together make 20.

[26]Pacioli arrives at the equation $190x+x^2=50$ instead of $190x+x^2=950$ and does not calculate the roots. The correct solution involves a negative: $105+5\sqrt{399}$ and $-95-5\sqrt{399}$.

III.2.27 [PPM1227]

Make me 2 parts of 10 such that each multiplied by itself and joining 10 above the multiplication of of the smaller one, the one makes as much as the other.

$$\begin{cases} a+b=10 \\ a^2=b^2+10 \end{cases} \tag{39}$$

(solution: $5\frac{1}{2}$ and $4\frac{1}{2}$)

[marginal problem]
Make me 2 parts of 10 such that the sum of their squares divided by the difference there is between the one and the other, 26 results from it. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{a^2+b^2}{a-b}=26 \end{cases} \tag{40}$$

(solution: 6 and 4)

III.2.28 [PPM1228]

Make me 2 parts of 10 such that multiplying the one by 2 and the other by 3 and these two multiplications joined together make 10.

$$\begin{cases} a+b=10 \\ 2a+3b=10 \end{cases} \tag{41}$$

(solution: 20 and $(10-20)$)[27]

III.2.29 [PPM1229]

Make me 2 parts of 10 such that dividing the one by 2 and the other by 3 and these outcomes joined together make 10.

$$\begin{cases} a+b=10 \\ \frac{a}{2}+\frac{b}{3}=10 \end{cases} \tag{42}$$

(solution: 40 and $(10-40)$)[28]

III.2.30 [PPM1230]

Make me 2 parts of 10 such that dividing the square of the larger by the square of the smaller 2 results from it. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{a^2}{b^2}=2 \end{cases} \tag{43}$$

[27] Sesiano discusses this negative solution in [42] p. 145).

[28] Sesiano discusses this negative solution in [42] p. 145).

(incorrect solution: $\sqrt{200}-10$ and $10-\sqrt{200}$)[29]

III.2.31 [PPM1231]

Make me 2 parts of 10 such that multiplying the larger by itself makes as much as the smaller by 10.

$$\begin{cases} a+b=10 \\ a^2=10b \end{cases} \tag{44}$$

(solution: $\sqrt{125}-5$ and $15-\sqrt{125}$)

[similar problem]

Make me 2 parts of 10 such that the smaller is a part of the larger as the larger is of everything.

$$\begin{cases} a+b=10 \\ \frac{a}{b}=\frac{b}{10} \end{cases} \tag{45}$$

III.2.32 [PPM1232]

Make me 2 parts of 10 such that dividing the larger by the smaller and then the smaller by the larger and then these two results joined together makes 9.

$$\begin{cases} a+b=10 \\ \frac{a}{b}+\frac{b}{a}=9 \end{cases} \tag{46}$$

(solution: $5+\sqrt{14\frac{10}{11}}$ and $5-\sqrt{14\frac{10}{11}}$)

III.2.33 [PPM1233]

Make me 2 parts of 10 such that dividing 10 by each of these and the outcomes joined together makes 10 exactly.

$$\begin{cases} a+b=10 \\ \frac{10}{a}+\frac{10}{b}=10 \end{cases} \tag{47}$$

(solution: $5+\sqrt{15}$ and $5-\sqrt{15}$)

III.2.34 [PPM1234]

Make me 3 parts of 10 such that multiplying the first against the third makes as much as the second by itself.

$$\begin{cases} a+b+c=10 \\ ac=b^2 \end{cases} \tag{48}$$

[29] Pacioli makes a mistake in subtracting the root from 10. The correct solution involves a negative: $\sqrt{200}-10$ and $20-\sqrt{200}$.

(solution: $1\frac{3}{7}$, $2\frac{6}{7}$ and $5\frac{5}{7}$)[30]

III.2.35 [PPM1235]

Make me 3 parts of 10 such that multiplying the first by the second and and the sum which results[31] multiplied with the third makes 6. I ask for the parts.

$$\begin{cases} a+b+c=10 \\ abc=6 \end{cases} \tag{49}$$

(solution: $4\frac{1}{2}+\sqrt{14\frac{1}{4}}$, $4\frac{1}{2}-\sqrt{14\frac{1}{4}}$ and 1)[32]

[reduced problem]
Make me 2 parts of 9 such that the one multiplied by the other makes 6.

III.2.36 [PPM1236]

Make me 2 parts of 10 such that multiplying the one against the other and dividing by the difference there is between one part and the other the root of 18 comes out. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{ab}{a-b}=\sqrt{18} \end{cases} \tag{50}$$

(solution: $\sqrt{43}+5-\sqrt{18}$ and $5+\sqrt{18}-\sqrt{43}$)

III.2.37 [PPM1237]

Make me 2 parts of 10 such that taking the roots of each joined together make $3\frac{1}{2}$.

