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INTERPRETING A NEWLY DISCOVERED MATHEMATICAL DOCUMENT WRITTEN AT THE BEGINNING OF THE HAN DYNASTY IN CHINA (BEFORE 157 B.C.E.) AND EXCAVATED FROM TOMB M77 AT SHUIHUDI（睡虎地）

Karine Chemla 林力娜
REHSEIS, UMR 7219 SPHERE, CNRS
University Paris Diderot, Sorbonne Paris Cité
France
chemla@univ-paris-diderot.fr

MA Biao 馬　彪
Faculty of Humanities
Yamaguchi University
Japan[1]
biaoma@yamaguchi-u.ac.jp

1. Introduction: New evidence on the mathematics of early China

Recently many mathematical documents from early imperial China have come to light. In 1984, a new book was excavated at Zhangjiashan 張家山: the *Book of mathematical procedures* (筭數書 *Suanshushu*), and several publications have already made its content available to a Western readership.[2] However, this was only the first of a series of recent discoveries. In the last few years, two new mathematical manuscripts have emerged together with other documents and artifacts. The first pieces of information regarding the archeological context and the content of these mathematical documents have just been

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[2] A first critical edition with annotations was published in 2001: [PENG Hao 彭浩, 2000]. The first translation into a foreign language was: [Jochi Shigeru 城地茂 2001]. Two translations into English have appeared: [Cullen, 2004]; [Dauben, 2008]. A critical edition and translation into Japanese and Chinese has also appeared: [張家山漢簡『算數書』研究会編 Chôka san kankan Sansûsho kenkyûkai. Research group on the Han bamboo strips from Zhangjiashan Book of Mathematical Procedures, 2006].

published. To begin with, let us give a summary of the information available on them.

According to the first publications providing detail about the discoveries, the earliest of the two mathematical books found dates from the Qin dynasty (221 BCE—206 BCE) and was entitled Shu 數 (*Mathematics*).[3] This book was bought in December 2007 on the Hong Kong antiquities market, being the product of illegal excavation. It is now kept in the Yuelu Academy 嶽麓書院 (Hunan University) along with other documents that were bought in the same lot (more than 1300 bamboo strips[4] altogether) and they are believed to come from the same site [Chen Songchang 陳松長, 2009, 75]. In August 2008, a collector from Hong Kong bought a second lot (more than 30 strips) and gifted it to the Yuelu Academy 嶽麓書院 (Hunan University). On the basis of the shape of the strips, the writing and the content, the two lots are believed to have come from the same tomb [Chen Songchang 陳松長, 2009, 75]. Several papers devoted to different aspects of the book *Mathematics* appeared in 2009.[5] [Xiao Can 肖燦 and Zhu Hanmin 朱漢民, 2009b, 48] suggest that *Mathematics* is a prototype of *The Nine Chapters on mathematical procedures*, the first book devoted to mathematics (1st century CE) and handed down through the written tradition. We shall not discuss this idea here, but mention it because it gives a rough idea of the content of *Mathematics*.[6]

By contrast, the second mathematical book that was found was excavated by a group of archeologists, so its archeological context is known. According to the excavation report and the first papers published by the archeologists working on this source material, the book, excavated from tomb M77 at Shuihudi (Yunmeng county, Hubei province), was copied at the beginning of the Han dynasty in China, before 157 B.C.E. The book contains 216 strips that are still in good condition and form a roll. The title of the book, 算術 *Suanshu* (*Mathematical procedures*), can be found on the verso of strip 1.[7]

[3] [Chen Songchang 陳松長, 2009, 75, 85], respectively. The title is on the verso of strip 0956, whose photograph is reproduced on p.83. [Chen Wei 陳偉, 2009] questions the reading Chen Songchang offers for some of the characters in this set of documents from the Yuelu Academy.

[4] Let us remind the reader of the fact that the mathematical manuscripts that were excavated were all written on bamboo strips. Originally, the strips were linked to each other by vegetable fiber cords to make a book. However, the cords decayed during the two thousand years they were buried underground. A key operation that archeologists face in order to publish critical editions of these documents is to attempt to recover the order in which the strips were originally arranged.

[5] [Xiao Can 肖燦 and Zhu Hanmin 朱漢民, 2009b, Zhu Hanmin 朱漢民 and Xiao Can 肖燦, 2009], discuss the overall content of *Mathematics* and also focus on the geometric knowledge that the book contains. [Xiao Can 肖燦 and Zhu Hanmin 朱漢民, 2009a] discuss the units of measurement with which quantities of grain were measured in Qin China.

[6] For a critical edition and a translation into French of *The Nine Chapters*, compare [Chemla and Guo Shuchun, 2004].

[7] [Hubei sheng wenwu kaogu yanjiusuo 湖北省文物考古研究所 and Yunmeng xian bowuguan 雲夢縣博物館, 2008]

For the moment, we know very little about the content of this mathematical document. However in [Hubei sheng wenwu kaogu yanjiusuo 湖北省文物考古研究所 and Yunmeng xian bowuguan 雲夢縣博物館, 2008], on plate 15, a photo of several strips was published. The strips contain several tables, for which we plan in this article to offer a critical edition and a translation with annotations. In addition to discussing the interpretation of this piece of the text, we shall cast light on some of the relationships that tie this excerpt from *Mathematical procedures* to other mathematical documents, whether they be other documents excavated or books handed down through the written tradition. After analyzing these parallels, we shall offer some ideas about the relationships between the earliest extant mathematical documents from China and about the transmission of mathematical knowledge from the 2nd century B.C.E. through the 8th century.

2. A preliminary: the *shilü* 石率 procedure in the *Book of mathematical procedures*[8]

Before we turn to examining the passage in *Mathematical procedures* which we are interested in, it will be useful to recall some details relating to a procedure found in the *Book of mathematical procedures*. The reason for this is that we shall suggest that part of our excerpt relates closely to a subprocedure in this procedure and can be understood in relation to it. Let us thus, to begin with, focus on the procedure for the shilü operation "*lü*-ing with the *shi*."[9] We shall however, at least for the moment, leave *lü* untranslated. The shilü procedure is expressed in an abstract way, outside the context of a particular problem. For the sake of simplicity, we shall present it in relation to the single problem that is solved by an explicit reference to it (bamboo strips 76-77, [PENG Hao 彭浩, 2001,

[8] On the pronunciation of 石 as shi or dan, see the forthcoming paper by Ma Biao, based on a conference presented at the University Paris Diderot on February 16, 2010, "Reading shi or reading dan for a unit of capacity: the history of a divergence between Chinese and Western scholarships on the pronunciation of the name of a unit measure." Note that below, we shall meet with another character which reads shi and means "ten." To avoid confusion, when shi refers to this character, we shall systematically add the meaning "ten" to it.

[9] Here we rely on the interpretation of the text published in [Chemla, 2006, Chemla, 2010]. For details, the reader is referred to these publications. Here, we simply state the conclusions. The critical edition of the procedure can be found in [PENG Hao 彭浩, 2001, 73—74], bamboo strips 74-75. Note that in this procedure, *shi* refers to the highest unit of measurement for measuring units of weight and for measuring units of capacity. Smaller units differ, depending on whether they relate to weight or to capacity. As for the meaning of *lü*, let us state for the sake of simplicity that it refers to the computation of the price for one unit, here the *shi*, of something, given the price for another quantity of the same thing. *Lü* is used here verbally. The reader will find a discussion of the meanings of the concept and its evolution in [Chemla and Guo Shuchun, 2004, 956—959].