[alternative problem]
Make me 2 parts of $3\frac{1}{2}$ so that multiplying each by itself and these multiplications joined together make 10. I ask for the parts.

$$\begin{cases} a+b=3\frac{1}{2} \\ a^2+b^2=10 \end{cases} \tag{51}$$

(solution: $1\frac{3}{4}+\sqrt{1\frac{15}{16}}$ and $1\frac{3}{4}-\sqrt{1\frac{15}{16}}$)

III.2.38 [PPM1238]

Make me 2 parts of 10 such that taking the roots of each joined together makes $3\frac{1}{2}$.

$$\begin{cases} a+b=10 \\ \sqrt{a}+\sqrt{b}=3\frac{1}{2} \end{cases} \tag{52}$$

[30] Same problem as PPM1223.

[31] Evidently, this should be "and the product which results".

[32] Pacioli assumes $a=1$ to resolve the indeterminacy and then solves it by algebra.

(solution: $1\frac{3}{4}+\sqrt{1\frac{15}{16}}$ and $1\frac{3}{4}-\sqrt{1\frac{15}{16}}$)

[similar problem]

Make me 2 parts of 13 such that taking the root of each joined together makes 5. I ask for the parts.

$$\begin{cases} a+b=13 \\ \sqrt{a}+\sqrt{b}=5 \end{cases} \tag{53}$$

(solution: 4 and 9)

III.2.39 [PPM1239]

Make me 2 parts of 10 such that dividing the larger by the smaller the difference comes out.

$$\begin{cases} a+b=10 \\ \frac{a}{b}=a-b \end{cases} \tag{54}$$

(solution: $7\frac{1}{4}+\sqrt{2\frac{9}{16}}$ and $2\frac{3}{4}-\sqrt{2\frac{9}{16}}$)

III.2.40 [PPM1240]

Make me 2 parts of 10 such that the smaller one is the root of the larger one. I ask for the parts.

$$\begin{cases} a+b=10 \\ a=\sqrt{b} \end{cases} \tag{55}$$

(solution: $\sqrt{10\frac{1}{4}}-\frac{1}{2}$ and $10-\sqrt{10\frac{1}{4}}+\frac{1}{2}$)

[similar problem]

Make me 2 parts of 12 such that the smaller one is the root of the larger one. (solution: 9 and 3).

III.2.41 [PPM1241]

Make me 2 parts of 10 such that dividing the larger by the smaller and the smaller by the larger and these outcomes detracted, the one from the other, 10 is the remainder. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{a}{b}-\frac{b}{a}=10 \end{cases} \tag{56}$$

(solution: $6-\sqrt{26}$ and $4+\sqrt{26}$)

[similar problem]

Make me 2 parts of 12 such that dividing the larger by the smaller and the smaller by the larger and these results detracted, the one from the other, $1\frac{1}{2}$ is the remainder. (solution: 8 and 4).

III.2.42 [PPM1242]

Make me 2 parts of 10 such that taking one third of the root of one and one fourth of the root of the other joined together makes $3\frac{1}{2}$. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{1}{3}\sqrt{a}+\frac{1}{4}\sqrt{b}=3\frac{1}{2} \end{cases} \tag{57}$$

(no solution given)[33]

[similar problem][34]
Make me 2 parts of $3\frac{1}{2}$ such that the one multiplied by 4 and the other by 3 and these two multiplied by itself and then these 2 results joined together makes 10.

$$\begin{cases} a+b=3\frac{1}{2} \\ (3a)^2+(4b)^2=10 \end{cases} \tag{58}$$

(no solution given)[35]

[similar problem]
Make me 2 parts of 25 such that taking on third of the root of one and one fourth of the root of the other joined together makes 2. I ask for the parts. (solution: 16 and 9).

III.2.43 [PPM1243]

Make me 2 parts of 10 such that three fourths of the root of one joined with two thirds of the root of the other make $3\frac{1}{2}$. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{3}{4}\sqrt{a}+\frac{2}{3}\sqrt{b}=3\frac{1}{2} \end{cases} \tag{59}$$

(solution: not given)

[similar problem]
Make me 2 parts of 25 such that taking three fourths of the root of one and joined with two thirds of the root of the other joined together makes 5. I ask for the parts. (solution: 16 and 9)

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[33]This problem has no real solutions.

[34]Pacioli claims this problem is equivalent with the previous one, which is mistaken.

[35]This problem has no real solutions.

III.2.44 [PPM1244]

Make me 2 parts of 10 such that three fourths the root of one make as much as two thirds of the root of the other. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{3}{4}\sqrt{a}=\frac{2}{3}\sqrt{b} \end{cases} \tag{60}$$

(solution: not given)[36]

[similar problem]
Make me 2 parts of 145 such that three fourths of the root of one make as much as two thirds of the root of the other. I ask for the parts. (solution: 81 and 64)

III.2.45 [PPM1245]

Make me 2 parts of 10 such that taking the root of each joined together this makes as much as the difference there is between one part and the other. I ask for the parts.

$$\begin{cases} a+b=10 \\ \sqrt{a}+\sqrt{b}=a-b \end{cases} \tag{61}$$

(solution: not given)[37]

[similar problem]
Make me 2 parts of 25 such that that taking the root of each joined together this makes as much as the difference there is between one part and the other. I ask for the parts. (solution: 16 and 9)

III.2.46 [PPM1246]

Make me 2 parts of 10 such that dividing the larger by the smaller and that what results multiplied by itself makes 2 less than the larger. I ask for the parts.

$$\begin{cases} a+b=10 \\ \left(\frac{a}{b}\right)^2=a-2 \end{cases} \tag{62}$$

(solution: not given)[38]

[similar problem]
Make me 2 parts of 10 such that dividing the larger by the smaller a that what results multiplied by itself makes $3\frac{3}{4}$ less than the larger. I ask for the parts. (solution: 6 and 4)

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[36] This problem has no real solutions.

[37] This problem has no real solutions.

[38] This problem has no real solutions.

III.2.47 [PPM1247]

Make me 5 parts of 10 such that multiplying each of the 5 by itself[39] by 10 and these 5 multiplications joined together make 90 exactly. I ask for the parts.

$$\begin{cases} a+b+c+d+e=10 \\ 10a+10b+10c+10d+10e=90 \end{cases} \tag{63}$$

(solution: impossible)

III.2.48 [PPM1248]

Make me 2 parts of 10 in such way that the one multiplied by the root of 8 makes as much as the other multiplied by 3. I ask for the parts.

$$\begin{cases} a+b=10 \\ \sqrt{8}a=3b \end{cases} \tag{64}$$

(solution: $\sqrt{7200}-80$ and $90-\sqrt{7200}$)

[similar problem]

Make me 2 parts of 5 such that the one multiplied by the root of 4 makes as much as the other multiplied by 3. I ask for the parts. (solution: 3 and 2)

III.2.49 [PPM1249]

Divide 100 by a quantity such that what results secure and next divide it by 5 more than with what it was divided the first time and that what results secure and next divide it by 6 less than divided the second time an that what results secure and these three results joined together it makes 20 exactly. I ask in what quantity it was divided the first time, the second time and the third time.

$$\frac{100}{a}+\frac{100}{a+5}+\frac{100}{a+5-6}=20 \tag{65}$$

(solution: impossible)[40]

[alternative problem]

I divide 100 by a quantity such that what results secure and next divide it by 5 more than the first and that what results secure and next divide by 6 less than divided the second time an that what results secure and these three results joined together it makes $14\frac{5}{19}$ exactly. I ask in what quantity it was divided the first time, the second time and the third time.

$$\frac{100}{a}+\frac{100}{a+5}+\frac{100}{a+5-6}=14\frac{5}{19} \tag{66}$$

(solution: 20, 25 and 19)

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[39] From the text it becomes clear that 'by 10' is really intended here.

[40] This problem has no real solutions.

III.2.50 [PPM1250]

Divide 100 by a quantity and that what results place aside and next divide it by 5 more than the first time and that what results makes one less than what results the first time. I ask in what quantity it was divided the first time, and then next.

$$\frac{100}{a} - \frac{100}{a+5} = 1 \tag{67}$$

(solution: 20 and 25)

III.2.51 [PPM1251]

Divide 100 by a quantity and next divide it by a quantity plus 3 and these two outcomes together make 20. I ask by what it was divided the first time and by what next.

$$\frac{100}{a} + \frac{100}{a+3} = 20 \tag{68}$$

(solution: $3\frac{1}{2} + \sqrt{24\frac{1}{4}}$ and $6\frac{1}{2} + \sqrt{24\frac{1}{4}}$)

III.2.52 [PPM1252]

Divide 100 by a quantity and next divide 120 by 3 more than by what 100 was divided. And when the second outcome is detracted from the first it leaves 5. I ask by what 100 was divided and by what 120.

$$\frac{100}{a} - \frac{120}{a+3} = 5 \tag{69}$$

(solution: 5 and 8)

III.2.53 [PPM1253]

Multiply 10 by a quantity and that what results we secure and next multiply it by 2 more that the first and that what results we secure and next multiply it by 6 less than the second and next join these three multiplications together and we find that it is 10. I ask by what it was multiplied the first time and by what the second and by what the third.

$$10a + 10(a+2) + 10(a+2-6) = 10 \tag{70}$$

(solution: 1, 3 and -3)