74]). The problem reads as follows:

> "Trading salt Suppose one has 1 *shi* 4 *dou* 5 *sheng* 1/3 *sheng* salt and that when trading it, one obtains 150 cash. If one wants to "*lü*" it (that is, to determine the unit price of salt) with the *shi* (that is, with respect to the higher capacity unit), how much cash does this make? One says: 103 cash 95[2]/43[6] cash[10] ."
>
> 賈鹽 今有鹽一石四斗五升少半升，賈取錢百五十，欲石術（率）之，為錢幾何∟?曰：百三錢四百卅（三十）$\boxed{六}$分錢九十五[二]。/76/.

In other words, for a given amount of cash, one trades an amount of salt, which is expressed with several units of capacity and a fraction. The question is: how much cash corresponds to a given unit of capacity, here the *shi*? The idea put into play in the algorithms for solving this category of problems in the *Book of Mathematical Procedures* is to apply the rule of three. In modern terms, the algorithm can be represented by the formula:

$$\frac{cash\, multiplied\, by\, 1\, unit\, (shi)}{quantity\, bought}$$

The algorithm first converts, simultaneously and in the same way, the unit (1 *shi*) and the quantity bought –the former in the dividend and the latter in the divisor–, so as to turn them into integers. Only then are the operations —multiplication and division— executed. The outcome of these conversions can be represented by the following formula:

$$\frac{cash\ multiplied\ by\ 1\ unit\ (shi)\ expressed\ in\ the\ same\ unit\ as\ the\ divisor}{quantity\ bought\ expressed\ with\ respect\ to\ a\ unit\ with\ which\ the\ quantity\ becomes\ an\ integer}$$

As for the sequence of conversions, it amounts to the following operations:

$$\frac{cash\ multiplied\ by\ 1\ unit\ (shi)}{quantity\ bought} = \frac{cash\ multiplied\ by\ 1\ unit\ u_1}{q_1u_1 + q_2u_2 + \frac{m}{n}u_2}$$
$$= \frac{cash\ multiplied\ by\ n.1\ unit\ u_1}{nq_1u_1 + nq_2u_2 + mu_2}$$

and if $u_1 = k_1u_2$

[10] The "6" which is between brackets is not legible in the text. There is the space for a missing character and one can suggest the restoration on the basis of computation. In the Chinese text, we indicate the fact by a square around the character. As for the "2," the scribe has wrongly copied "5." We keep the original text, but we indicate the correct digit between squared brackets, in the translation as well as in the Chinese text. These remarks are based on [PENG Hao 彭浩, 2001, 74], footnote 2.

$$= \frac{cash\ multiplied\ by\ n.k_1u_2}{nq_1k_1u_1 + nq_2u_2 + \ mu_2}$$

This sequence of conversions is described in the text of the algorithm associated with this particular problem as follows:

> "Procedure: One triples the quantity of salt, which is taken as divisor. One also triples the quantity of *sheng* of 1 *shi*, and with the cash, one multiplies it, which is taken as dividend."
>
> 术(術)/76/曰：三鹽之數以為法，亦三一石之升數，以錢乘之為實。/77/

Let us now consider the general algorithm solving this category of problem, which Peng Hao has chosen to place directly before the specific problem and procedure just mentioned. As has already been said, this general text does not appear to be associated with any specific problem. Let us translate it before suggesting an interpretation within the framework of the example of the previous problem. The text reads:

> "*Lü*-ing with the *shi* Procedure for *lü*-ing with the *shi*: One takes what is exchanged as divisor. One multiplies, by the cash obtained, the quantity of 1 *shi*, which is taken as dividend. Those for which, in their lower (rows), there is a half, one doubles them; (those for which there is) a third, one triples them. Those for which there are *dou* and *sheng*, *jin, liang* and *zhu*, one also breaks down all their upper (rows), one makes the (rows) below join them, (yielding a result) which is taken as divisor. What the cash was multiplying is also broken down like this."
>
> 石衛（率） 石衛 (率)之术（術）曰：以所買=（賣）為法，以得錢乘一石數以為實。其下有半者倍之，少半者三之，有斗、升、斤、兩、朱（銖）者亦皆//破其上，令下從之以為法。錢所乘亦破如此。／７４-７５／

The interpretation of the text that we use depends, not only on the problem quoted above, but also on hypotheses regarding the use of the calculating table to which the *Book of Mathematical Procedures* refers (see Figures 2-7 below).[11] Step by step**:**

1. "One takes what is exchanged as divisor. One multiplies, by the cash obtained, the quantity of 1 *shi*, which is taken as dividend." 以所買＝（賣）為法，以得

[11] In these figures, we use Arabic numbers in place of the configurations of counting rods with which, in ancient China, figures were written down on the calculating table. On the system of counting rods and the use of a calculating table to compute elementary operations with them, compare [Lam and Ang, 2004, 20-47, Martzloff, 1987, 169-171, 204-205, Qian Baocong 錢寶琮, 1964, 7-12]. On the principles constraining the use of the calculating table with counting rods, see [Chemla, 1996].

錢乘一石數以為實 。

The terms of dividend and divisor refer to, respectively, the middle and the lower rows of the calculating table. When the division is executed, the quotient is progressively placed in the higher row. In the case of the procedure analyzed, what is placed in the middle row are the terms of the multiplication, placed according to the usual layout for a multiplication (see below). Each row can become the space in which an operation can be set up. Here the multiplicand and multiplier are placed in sub-rows of the middle row, as per usual for a multiplication: the multiplier is in the higher sub-row, the multiplicand in the lower one. However, although the terms of the operations are set up, neither the division nor the multiplication seem to be executed at this point, since several terms will undergo conversion before the main operations are carried out (see below). Exactly the same thing occurred in the sequence of formulas above: it presented multiplications and divisions, and modified their terms before they were executed. This phenomenon also appears in the text of the algorithm for division examined above.

Quotient	Below — not indicated	any longer
Dividend	1 *shi* multiplied by 150 cash	
Divisor	1 *shi* 4 *dou* 5 *sheng* 1 3 *sheng*	 Upper Middle Lower

Figure 1: The first step in the use of the calculating table

Finally, the quantity placed in the position for the divisor comprises several units and a fraction. In the interpretation suggested here, the lower unit associated with an integer is placed in the middle sub-row of the lower position, whereas the larger units are placed in the sub-rows above it, and the fractions horizontally (numerator on the left, denominator on the right) in the sub-rows under it[12]. The initial configuration thus resembles Figure 1.

[12] This interpretation of the term "below *xia* 下," which occurs here in the text of the *Book of mathematical procedures*, with the related layout of the fraction, is the same as the interpretation that [Li Jimin, 1990, 87—91] and Chemla's introduction to Chapter 4 [Chemla and Guo, 2004, 313—322] both give for the same term in the procedure "Small width 少廣 *shaoguang*" in *The Nine Chapters*. Despite these common points, there are slight differences between these two interpretations of "Small width," on which we need not comment here. There are reasons to believe that the two procedures "*Lü*-ing with the *shi*" and "Small width" are related. They share the fact that their layouts put different units of measurement that enter

2. "Those for which, in their lower (rows), there is a half, one doubles them; those for which there is a third, one triples them." 其下有半者倍之，少半者三之。

The text now turns to the examination of cases in which the quantity exchanged includes fractions. Later, it prescribes what to do in cases where the quantity contains more than one unit from a series of units of measurement. In other words, the text encompasses several types of cases and gives sequences of actions to be followed depending on the particular case encountered. Both types of case are illustrated by the problem which we examined above.

In cases where there are fractions, one has to multiply the quantity in the divisor position (i.e., each of the rows constituting it), by the denominators of these fractions. This operation is prescribed in a new indirect way; that is, by a simple enumeration of two paradigmatic cases and the specific action that they require. A similar type of prescription will be chosen in the following sentences. If there is no fraction, the practitioner skips this sentence when deriving actions from the text. However, the sentence must, in any case, be read. For our example, the sentence prescribes actions that lead to the configuration in Figure 2:

Dividend	1 *shi* multiplied by 150 cash	
Divisor	3 *shi* 12 *dou* 15 *sheng* 1 (3) *sheng*	 Upper Middle Lower

Figure 2: The second step in the use of the calculating table

The next step contains the first key step that will be of interest for us below:

3. "Those for which there are *dou* and *sheng*, *jin*, *liang* and *zhu*, one also breaks down all their upper (rows)." 有斗、升、斤、兩、朱(銖)者亦皆//破其上。

As above, the general possibility that there is more than one unit in the quantity exchanged is expressed by an enumeration of two specific cases. Each of these cases is itself formulated as an enumeration: The quantity would have either two units from the series of capacity units or three from the series of weight units, both enumerations listing

the quantity statement in different lines of the calculating table on which computations were carried out.

units smaller than the *shi,* which both series have as their largest unit.

The prescription uses the expression “one also breaks down….”[13] The fact that the text underlines it by “also” implies that the operation meant is a multiplication, as in Sentence 2 above. This fact highlights why, even if there is no fraction in the quantity by which one divides, the practitioner still needs to read the related sentence to make sense of the following one. What needs to be multiplied is made clear: the operation is to be executed on “all the upper (rows)” (皆…其上) in the quantity placed in the position of the divisor, that is, “all the rows” above the middle one, in which the smaller unit is placed. We see here that the layout leads to differentiating the “upper rows” from the middle row, in which the amount in *sheng* is kept, in a way that is essential for the prescription as carried out in the text to work correctly. This leads to the configuration in Figure 3:

Dividend	1 *shi* multiplied by 150 cash	
Divisor	300 120 15 1	 Upper Middle Lower

Figure 3: The third step in the use of the calculating table

Through the operation of “breaking down” the higher units, the quantities placed in the upper rows are converted into *sheng* (or, to be more precise, into thirds of *sheng*, which is the new basic unit, after the multiplication by 3). Sentence 4 simply prescribes adding up all the lower rows to the sum of all these quantities disaggregated into the same units in the divisor position, all rows having by this point been converted into the same unit. The prescription reads:

[13] The fact that the verb *po* 破 can be understood as having the same meaning as *pou* 剖 can be established on the basis of 李善 Li Shan’s commentary (Tang dynasty) on Mu Hua 木華’s rhapsody entitled “The sea” in the framework of Li Shan’s commentary on the *Wen xuan* 文選 *Selections of refined literature*. There Li Shan comments on *pou*, writing: “*pou* is like *po*.” In addition，the Jin dynasty commentator Du Yu 杜预， in his commentary on the *Master Zuo’s Commentary on the Spring and Autumn Annals* commented on the character *pou* 剖, as occurring in the Year 14 of Duke Xiang 襄公, “cut in its middle is *pou* 中分為剖.”

4. "One makes the (rows) below join them, (yielding a result) which is taken as divisor."[14]
   令下從之以為法。

The last two steps will prove essential for our argument below. The fifth and final sentence of the procedure prescribes converting the 1 *shi* in the dividend as was done before:

5. "What the cash was multiplying is also broken down like this."
   錢所乘亦破如此.

Let us discuss the interpretation of this sentence piece by piece. "What the cash was multiplying" designates the "1 *shi*" by the operation involving it in Sentence 1. However, this operation, by means of which the value "1 *shi*" is indicated, was not executed then, since one of its terms is now to be modified.[15] The procedure used to modify this "1 *shi*" is the one that modified the quantity in the divisor, and it is signified as "like this." The prescription simply indicates that the procedure to be applied to 1 *shi* is the same one needed to apply to the quantity in the divisor, depending on the fractions and list of units contained in the value originally taken as divisor. In our example, the procedure involves multiplying by 3 and converting into *sheng*. It yields the configuration in Figure 4:

Dividend	300 multiplied by 150 cash	
Divisor	436	

Figure 4: The fourth step in the use of the calculating table

We can now go back to the excerpt from *Mathematical procedures* on which we concentrate in this article.

[14] Note that the same term "divisor" designates different values at different points in the flow of computations, depending on what is to be found in the lower row in which the divisor is put, when the "divisor" is called for. This is one of the many examples of the use of the "assignment of variables" in ancient Chinese algorithm texts.

[15] The 1 by which the amount of cash was supposed to be multiplied will now be modified. This explains why we have suggested not executing the multiplication immediately.

•一乘一=也 十乘千萬也 廿四朱一兩 以石求斤十而倍之六之

一乘十=也 十乘萬十萬也 十六兩一斤 以斤求兩十倍八之

一乘百=也 百乘千十萬也 卅斤一鈞 以兩求朱十而三八之

一乘千=也 百乘萬百萬也 四鈞一石

一乘萬=也 千乘萬千萬也 劵十朱夫反十

十乘十百也 萬乘萬==也 劵朱升之

十乘百千也 百乘百萬也

Figure 5a: Seven bamboo strips from *Mathematical procedures* (transcription)

• 一乘一，一也。 十乘千，萬也。 廿四朱（銖）一兩。 以石求斤，十，[十]＊而倍之，六之。

一乘十，十也。 十乘萬，十萬也。 十六兩一斤。 以斤求兩，十，倍，八之。

一乘百，百也。 百乘千，十萬也。 卅斤一鈞。 以兩求朱（銖），十而三，八之。

一乘千，千也。 百乘萬，百萬也。 四鈞一石。

一乘萬，萬也。 千乘萬，千萬也。 券十朱（銖）夫反十。

十乘十，百也。 萬乘萬，萬萬也。 券朱（銖）升之。

十乘百，千也。 百乘百，萬也。

＊與下文的「十」相對比可知，原文此處的「十」下面脱落了一个重文符號，所以釋文中補出一個「十」字，爲了與原文相區別，以[十]表示。

Figure 5b: Seven bamboo strips from *Mathematical procedures* (edition)

3. Critical edition and translation of an excerpt from *Mathematical procedures*

On the basis of the previous discussion, we are now in a position to offer a translation and interpretation of the excerpt from *Mathematical procedures* that was published in Plate 15 of [Hubei sheng wenwu kaogu yanjiusuo 湖北省文物考古研究所 and Yunmeng xian bowuguan 雲夢縣博物館, 2008]. The critical edition is presented in Figure 5, with a layout similar to what the bamboo strips show. The reader is reminded of the fact that, usually bamboo strips were read as Chinese books written vertically today are read. The reader starts reading from right to left and from the top down.

To begin, let us describe this layout (fig. 5a, 5b).[16] The text that we are translating is composed of 7 bamboo strips, which are arranged in four superposed parts. This feature correlates to the fact that these strips are not to be read in the usual way. Even though they are read from right to left, they are not to be read from the upper first character of the first column to the right to the lowest character of the strip before moving to the next strip to the left. More precisely, the first part in the text —the upper one—is to be read first, from right to left and top to bottom before moving to the second part of the text, below it, and so on. Note that the published photo contains three other strips, whose layout is completely different. These three strips are to be read in the usual manner from right to left and top to bottom, since the text appears on them in a continuous way. Due to the fact that the layout of the translated text is completely different, we know that the seven strips on which we are concentrating form a separate segment within the whole book. Moreover, these seven strips start with a black dot placed in the upper margin of the very first strip. This feature further indicates that the text starting at this point constitutes a new section of the whole book.[17]

Let us go back to the layout of these seven bamboo strips. In fact, first two parts of our text —counted from the top— compose one unit of the text, whereas the next two parts probably each constitute a unit. This description is based on the meaning of the text. However, it also relates to physical features of the strips. One can see traces of cords that once linked the strips together at the top of the upper part of the text. Two other sets of similar cord traces can be seen: a first set appears immediately after the second part of the text, that is, after the first unit of the text. The second set appears after the fourth part of the text. However, at the end of the third part of the text, there is a blank column, which indicates that the fourth part is conceived of as being separate to the third part. On the

[16] Note that we describe the set of bamboo strips, arranging them from right to left, which is the order in which, in our view, they should be read in Chinese. However, this order is the reverse of the one shown in the photo of the bamboo strips given in Plate 15.