[similar problem]

$$10a + 10(a+2) + 10(a+2-6) = 220 \tag{71}$$

(solution: 8, 10 and 4)

[similar problem]

$$10a + 10(a+2) + 10(a+2-6) = 221 \tag{72}$$

(solution: $8\frac{1}{30}$, $10\frac{1}{30}$ and $9\frac{1}{30}$)

III.2.54 [PPM1254]

Divide 10 by a quantity and that what comes out secure and next divide 12 by another quantity[41] and that what results secure and these two outcomes joined together make 4. I ask what are the divisors.

$$\frac{10}{a} + \frac{12}{a} = 4 \tag{73}$$

(solution: $5\frac{1}{2}$)

III.2.55 [PPM1255]

Divide 100 by a quantity and that what comes out detracted from 100 and then dividing what remains in one more than the first was divided results in 4 less than what resulted the first time. I ask which number is the divisor.

$$\frac{100}{a} = \frac{100 - \frac{100}{a}}{a+1} + 4 \tag{74}$$

(solution: $\sqrt{50\frac{1}{4}} - \frac{1}{2}$)

[similar problem]

Divide 60 by a quantity and that what comes out detracted from 60 and then dividing what remains in one more than the first was divided results in 6 less than what resulted the first time. I ask what is the divisor.

$$\frac{60}{a} = \frac{60 - \frac{100}{a}}{a+1} + 6 \tag{75}$$

(solution: 4)

III.2.56 [PPM1256]

Divide 100 by a quantity and that what results secure and next divide 130 by this quantity and 3 more that what comes out multiplied against what resulted the first time, it makes 101. I ask by what 100 and 130 was divided.

$$\left(\frac{100}{a}\right)\left(\frac{100}{a+3}\right) = 101 \tag{76}$$

(solution: $\sqrt{130\frac{389}{404}} - 1\frac{1}{2}$ and $\sqrt{130\frac{389}{404}} + 1\frac{1}{2}$)

[41] Pacioli uses the same unknown for the two divisors, so this should be read as "by the same quantity".

III.2.57 [PPM1257]

Divide 100 by a quantity and secure that what comes out and next multiply this quantity by six more than it was divided and detract the sum it makes from 100 and next return and divide it by the quantity which divided it in the first place and next divide the multiplication of the quantity by 6 in 3 and the outcomes of the one division and the other are equal. I ask what are the divisors.[42]

[similar problem]

$$\frac{100-6x}{x} = 3x - x \tag{77}$$

(solution: $\sqrt{52\frac{1}{4}} - 1\frac{1}{2}$ and $\sqrt{209} - 3$)

III.2.58 [PPM1258]

Divide 100 fl. by a number of men and that what falls to ones share is give two times worth to another man such that the one finds himself with as much as the other. I ask how much get the first men and how much falls to ones share.

$$\frac{100-2x}{x} = 2x \tag{78}$$

(solution: $\sqrt{50\frac{1}{4}} - \frac{1}{2}$)

III.2.59 [PPM1259]

Make me 2 parts of 10 such that the difference between the one part and the other multiplied by itself makes $20\frac{1}{4}$. I ask for the parts.

$$\begin{cases} a+b=10 \\ (a-b)^2 = 20\frac{1}{4} \end{cases} \tag{79}$$

(solution: $7\frac{1}{4}$ and $2\frac{3}{4}$)

III.2.60 [PPM1260]

Divide 100 fl. by a number of men and secure that what comes out and next divide it by 2 more men than it was divided first and that what comes out of this second division together with that what comes out of the first, it makes 10. I ask how many men comes out of this.

$$\frac{100}{a} + \frac{100}{a+2} = 10 \tag{80}$$

(solution: $9+\sqrt{101}$ and $11+\sqrt{101}$)

[42] As Pacioli does not provide a solution, its is not fully clear what is intended with the problem.

III.2.61 [PPM1261]

Divide a number by one more and it comes out that it makes one less. I ask what is this number.

$$\frac{a}{a+1} = a - 1 \tag{81}$$

(solution: $\frac{1}{2} + \sqrt{1\frac{1}{4}}$)

[similar problem]

Find a number which divided by one more $3\frac{1}{5}$ results from it. I ask this number.

$$\frac{a}{a+1} = 3\frac{1}{5} \tag{82}$$

(solution: $\frac{4}{5} - 3\frac{1}{5}$)[43]

III.2.62 [PPM1262]

Divide 100 fl. amongst a number of men and what falls to ones share joined above the number of men makes 25.

$$\frac{100 + a^2}{a} = 25 \tag{83}$$

(solution: 5)

III.2.63 [PPM1263]