[17] This use of the upper dot seems to be a feature in the manuscript to identify sections in the text. This can be perceived from the three other bamboo strips whose photo is published. They also form a section and the first in them starts with a dot placed in the upper margin.

basis of this description, we shall translate the text as follows: we first translate the upper part of the bamboo strips, from right to left, then the second part, from right to left. This will give us the first unit of the text. We then move to the third and fourth parts of the text, each read from right to left. They constitute the two following units of the text.[18] Note that, thanks to the layout of the text, we are certain that we have complete text units.

For the sake of the discussion, we shall introduce the following numbering. Each part of the text is divided into sentences that each appear in a column. We shall associate either a letter or a number to the sentences appearing in the successive columns and within a given part of the text. Each of the sentences appearing in a column of the upper part will receive a letter from "a" to "g." The sentences of the following part, which belong to the same unit, will be identified by the letters "h" to "n," according to the column in which they appear. The sentences placed in the successive columns of the third part will be given capital letters, from "A" to "F" respectively. As for the sentences in the lower part, we shall use "I," "II," "III," respectively. We shall introduce this notation between brackets in the translation. Needless to say, they do not belong to the original document, but are only meant for our discussion.

Let us present how we propose to translate the text written on these seven bamboo strips, before we provide the explanations accounting for our critical edition and our interpretation:

" (a) One multiplied by one, one.
(b) One multiplied by ten, ten.
(c) One multiplied by a hundred, a hundred.
(d) One multiplied by a thousand, a thousand.
(e) One multiplied by ten thousand, ten thousand.
(f) Ten multiplied by ten, a hundred.
(g) Ten multiplied by a hundred, a thousand.

[19]

(h) Ten multiplied by a thousand, ten thousand.
(i) Ten multiplied by ten thousand, a hundred thousand.

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[18] Note that we have here two different phenomena. The first part of the text which is composed of two registers on the seven bamboo strips is written first continuously in the first register and then continues in the same way in the second register. The two registers are separated by a space running across the seven bamboo strips. Moreover, the meaning of the text recorded in these two registers leaves no doubt as to how the text is to be read. The continuous text was thus written in two registers, even though it could have been written in a single one. This way of writing the text is similar to the way in which the lower two registers are physically organized. However, the two texts recorded in these registers are independent pieces.

[19] The empty line corresponds to the fact that the upper register is separated from the second register following it by an empty space extending across all the strips. The same holds true for the subsequent empty lines.

(j) A hundred multiplied by a thousand, a hundred thousand.[20]
(k) A hundred multiplied by ten thousand, a million.
(l) A thousand multiplied by ten thousand, ten million.[21]
(m) Ten thousand multiplied by ten thousand, a hundred million.
(n) A hundred multiplied by a hundred, ten thousand.[22]

(A) Twenty-four *zhu* (make) one *liang*.[23]
(B) Sixteen *liang* (make) one *jin*.
(C) Thirty *jin* (make) one *jun*.
(D) Four *jun* (make) one *shi*.
(E) To break down (the quantity obtained expressed in terms of *shi*, *jun*, *jin*, *liang*) into tens of *zhu*, going backwards, one ten-ifies.[24]
(F) The *zhu* into which one broke down (the other units), one makes them go up (adding them to one another) (alternatively: (one places) them as *sheng* (on the calculating table))[25].
(empty column)
(I) If having *shi*, one looks for *jin*, ten-ifying (the quantity of *shi*), one decuples

[20] This operation shares the same result with the previous one. However, if we rely on the pattern established in sentences (a) and (f), one may surmise that the text has between these two sentences the following sentence: “百乘百，萬也 A hundred multiplied by a hundred, ten thousand.” Note that sentence (m) also shares the same pattern. It seems that the copyist forgot this sentence, while making the copy, and that, realizing his omission, he added the sentence at the end of the table, as sentence (n). However, one may also consider that such an omission occurs twice, since there is no sentence “千乘千，百萬也 A thousand multiplied by a thousand, a thousand thousand,” corresponding to the pattern described above. It may thus very well be that the copyist deleted two sentences of the same type.

[21] See footnote 20.

[22] See footnote 20.

[23] Sentences (A) to (D) give the relationships between the various units of weight. The same relationships are expressed in another format in the *Book of mathematical procedures*. There, one reads in bamboo strip 47: “24 *zhu* is 1 *liang*. 384 *zhu* is 1 *jin*. 11520 *zhu* is 1 *jun*. 46080 *zhu* is 1 *shi*. 廿四朱(銖)一兩∟ 。三百八十四朱(銖)一斤∟ 。萬一千五百廿朱(銖)一鈞∟ 。四萬六千八十朱(銖)一石。” We shall discuss these different formats in another publication. The same system of weight measuring units is described in the official history of the former Han dynasty, the *History of the Han*, compiled by Ban Gu and completed by his sister Ban Zhao shortly after his death. Without entering into a complex explanation, let us state that the latter book was written *ca* 92 CE [Hulsewé, 1993a]. The chapter entitled “Monograph on pitch pipes and the calendar” describes the whole system of measuring units supposedly used during the former Han dynasty (206 BCE—9 CE). On this system of measuring units, we shall develop this idea in another article.

[24] On the meaning of this sentence, and especially “tens of *zhu*” and the neologism “ten-ify,” see the discussion below.

[25] The alternatives are discussed below.

and then doubles it and sextuples it.

(II) If having *jin*, one looks for *liang*, ten-ifying (the quantity of *jin*), one doubles it and octuples it.

(III) If having *liang*, one looks for *zhu*, ten-ifying (the quantity of *liang*), one triples it and octuples it.

(empty column)

(empty column)

(empty column)

(empty column)."

Clearly, the excerpt translated above deals with basic computations in a decimal system and with the conversion of quantities expressed with respect to different units of weight measurement into one another. Note that, in the system of measurement units used in the earliest extant bamboo strips and attested to —with a slight change— in the *History of the Han*, it was only the system of weight measurement that was not decimal.[26] Let us now discuss our interpretation and the key features of the text.

How computation was carried out in the background of *Mathematical procedures* 算術.

The newly discovered manuscript evidence about mathematical practices in Qin times contains a key element in the history of mathematics in China, which will serve as a background to our interpretation of the excerpt which we are concentrating on here. On bamboo strip J25 of *Mathematics* 數,[27] we find the verb *zhi* 置 "to put," designating the fact that some quantities must be placed on a computing tool. The same word is used in all subsequent Chinese mathematical documents until the 14th century to designate the use of counting rods to represent numbers in a place-value decimal system on a calculating table.[28] The use of this system implies that the representation of numbers with rods can be changed and can be moved. If *zhi* indicates the same system as early as the Qin document *Mathematics*, it was most probably also used in relation to *Mathematical procedures*. We shall see that our interpretation of the excerpt from *Mathematical procedures* relies on the hypothesis that the representation of the quantities being dealt with can be moved on the calculating table. This fact further reinforces the hypothesis that what was to be placed was a physical representation of the numbers. Moreover, we

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[26] [Ban Gu 班固 et al., 1962, 966—972]. In fact, non-decimal measurement units for length and area were used, as can be seen for instance in mathematical sources. These units included *li* 里, *bu* 步, *mu* 畝, *qing* 頃. However, interestingly enough, these units are not mentioned in the chapter of the *History of the Han* evoked above, in which one is supposed to find a description of the official system of measurement units used during the former Han dynasty. We shall come back to these units below and also in the other publication mentioned in footnote 23.