Divide 20 by a quantity and secure what comes out. Next divide it by this quantity and 2 more and these two outcomes joined together make 9. I ask what are the divisors.

$$\frac{20}{a} + \frac{20}{a+2} = 9 \tag{84}$$

(solution: $1\frac{2}{9} + \sqrt{5\frac{76}{81}}$ and $3\frac{2}{9} + \sqrt{5\frac{76}{81}}$)

III.2.64 [PPM1264]

Make me 2 parts of 10 such that the larger divided by the smaller $7 + \sqrt{7}$ results from it. I ask the parts.

$$\begin{cases} a + b = 10 \\ \frac{a}{b} = 7 + \sqrt{7} \end{cases} \tag{85}$$

(solution: $1\frac{23}{57} - \sqrt{\frac{700}{3249}}$ and $8\frac{34}{57} + \sqrt{\frac{700}{3249}}$)

[43] Pacioli does not calculate this further. The solution is negative: $-1\frac{5}{11}$.

III.2.65 [PPM1265]

Make me 3 parts of 10 such that the first is to the second as the second is to the third and multiplying the first by 3, the second by 4 and the third by 5 and these outcomes joined together make 35 exactly. I ask the parts.

$$\begin{cases} a+b+c=10 \\ \frac{a}{b}=\frac{b}{c} \\ 3a+4b+5c=35 \end{cases} \tag{86}$$

(solution: $\sqrt{8\frac{1}{36}}-\sqrt{9\frac{1}{36}}$, $\sqrt{36\frac{1}{9}}-3\frac{1}{3}$ and $4\frac{1}{6}-\sqrt{9\frac{1}{36}}$)[44]

III.2.66 [PPM1266]

Make me 4 parts of 10 such that the first is to the second as the second is to the third and the third is to the fourth and multiplying the first by 8 and the other by 4 and the other by 3 and the other by 1, these multiplications joined together make 16 exactly. I ask the parts.

$$\begin{cases} a+b+c+d=10 \\ \frac{a}{b}=\frac{b}{c}=\frac{c}{d} \\ 8a+4b+3c+d=16 \end{cases} \tag{87}$$

(solution: $\frac{2}{17}$, $\frac{8}{17}$, $1\frac{15}{17}$ and $7\frac{9}{17}$)

III.2.67 [PPM1267]

Make me 2 parts of 10 such that dividing the larger by the smaller and then the smaller by the larger and then these two outcomes joined together makes the root of 16 exactly. I ask the parts.

$$\begin{cases} a+b=10 \\ \frac{a}{b}+\frac{b}{a}=\sqrt{16} \end{cases} \tag{88}$$

(solution: $5+\sqrt{14\frac{10}{11}}$ and $5-\sqrt{14\frac{10}{11}}$)

III.2.68 [PPM1268]

Make me 2 parts of 10 such that multiplying the one against the other makes as five times as much as dividing the larger by the smaller. I ask the parts.

$$\begin{cases} a+b=10 \\ ab=5\left(\frac{a}{b}\right) \end{cases} \tag{89}$$

(solution: $\sqrt{5}$ and $10-\sqrt{5}$)

[44] Pacioli's solution is correct but this final result becomes corrupted in the text. Read: $9\frac{1}{6}-\sqrt{9\frac{1}{36}}$, $\sqrt{36\frac{1}{9}}-3\frac{1}{3}$ and $4\frac{1}{6}-\sqrt{9\frac{1}{36}}$.

III.2.69 [PPM1269]

Make me 2 parts of 12 such that multiplying the first by itself and joining 13 and multiplying the other by itself and detracting 11 and taking the root of the one and the other joined together makes 12. I ask the parts.

$$\begin{cases} a+b=12 \\ \sqrt{a^2+13}+\sqrt{b^2-11}=12 \end{cases} \tag{90}$$

(solution: 6 and 6)

III.2.70 [PPM1270]

Make me 2 parts of 12 such that multiplying each by itself and detracting 7 from the multiplication of the first and the multiplication of the second with 8 joined above and taking the root of each part and joined together makes 12 exactly. I ask the parts.

$$\begin{cases} a+b=12 \\ \sqrt{a^2-7}+\sqrt{b^2+8}=12 \end{cases} \tag{91}$$

(solution: $5\frac{73}{80}$ and $6\frac{73}{80}$)

III.2.71 [PPM1271]

Make me 2 parts of 10 such that multiplying the first by itself and detracting from 100 the root of this we secure. Next multiplying the other by itself and detracting from 101 and taking the roots of each of these remainders joined together makes 14 exactly. I ask the parts.

$$\begin{cases} a+b=10 \\ \sqrt{100-a^2}+\sqrt{101-b^2}=14 \end{cases} \tag{92}$$

(solution: $4\frac{291}{296}+\sqrt{14\frac{142904}{47616}}$ and $5\frac{5}{296}-\sqrt{14\frac{142904}{47616}}$)

III.2.72 [PPM1272]

Make me 2 parts of 10 such that dividing the larger by the smaller and that what comes out multiplied with the difference there is between one part and the other, it makes as much as joining 10 with the difference. I ask for the parts.

$$\begin{cases} a+b=10 \\ \frac{a}{b}(a-b)=(a-b)+10 \end{cases} \tag{93}$$

(solution: $6\frac{1}{4}-\sqrt{14\frac{1}{16}}$ and $3\frac{3}{4}+\sqrt{14\frac{1}{16}}$)[45]

[45] Pacioli solves the problem by taking x and $(10-x)$ for the two parts and arrives correctly at the equation $100+4x^2=50x$ but fails to give the correct answer. The solution is $\frac{15}{2}$ and $\frac{5}{2}$.

III.2.73 [PPM1273]

Make me 3 parts of 10 which are such that the first is to the second as the second is to the third and multiplying each by itself and these multiplications joined together make 40 exactly. I ask for the parts.

$$\begin{cases} a+b+c=10 \\ \frac{a}{b}=\frac{b}{c} \\ a^2+b^2+c^2=40 \end{cases} \tag{94}$$

III.2.74 [PPM1274]

Make me 2 parts of 10 such that multiplying the one with the other and dividing the multiplication by the root of the difference, the root of the difference results from it. I ask the parts.

$$\begin{cases} a+b=10 \\ \frac{ab}{\sqrt{a-b}}=\sqrt{a-b} \end{cases} \tag{95}$$

(solution: not given)

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References

[1] Arrighi, Gino (1966) "Spigolature di arithmetica medievale", *Physis*, 8, pp. 307-316.

[2] Arrighi, Gino (ed.) (1967) *Antonio de' Mazzinghi. Trattato di Fioretti, secondo la lezione del codice L.IV.21 (sec. XV) della Biblioteca degli Intronati di Siena*, Domus Galilaeana, Pisa.

[3] Arrighi, Gino (ed.) (1970) *Trattato d'abaco.* Dal Codice Ashburnhamiano 280 (359*-291*) della Biblioteca Medicea Laurenziana di Firenze, Domus Galileana, Pisa.

[4] Arrighi, Gino (ed.) (1987) *Paolo Gherardi, Opera mathematica: Libro di ragioni Liber habaci. Codici Magliabechiani Classe XI, nn. 87 e 88 (sec. XIV) della Biblioteca Nazionale di Firenze*, Pacini-Fazzi, Lucca.

[5] Bartoli Langeli, Attilio (2007) "Descrizione del manoscritto" in *Gli scacchi di Luca Pacioli*, pp. 75-99.

[6] Biagioli, Mario (1989) "The Social Status of Italian Mathematicians 1450-1600", *History of Science*, 27, pp. 41-95.

[7] Boncompagni, Baldassarre (1879), "Appendice di documenti inediti relativi a Fra Luca Pacioli", *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche*, 12 (1879), pp. 420-438.

[8] Boncompagni, Baldassarre (1857-62) *Scritti di Leonardo Pisano, matematico del secolo decimoterzo (2 vols.), I.: Liber Abbaci di Leonardo Pisano, II.: Leonardi Pisani Practica geometriae ed opuscoli*, Tipografia delle Scienze Matematiche e Fisiche, Rome.

[9] Calzoni, Giuseppe and Gianfranco Cavazzoni (eds.) (1996) *Tractatus Mathematicus ad Discipulos Perusinos*, Città di Castello, Perugia, (Reprinted in 2007 on 900 copies)

[10] Calzoni, Giuseppe (1992) "Linsegnamento della matematica applicata agli affari nel secolo XV a Perugia: linedito Tractatus mathematicus ad discipulos perusinos", *Rivista di Ragioneria e di Economia Aziendale*, 3-4, 1992, pp. 9-13.

[11] Cavazzoni, Gianfranco (1998) "Tractatus Mathematicus ad Discipulos Perusinos", in E. Giusti (ed.) *Luca Pacioli e la mathematica del Rinascimento*, Petruzzi Editore, Castello, pp. 199-208.

[12] Cardano, Girolamo (1545) *Ars Magna sive de regulis algebraicis*, Johann Petreius, Nürnberg. (English translation by Witmer, R. T. (1968) *Ars Magna or the Rules of Algebra*, M.I.T. Press, Cambridge, Mass., Reprinted by Dover Publications, New York, 1993).

[13] Curtze, Maximilian (ed.) (1902) "Der Briefwechsel Regiomontan's mit Giovanni Bianchini, Jacob von Speier und Christian Roder", Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance 1, in : *Abhandlungen zur Geschichte der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen begrndet von Moritz Cantor*, 12, pp. 185-336.