[27] [Zhu Hanmin 朱漢民 and Xiao Can 肖燦, 2009, 58].

[28] See footnote 11 for references on the system.

shall see that the excerpt grants fundamental importance to converting quantities into a decimal system. This feature echoes the table placed at the beginning of the translated passage, which describes the multiplication between quantities expressed with respect to a decimal system. So, we can be sure that what was "placed" was something physical and something written according to a decimal system. Whether that was the counting rod system or not, we cannot be sure. However, this is highly probable, as other evidence indicates that this system appeared no later than the 3rd century BCE onwards.[29]

Let us recall the earliest evidence on the basis of which previous authors inferred the existence of the counting rod system at the latest from the 3rd century BCE onwards. First, A. Volkov emphasized that the title of the manuscript recently excavated from a tomb sealed *ca* 186 BCE, the *Book of mathematical procedures,* contains the character 筭.[30] The main piece of evidence about the meaning of this character is to be found in the earliest dictionary of Chinese characters, the *Shuowen jiezi, Explaining graphs and interpreting characters,* completed by Xu Shen in 100 CE. Let us quote the dictionary's entry for the character *suan* 筭:

> "筭，長六寸，所以[31] 計曆數者。从竹弄。言常弄乃不誤也。
>
> *suan* 筭[32] are 6 *cun* long; they are a thing used to compute the numbers in the calendar. (The character *suan*) is composed of the (characters) *zhu* (i.e., bamboo) and *nong* (i.e., operate with, handle). This means: if one always operates (with them), then one does not make mistakes." [Duan Yucai 段玉裁, 1815(1981), 198]

The eighteenth century commentator on the *Shuowen jiezi,* Duan Yucai, quotes the chapter "Monograph on pitch-pipes and the calendar 律曆志" from the *History of the Han* 漢書, to explain this entry. The quotation reads as follows:

"The method for making the counting rods is to use bamboo sticks (of a cylindrical form)[33] with a diameter of one *fen* and a length of 6 *cun*. Two hundred and seventy-one

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[29] On the fact that we have evidence of the counting rod system indicating that the system existed no later than the third century BCE, see [Volkov, 2001]. The earliest evidence from sources handed down by the written tradition are discussed in [Li Yan 李儼, 1954, 156-160, Needham and Wang Ling, 1959, 70-72, Wang Ling, 1956, 77-83].

[30] Compare [PENG Hao 彭浩, 2001, Plate 1].

[31] The original character here is the ancient form of 以， which is similar to 目. But we could not reproduce the exact ancient form.

[32] The entry is a character, as the dictionary deals with characters. The meaning of the entry is "Counting rods." The translation of the entry of the *Shuowen jiezi* takes up [Volkov, 2001]'s interpretation with slight modification.

[33] The shape of a body was usually indicated in ancient China by means of the names of its key

rods generate a hexagon,[34] making a bundle." "算法用竹，徑一分，長六寸，二百七十一枚而成六觚，為一握。"[35]

Duan Yucai concludes:

> "This (character) thus means counting rods."[36]

On the basis of these quotations, we can infer that first century sources consider that *suan* 筭 refers to the counting rods used for computation. Excavated artifacts seem to confirm that the character 筭 referred to rods that were used to compute as early as the beginning of the Han dynasty (time period of the Emperor Wen, r. 179 BCE—156 BCE). This conclusion can be derived from a game set, found in tomb 3 at Mawangdui. The set includes a set of ivory rods, on which was inscribed the name of "象筭 ivory counting rods" and which may have been used to compute the players' winnings [He Jiajun, 2004, 98—99]. The fact that this character was used in the title of the *Suanshushu* together with the fact that the *Suanshushu* uses the verb 置 *zhi* "one puts" indicate that the book refers to a counting instrument based on counting rods. This seems to indicate that rods were used quite early. However, so far, it is not possible to determine whether these ancient rods from Mawangdui were already used for computing with a decimal place-value system or whether they were only used for divination. This is why we do not take it into account in our discussion. Other Han sources handed down through the written tradition confirm that the system to which the *Suanshushu* alludes existed quite early. Let us mention the reference made to the use of counting rods for computation in the book that was to become the first official history, *Records of the Grand Scribe* (史記 *Shiji*), which was almost completed when its author Sima Qian died in 86 BCE [Hulsewé, 1993b]. It is to be found in the collective biography of men who worked to increase the production of goods (Chapter *Huozhi liezhuan* 貨殖列傳)："When the Qin (kingdom) destroyed the Zhao kingdom, Mr Zhuo was sent to the Shu kingdom. (…) He arrived at a mountain (which had) iron, smelted and cast (iron), and, <u>operating the counting rods</u> (i.e., counting prices and profit), he sold to people from Dian and Shu,[37] so that he became rich to the extent that he had a thousand servants. 秦破趙，遷卓氏之蜀（…）即鐵山鼓鑄，<u>運</u>

dimensions. This is how we come to the interpretation that diameter and length designate rods having a cylindrical shape.

[34] The section of the body constituted by the 271 sticks is a hexagon. 271 is the 10th centered hexagonal number.

[35] Compare the critical edition of the *History of the Han*, vol. 4 [Ban Gu 班固 et al., 1962, 956]. In this text, the character written by Xu Shen as 筭 is written 算.

[36] [Duan Yucai 段玉裁, 1981, 198].

[37] We suggest interpreting 傾 *qing* as 賣 *mai*, because of the rewriting of the sentence carried out in the *History of the Han* for the same passage [Ban Gu 班固 et al., 1962, 3690].

籌算，傾滇、蜀民，富至僮千人 。”［Sima Qian 司馬遷, 1959, 3277］Here the use of computation with rods is associated with the activity of accounting. It is interesting that counting rods of the size described in the statement in the *History of the Han* quoted earlier were excavated from Han tombs (see photo on p.1 of plates in [WU Wenjun 吳文俊, 1998], volume 1). In conclusion, given the fact that the *Suanshushu* dates more or less from the same time as the book *Mathematical procedures* from which we are translating a passage, it is not far-fetched to surmise that this book also refers to the counting rod system. Probably, when the whole text is published, we shall have a clearer idea on this topic.

Multiplying between powers of 10.

The meaning of the first part of the text (sentences (a) to (n)) is clear. It describes how to multiply powers of 10 with each other. Since the first term of the multiplication is always smaller than the second, there is no redundancy in the table.[38] As for the remaining part, we shall argue that it describes how to convert a quantity of *zhu* expressed according to a decimal system into the measure of a weight expressed with respect to the whole scale of measurement units for weights and conversely, a weight expressed with respect to the whole scale of measurement units into a quantity of *zhu* expressed according to a decimal system. Moreover, sentences (I) to (III) provide a way of carrying out the “reverse procedure,” relying only on sequences of multiplications by a one-digit number. Let us now see how we have interpreted the key terms of the passage to reach this conclusion.

***Shi* 十 “ten,” “ten-ify,” “tens of.”**

The key character of the whole passage is the character *shi* 十, which usually means “ten.” The meaning “ten” is found, for example, in sentence (a) of our text. If we focus on the second half of the text (sentences (A) to (F) and (I) to (III)), and if we put aside the occurrence of *shi*-ten in sentence (B) in the expression “*shiliu* sixteen,” we see that the character *shi*-ten occurs five times. Let us observe the three occurrences in sentences (I) to (III). Clearly the three sentences are parallel to each other. Sentence (B) states that “Sixteen *liang* (make) one *jin*”. So we know that if we have a quantity expressed in *jin* and want to convert it into *liang,* we need to multiply it by 16. However, sentence (II) says “*shi*-ten, one doubles it and octuples it.” Multiplying by 2 and then by 8 is the same as multiplying by 16: why do we have “*shi*-ten” in the sentence? The same problem recurs in sentence (III), where one only needs to multiply by 3 and by 8 to convert *liang*

[38] Note that a similar list of multiplications is given in the *Book of mathematical procedures*, in bamboo strips 11 and 12, compare [PENG Hao 彭浩, 2001, 41]. In Plate 22 of [張家山漢簡『算數書』研究会編 Chôka san kankan Sansûsho kenkyûkai. Research group on the Han bamboo strips from Zhangjiashan *Book of Mathematical Procedures*, 2006], one sees that the list is given as a sequence of sentences and not in table form. Later on, we no longer find such multiplication lists in the sources.

into *zhu*. However, in the sentence, again we have an additional "*shi*-ten," which one cannot explain as a multiplication by 10. One may think that the text is corrupt. However, this would entail it being corrupt in several places, all linked to the "*shi*-ten" character. This is all the more difficult as sentences (I) to (III) seem to be three parallel sentences. Moreover, in sentence (E) we also have two occurrences of the same *shi*-ten character which would not be easy to interpret as "ten." This is why we believe that in this text, *shi*-ten has another technical meaning. Bringing this to light will help us interpret all the occurrences of the word in the passage examined.

Before we describe our suggestion for the interpretation of *shi*-ten*,* let us examine sentence (I) more closely. At first sight, one may be tempted to interpret it as meaning that in order to convert a quantity expressed in *shi* into one expressed in *jin*, one should "decuple and then double it and sextuple it." This sequence of multiplication would amount to multiplying by 120.[39] We do find parallel prescriptions for a sequence of multiplications in the *Book of mathematical procedures*[40] and in the *Mathematical classic by Xiahou Yang* 夏侯陽算經.[41] However, if we keep this interpretation, then the parallel

[39] One may wonder why the text has a prescription for converting *shi* into *jin* and does not have a prescription for converting *shi* into *jun* or *jun* into *jin*. Some remarks may be helpful here. The former conversion would require that one multiplies by 4. However, this is already a multiplication by a single-digit number. So, if the purpose of sentences (I) to (III) is to convert a multiplication into a sequence of multiplications by single-digit numbers, there would be no need to provide a procedure for this. The latter case is much more interesting for us. Indeed, to convert *jun* into *jin*, one would have to multiply the quantity of *jun* by 30. This requires multiplying by 3 and by 10. However, one may surmise, from the fact that the procedure is *not* given in the final part of the text, that this multiplication is considered as being a multiplication by a single-digit number. In other words, in this context, multiplying by 10 would not be considered a multiplication worth a specific mention. If this were the case, the "*shi* ten" occurring in sentence (I) would have to be interpreted in a different way. For the moment, we do not place ourselves within the context of such an interpretation.

[40] In bamboo strip 55, to carry out a multiplication by 50, one reads "有（又）十而五之以為實 One further decuples (these numbers) and quintuples them to make the dividends" [PENG Hao 彭浩, 2001, 64]. In footnote 4, [張家山漢簡『算數書』研究会編 研究会編 Chôka san kankan Sansûsho kenkyûkai. Research group on the Han bamboo strips from Zhangjiashan *Book of Mathematical Procedures*, 2006, 64] suggests that the multiplication by 50 was divided into two phases probably because the book relies on the counting rod system for computation. The authors suggest that this fact may have entailed that the multiplication by 50 is divided into, first, moving the representation of the number to be multiplied by one column and then multiplying the result by 5. In section 4.2, we shall show that this is only one particular case of a much more general phenomenon.

[41] Qian Baocong established that the *Mathematical classic by Xiahou Yang* was a book from the 8th century, which, in the 11th century, was introduced by mistake into classical texts, in the edition of the 7th century collection of the *Ten Classics of Mathematics* prepared under Li Chunfeng's supervision, compare [Qian Baocong 錢寶琮, 1963, 551]. [Berezkina 1985] offers a translation of this classic. In the first chapter

shi-ten characters in sentences (II) and (III) would not be easy to interpret. This is why, in our critical edition of the mathematical tables from tomb M77, we suggest that there was a *shi*-ten in sentence (I) parallel to that in sentences (II) and (III), and that it was most probably followed by a mark indicating that it was to be repeated.[42] The second *shi*-ten indicates the multiplication to be carried out, whereas the first one takes a meaning similar to the *shi*-ten in sentences (II) and (III) as well as in sentence (E).

What can thus be the technical meaning of *shi*-ten in all these occurrences in which the character cannot be interpreted as "ten" or "decuple"? We suggest that *shi*-ten designates the fact of converting a quantity expressed in a non-decimal system into an amount of a lower unit expressed with respect to a decimal system. We introduce the neologism "ten-ify" to designate this technical meaning. We are aware of the fact that such a meaning has not yet been documented. However, this is the only way we could find to make sense of all these occurrences. We thus present this tentative solution to the puzzle. Indeed, in sentences (I) to (III), the idea is that, through carrying out the multiplications prescribed, one converts an amount in a given unit into a decimal amount in a lower unit. In other words, the multiplications prescribed allow the carrying out of an operation whose technical meaning is *shi*-ten. This suggestion will provide one of the necessary clues to interpret sentences (E) and then (F). The second *shi*-ten in sentence (E) can be interpreted as the prescription of carrying out the operation *shi*-ten, for which details are provided in sentences (I) to (III). We shall come back to this point below. As for the first *shi*-ten in sentence (E), we interpret it as referring to a goal of expressing a quantity, originally formulated according to a sequence of weight measurement units, with respect to the lowest unit —the *zhu*. In that case, the goal is formulated as an amount of *zhu* expressed in a decimal system. This is what we translate as "tens of *zhu*." Before we explain further what we mean by this expression, it will prove useful to focus on another difficult character which occurs twice in sentences (E) and (F), namely *quan* 券.

***Quan* "break down".**

The interpretation of the character 券 *quan* is another key clue in our interpretation. Indeed, we suggest taking the character with its etymological meaning of "breaking down." This interpretation is supported by an occurrence of the character in a chapter of the *Book of Master Guan* (*Guanzi* 管子). The chapter in question is the second in the section of the book entitled "Weighing (*Qingzhong* 輕重)." This section is thought to

of the book, one reads "斤之求兩, 二而八之 If having *jin*, one looks for *liang*, one doubles and then octuples the (quantity of *jin*)." Below, we shall come back to this passage and show how closely it is related to the excerpt we are translating.

[42] Such a mark is quite common in manuscripts of that period, as is shown everywhere in the *Book of mathematical procedures*. Moreover, scribes regularly forgot to copy the mark. We have an example of such an omission in the section of the *Book of mathematical procedures* entitled "Rules about millet *chenghe* 程禾." [PENG Hao 彭浩, 2001, 80], footnote 3.

have been completed at the latest at the beginning of the Common Era. It contains an occurrence of the character *quan* which the Tang dynasty commentator Yin Zhizhang 尹知章 explains as follows: "When one shares it, it is called *quan*; when one brings it together, it is called *qi* 分之曰券，合之為契。" [Ma Feibai 馬非百, 1979, 609]

With this clue, it turns out that sentences (E) and (F) echo the procedure from the *Book of mathematical procedures* translated in the previous section. In fact, they can be interpreted as referring to a sub-procedure equivalent to its key step. The subprocedure in question is described within the "*shilü*" procedure, as follows:[43] "有斗、升、斤、兩、朱（銖）者亦皆破其上，令下從之為法。 Those for which there are *dou* and *sheng*, *jin, liang* and *zhu*, one *also* breaks down *all* their upper (rows) (i.e., using bamboo strip 47) and makes the (rows) below join them (i.e., the rows below respectively joining the middle row where the results of breaking down into *sheng* were placed), which is taken as divisor."