[14] Davis, Margaret Daly (1977) *Piero della Francesca's mathematical treatises: the Trattato d'abaco and Libellus de quinque corporibus regularibus*, Speculum artium 1, Longo Ravenna.

[15] Derenzini, Giovanna (1998) "Il codice Vat. Lat. 3129 di Luca Pacioli", in Enrico Giusti (ed.) *Luca Pacioli e la matematica del Rinascimento*, Petruzzi Editore, Città di Castello, pp. 169-92.

[16] Franci, Rafaella (ed.) (1983) *Maestro Gilio. Questioni d'algebra, dal codice L.IX.28 della Biblioteca Comunale di Siena*, Quaderni del Centro Studi della Matematica Medioevale, 6, Università degli studi di Siena, Siena.

[17] Franci, Rafaella and Laura Toti Rigatelli (1985) "Towards a History of Algebra from Leonardo of Pisa to Luca Pacioli", *Janus*, 72 (1-3), pp. 17-82.

[18] Franci, Rafaella and Marisa Pancanti (eds.) (1988) *Anonimo (sec. XIV). Il Trattato d'algibra dal manoscritto Fond. Princ. II.V.152 della Bibliotheca Nazionale di Firenze*, Quaderni del Centro Studi della Matematica Medioevale, 22, Uni-

versità degli studi di Siena, Siena.

[19] Giusti, Enrico (1991) "L'Algebra nel *Trattato d'abaco* di Piero della Francesca: osservazioni e congetture", *Bollettino di Storia delle Scienze Matematiche*, 11 (2), pp. 55-83.

[20] Giusti, Enrico (1993) "Fonti medievali dell'*ALgebra* di Piero della Francesca", *Bollettino di Storia delle Scienze Matematiche*, 13 (2), pp. 199-249.

[21] Heeffer, Albrecht (2007) "Abduction as a Strategy for Concept Formation in Mathematics: Cardano Postulating a Negative" in Olga Pombo and Alexander Gerner (eds.) *Abduction and the Process of Scientific Discovery*, Coleco Documenta, Centro de Filosofia das Cincias da Universidade de Lisboa, Lisboa, pp. 179-194, 2007.

[22] Heeffer, Albrecht (2007) "The Genesis of the Algebra Textbook: From Pacioli to Euler" in Mahesh C. Joshi (ed.) *International Conference Advances in Mathematics: Historical Developments and Engineering Applications*, December 19-22, 2007, G.B. Pant University, Pantnagar India.

[23] Heeffer, Albrecht (2008) "A Conceptual Analysis of Early Arabic Algebra" in T. Street, S. Rahman and H. Tahiri (eds.) *The Unity of Science in the Arabic Tradition: Science, Logic, Epistemology and their Interactions*, Dordrecht: Kluwer Academic Publishers, pp. 89-129.

[24] Heeffer, Albrecht (2008) "Text production reproduction and appropriation within the abbaco tradition: a case study", *Sources and Commentaries in Exact Sciences*, 9, pp. 101-145.

[25] Høyrup, Jens (2002) *Lengths, widths, surfaces. A Portrait of Old Babylonian Algebra and Its Kin*, Springer Verlag, Heidelberg.

[26] Høyrup, Jens (2007) "Review of Menso Folkerts, The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus. (Variorum Collected Studies Series CS811). Aldershot: Ashgate, 2006", *Aestimatio*, (2007) 4, pp. 122-141.

[27] Høyrup, Jens (2008) "The Tortuous Ways Towards a New Understanding of Algebra in the Italian Abbacus Schools ($14^{th} - 16^{th}$ centuries)", in O. Figueras, J. L. Corina, S. Alatorre, T.Rojano, A. Sepúlveda (eds), *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX*, Morelia, Mexico, July 17-21, 2008, vol. I. México: Cinvestav-UMSNH, 2008, pp. 1-15.

[28] Høyrup, Jens (2009) "How to transfer the conceptual structure of Old Babylonian Mathematics: solutions and inherent problems. With an Italian Parallel" in Beitrag zur Tagung Zur Übersetzbarkeit von Wissenschaftssprachen des Altertums Johannes Gutenberg-Universität Mainz, 27.29 Juli 2009.

[29] Høyrup, Jens (2010) "Hesitating progress the slow development toward algebraic symbolization in abbacus and related manuscripts, c. 1300 to c. 1550" in A. Heeffer and M. Van Dyck, *Philosophical aspects of Symbolic Reasoning in Early Modern Science and Mathematics*, College Publications, London (forth-

coming).

[30] Hughes, Barnabas (1986) “Gerard of Cremona’s Translation of al-Khwārizmī’s al-Jabr: A Critical Edition”, *Mediaeval Studies*, 48, pp. 211-263.