In fact, the subprocedure described in sentences (E) and (F) of the passage we are examining is the same as this except for some details to be pointed out below. However, the subprocedure is described with other words. In each context, the terms emphasize different aspects of the computation. One feature remains the same: even though the term used is different, in both contexts, we find the prescription expressed with terms that refer to "breaking down" higher units into lower ones. In the *Book of mathematical procedures,* the prescription is expressed by *po* 破 and it refers to the breaking down of either *dan* and *dou* into the lowest unit *sheng,* or *dan, jun, jin* and *liang* into the lowest unit *zhu.* Similarly, in *Mathematical procedures*, the procedure is prescribed with the term *quan* 券 and it refers to a global conversion of all units into *zhu* —the text deals only with units of weight.

***Fan* "going backwards."**

The indication of "going backwards" that we find in sentence (E) confirms these choices of interpretation. Indeed, it appears that "going backwards" refers to the fact of taking sentences (A) to (D) in the reverse order to the one that was followed previously. Note that this reversal corresponds to the reversal that occurs between sentences (A) to (D) and sentences (I) to (III). (A) deals with the relationship between *zhu* and *liang* whereas, below, this relationship is dealt with in sentence (III). Accordingly, (B) corresponds to sentence (II) whereas (C) and (D) correspond to sentence (I).

What can then be the use of sentences (A) to (D)? It appears that if one follows them in the order (A) to (D), one can convert an amount expressed in *zhu* according to a decimal system — that is, an amount similar to those dealt with in the first half of the text— into a quantity expressed with respect to the sequence of units *shi, jun, jin, liang, zhu*. Let us give an example to clarify the meaning. Suppose that, to begin with, we have the quantity 104953 *zhu*. Using sentence (A) (that is, dividing the quantity by 24) would

[43] Please see the previous section for the relevant explanations.

give 4373 *liang* 1 *zhu.* Then using sentence (B) (that is, dividing the quantity of *liang* by 16) would give 273 *jin* 5 *liang* 1 *zhu*. Using sentence (C) (that is, dividing the quantity of *jin* by 30) would give 9 *jun* 3 *jin* 5 *liang* 1 *zhu.* Lastly, using sentence (D) (dividing the quantity of *jun* by 4) would give 2 *shi* 1 *jun* 3 *jin* 5 *liang* 1 *zhu*. Note that the successive values are placed first in the row in which the *zhu* are stored, and then, progressively, a row higher at each step. Now, if one wants to "break down" the higher units "into tens of *zhu*," one needs to go backwards and use sentences (A) to (D) in the reverse order. Using (D) converts 2 *shi* into 8 *jun*, in addition to the 1 *jun* that one already has. Using (C) converts 8 *jun* and the 1 *jun* into, respectively, 240 *jin* and 30 *jin*, in addition to the 3 *jin* that one already has. And so on, until one goes back to the quantities of *zhu*, expressed with respect to a decimal system. Note that in this reverse phase, one goes also in the reverse direction on the table on which computations are carried out, that is to say, from the top row, in which the number of *shi* is stored, down to the row for the *zhu*. Moreover, one uses the clauses in the reverse direction, from the second value they contain up to the first value.[44] Needless to say, sentences (I) to (III) provide another option to "ten-ify" the quantities in *shi, jin* and *liang*, an option that proceeds through multiplications by 1-digit numbers. We shall come back below to this feature.

Let us compare the procedure that we identify in this sentence with the similar subprocedure within the *shilü* procedure in the *Book of mathematical procedures*, which we have outlined above. It seems that the conversion of the higher units into the lowest one —the *zhu*— is carried out in *Mathematical procedures* by means of accumulated multiplications. By contrast, most probably the step "one *also* breaks down *all* their upper (rows)" in *shilü* was carried out by using bamboo strip 47 of the *Book of mathematical procedures*, which states: "24 *zhu* (make) 1 *liang*. 384 *zhu* (make) 1 *jin.* 11520 *zhu* (make) 1 *jun*. 46080 *zhu* (make) 1 *shi*."

On one hand, the formulation appears to indicate that the successive units were placed in rows above each other on the table on which one computed with counting rods. We have already explained this feature above. On the other hand, it seems that the value recorded in each row was converted into *zhu* by multiplication by the corresponding value provided by bamboo strip 47. We thus have two forms of statement of the relationships between weight measurement units. In correlation with them, the procedures described respectively in the two books, although quite similar, are slightly different, and the difference between them comes from the fact that they make use of the statement as found in the corresponding book.

If we compare the formulations of the two corresponding steps in both books, we see that the prescription in the *Book of mathematical procedures* emphasizes the layout of the computation on the calculating table as well as the physical counterpart of the operation

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44 On the use of *fan* 反 in the earliest mathematical sources in Chinese, see Chemla's glossary in [Chemla and Guo Shuchun, 2004, 920]. This term is used when operations are carried out in a way reverse to, or in a reverse order to, the one that could be expected.

prescribed ("breaking down.") By contrast, *Mathematical procedures* stresses that we use the same list of equivalences between weight measurement units in a reverse order to the preceding order. Moreover, *Mathematical procedures* also stresses the fact that the results of breaking down into lower units yield a quantity expressed according to a decimal system. This emphasis echoes the numerous occurrences of the character *shi*-ten.

At the end of the two procedures, we seem to be confronted with the same result on the table on which computations were carried out, namely, in the various rows one has the quantity in *zhu* equivalent to the quantity formerly placed in a row and corresponding to one of the higher measuring units (see figure 3). We shall see that the final part of the two subprocedures appear to be the same, even though they are not formulated in the same way.

***Sheng* "to make move up" or "to (place) as *sheng* (on the calculating table)"**

Once one has all the various components of a quantity, which are placed in different rows, expressed in *zhu* —according to a decimal system—, the next step in the *shilü* procedure from the *Book of mathematical procedures* prescribes: "令下從之為法。one makes the (rows) below join them, which is taken as divisor." This statement can be understood in two ways. Either, the rows below, progressively, join the row above them until one reaches the upper row. Or, the rows below the middle row join the middle row, the *sheng* row, in which the results of the previous breaking down into *sheng* operations have been placed. Whatever the case, this sentence indicates that one takes the rows below to make them "follow" or, in other words, "add up to" the row(s) above them —precisely the one (or those) containing the units that were broken down. Each of the two interpretations of this step could be related to what sentence (F) in the text from tomb M77 prescribes.

According to the first interpretation of the statement in the *Book of mathematical procedures*, we would have an addition going upwards. This description of a movement upwards evokes the meaning of "moving up" that the character *sheng* 升 which occurs in sentence (F) of the text from tomb M77 can take. In this case, the parallel between the two procedures prompts us to interpret the *sheng* 升 character as referring to the movement upwards of the counting rods representing the numbers placed in the various rows, thereby leading to their accumulation in the same row above, that is to say, to their sum.

According to the second interpretation of the statement in the *Book of mathematical procedures*, the procedure implies that quantities converted into *zhu* are progressively and naturally placed together in the middle row, the one corresponding to the *zhu* measurement unit. The parallel between the two procedures in the respective books suggests that these rows may have been designated by the names of the various units in the series of units measuring capacity, which represented a decimal scale. The interpretation of *sheng*, the measurement unit of capacity, as a verb would read in this term the prescription of the sum of all the quantities converted in decimal numbers of *zhu* by referring to the operation of placing them together in the row associated to the *sheng*.

Each of these interpretations implies that at the end of the *sheng* operation, one has definitely obtained the quantity, originally expressed with respect to the scale of measurement units, as an amount of *zhu* expressed with respect to a decimal system. One has thereby returned to the quantity one had before sentence (A).