[31] Jayawardene, S. A., (1998) “Towards a biography of Luca Pacioli”, in Enrico Giusti (ed.) *Luca Pacioli e la matematica del Rinascimento*, Petruzzi Editore, Città di Castello, pp. 19-28.

[32] Levey, Martin (1966) *The algebra of Abu Kamil. Kitab, fi al-jabr wa’l- muqabala in a commentary by Mordecai Finzi*, The University of Wisconsin Press, Madison.

[33] Mancini, Girolamo (1915) “L’opera ”De corporibus regularibus” di Piero Franceschi detto Della Francesca, usurpata da fra Luca Pacioli”, *Memorie della R. Accademia dei Lincei, Classe di scienze morali, storiche e filologiche*, 14, pp. 441-580.

[34] Murray, Harold James Ruthven (1913) *A History of Chess*, Oxford University Press, London.

[35] Pacioli, Luca (1494) *Summa de Arithmetica Geometria Proportioni et Proportionalita*, Paganino de Paganini, Venice.

[36] Pacioli, Luca (2007) *Gli scacchi di Luca Pacioli. Evoluzione rinascimentale di un gioco matematico* Aboco museum editione, Sepolcro, 2007.

[37] Marisa Pancanti. (ed.) (1982) *Giovanni di Bartolo. Certi chasi nella trascelta a cura di Maestro Benedetto* secondo la lezione del Codice L.IV.2l (sec. XV) della Biblioteca degli Intronati di Siena. Quaderni del Centro Studi della Matematica Medioevale, 2, Servizio editoriale dell’università di Siena, Siena.

[38] Picutti, Ettore (1989) “Sui plagi matematici di frate Luca Pacioli”, *La Scienze*, 246, pp. 72-9.

[39] Pieraccini, Licia (ed.) (1983) M Biagio. *Chasi exenplari alla regola dell’algibra.* Nella trascelta a cura di Maestro Benedetto. Dal Codice L.VI.21 della Biblioteca Comunale di Siena, Quaderni del Centro Studi della Matematica Medioevale, 5, Università di Siena, Siena.

[40] Postma, Johanna and Anne J. van der Helm (2000) “La Riegola de Libro, Bookkeeping instructions from the mid-fifteenth century”, Paper for the 8th World Congress of Accounting Historians Madrid, Spain, 19-21 July 2000 (online at http://home.hetnet.nl/ annejvanderhelm/paper.html)

[41] Rosen, Frederic (1831) *The Algebra of Mohammed Ben Musa*, The Oriental Translation Fund, London (Reprinted by Olms Verlag, 1986).

[42] Sesiano, Jaques (1985) “The Appearance of Negative Solutions in Mediaeval Mathematics”, *Archive for History of Exact Sciences*, 32 (2), pp. 105-150.

[43] Sesiano, Jacques (1993) “La version latine médiévale de l’Algèbre d’Abū Kāmil” in Menso Folkerts and J. P. Hogendijk (eds.) (1993) *Vestigia Mathematica. Studies in medieval and early modern mathematics in honour of H. L. L. Busard*, Rodopi, Amsterdam, pp. 313-452.

[44] Sigler, Laurence (2002) *Fibonacci's Liber Abaci. A Translation into Modern English of Leonardo Pisano's Book of Calculation*, Springer, Heidelberg.

[45] Céu Silva, M. (2008) "The algebraic content of Bento Fernandes's Tratado da arte de arismetica (1555)", *Historia Mathematica*, vol. 35 (3) pp. 190-219.

[46] Simi, Annalisa (ed.) (1994) *Anonimo (sec. XIV). Trattato dell'Alcibra Amuchabile. Dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze*, Quaderni del Centro Studi della Matematica Medioevale, 22, Università degli studi di Siena, Siena.

[47] Taylor, Emett R. (1972) *No Royal Road. Luca Pacioli and his times*, The University of North Carolina Press, Chapell Hill.

[48] Ulivi Elisabetta (2007) "Antonio di Giovanni Mazzinghi ed il Petriboni, due abacisti del XV secolo", *Bolletino di storia delle scienze matematice*, 27 (1), pp. 1-49.

[49] Vasari Giorgio (1550) *Le vite de' più eccelenti pittori, scultori et architettori*, (reprint: Biblioteca dell'immagine, Pordenone, 1984)

[50] van Egmond, Warren (1978), "The Earliest Vernacular Treatment of Algebra: *The Libro di Ragioni* of Paolo Gerardi", *Physis*, 20, pp. 155-189.

[51] van Egmond, Warren (1980) *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abacus Manuscripts and Printed Books to 1600*, Monografia, 4, Florence: Istituto e Museo di Storia della Scienza.

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