This comment concludes the explanations clarifying our interpretation and provides the final piece of evidence on which we rely to support it. We are perfectly aware that this interpretation is tentative and will remain so until the full book is available to researchers. May this first attempt help colleagues who are collating *Mathematical procedures* to complete their work.

4. Discussion and Comparisons with Other Documents

4.1. The decimal system and the counting rod system

One of the main points of the interpretation presented in part 3 was the identification of the technical term "*shi* ten." We are not aware of any later text that used *shi*-ten with the same meaning of "converting into a decimal expression." The operation occurs within the context of the system of weight measurement units, which is not decimal. Its introduction seems to reveal an emphasis placed on the fact that writing an amount according to the lowest unit leads to writing it according to a decimal system, which can therefore be adequately transferred into a representation of numbers with counting rods, that is, with a place-value decimal system. It is probably meaningful that the intermediate step receives here some emphasis.

In relation to this feature, the text also contains both statements of relationship between the units of weight and a table for multiplying powers of 10 with each other. We have stressed the fact that in this respect *Mathematical procedures* is similar to the *Book of mathematical procedures*, as they both contain these elements.[45] Moreover, as we have argued above, it is highly probable that the two books share the feature of depending on the counting rod system.[46] These remarks indicate that the two documents present similarities. By contrast, even though *The Nine Chapters* contains references to the counting rod system, it does not contain any systematic statement of relationship between weight measurement units, nor any treatment of the multiplication between powers of 10.

[45] The *Book of mathematical procedures* contains nothing comparable for the other systems of measurement units, which were all decimal.

[46] So far, we know that *Mathematics* also contains the statement of relationships between the units of weight. Compare bamboo strips 0303 and 0458, [Xiao Can 肖燦 and Zhu Hanmin 朱漢民, 2009a, 424]. Moreover, we also know that *Mathematics* most probably refers to the counting rod system. We shall have to wait until the whole book is published to know whether it also contains strips devoted to multiplying powers of ten.

In what follows, in addition to discussing the features of the translated text, we shall focus on the relationships that the newly discovered documents present to each other as well as the relationships and contrasts that they show with mathematical works handed down.

4.2 Multiplying by 1-digit numbers

Sentences (I) to (III) of the excerpt translated above reveal an interest in converting a multiplication by a number having several digits into a sequence of multiplications by numbers having only one digit. In fact, such an interest recurs regularly in all the excavated manuscripts.

Let us illustrate this statement by the procedure given by the *Book of mathematical procedures* under the name "*Li tian* 里田 Fields in *li*." To begin with, we need to introduce some background elements regarding the measurement units. Several units of length are used in our sources and accordingly several units for area. Lengths can be measured in *chi* 尺 —the same name is used for the unit of length and the unit of area based on it, but the uses are not the same. Lengths can be measured in *bu* 步, and in the same way, *bu* is also the name of a unit for area based on the *bu*. Higher units for areas are based on the *bu*: 1 *mu* 畝 is 240 *bu*[47] and 1 *qing* 頃 is 100 *mu*.[48] Lastly, lengths can be measured in *li* —1 *li* being 300 *bu*.[49] As a unit of area, 1 *li* is 375 *mu*. The "Fields in *li*" procedure provides the means of computing the area of a rectangle[50] whose dimensions are given in *li* and further of converting the result into the other units of area (*qing* and *mu*). It can be translated as follows:

"里田　里田术（術)曰：里乘里₌，(里)也。廣、從(縱)各一里，即直(置)一，因而三之，有(又)三五之，即為田三頃【上十下一】[51](七十)五畝ㄥ。其廣、從(縱)不等者，先以里相乘，巳(已)，[52]/187/乃因而三之，有(又)三五之，乃成。(…)/188/

Fields in *li*　　　Procedure for the Fields in *li*: (an amount of) *li* is multiplied

[47] The relationship is stated in bamboo strip 159 of the *Book of mathematical procedures*, compare [PENG Hao 彭浩, 2001, 113] .

[48] The relationship can be deduced from the bamboo strip discussed below.

[49] The relationship can also be deduced from the bamboo strip discussed below.

[50] The shape of the rectangle is indicated by the name of the dimensions of the shape. On this point, see K. Chemla, "La langue mathématique et les problèmes de sa traduction," in [Chemla and Guo Shuchun, 2004, 101—102].

[51] Between the bold brackets, we describe the character used in the manuscript to write down "seventy."

[52] Here we follow the reading of the manuscript suggested by [張家山漢簡『算數書』研究会編 Chôka san kankan Sansûsho kenkyûkai. Research group on the Han bamboo strips from Zhangjiashan *Book of Mathematical Procedures*, 2006, 13].

by (an amount of) *li*.[53] If the width and the length each make 1 *li*, one then puts 1, and through *yin* 因 (method)[54], tripling this, and further three times quintupling this, one thus makes a field of 3 *qing* 75 *mu.* In the cases when width and length are not equal, one multiplies first the (amounts of) *li* by one another. After this is completed, through *yin* 因 (method)[55], tripling this, and further three times quintupling this, then it (the computation) is completed."

Clearly the multiplication by 375 of the area expressed in *li* to further convert it into *qing* and *mu* is carried out by a sequence of multiplications by either 3 or 5, that is, multiplications by a single digit number. We recognize here exactly the same concern as evidenced in the excerpt from *Mathematical procedures* translated above. A very similar procedure for the "Fields in *li*" operation is also included in *Mathematics*.[56] On one hand, this shows a relationship between the *Book of mathematical procedures* and *Mathematics*. On the other hand, this evidence reveals links in the practices to which all the manuscripts bear witness. What is worth noting is that the same concern occurs in different procedures throughout the set of excavated manuscripts. This doubtless relates to how computations were carried out at the time. It is important to stress that we no longer find such multiplication transformations in *The Nine Chapters*. For instance, we do find there a procedure for "Fields in *li*." However, its statement prescribes the operations to be carried out in a different way. Let us quote the procedure:

"里田
術曰：廣從里數相乘得積里。以三百七十五乘之，即畝數。
Fields in *li*
Procedure: the quantities of *li* of the width and the length being multiplied by one another, this yields the *li* of the product/area. One multiplies this (the result) by 375, hence the quantity of *mu*."

Whether the practitioner who used the procedure tacitly converted the multiplication by 375 into as series of simpler ones or multiplied the area directly by 375, we do not know. However, clearly the way of stating the procedure has changed, pointing out a difference between the mathematical practice revealed by the excavated manuscripts and that to which *The Nine Chapters* bears witness.

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53 It is difficult to determine whether *li* here refers to only "1 *li*" or, as below, "an amount of *li*." However, as the text below introduces two cases, we choose to interpret the first sentence of the general procedure as making a general statement .The other occurrences of the procedure in *Mathematics* and in *The Nine Chapters* seem to support this choice.

54 We shall come back to this *yin* below.

55 The same character as before *yin* appears. See below.

56 See bamboo strip 0947, in [Xiao Can 肖燦 and Zhu Hanmin 朱漢民, 2009b, 42].

If we go back to the manuscripts and focus on how the multiplication by single digit numbers is prescribed, we discover that the character *yin* 因 regularly occurs in these contexts. The technical expression reads as follows: "因而 n 之." This expression has usually been interpreted as another way of referring to a multiplication. However, it now appears that such expressions occur in specific contexts, especially when the multiplication is by a single digit number.[57] It is well known that subsequent mathematical books use the character 因 *yin* to designate specific procedures to multiply by a single digit.[58] It thus seems that as early as the time when the excavated documents were produced, the expression "因而 n 之" may have referred to a specific kind of multiplication. The same expression still occurs in *The Nine Chapters* and its commentaries.[59] This may indicate a permanence in practice. We shall come back to this hypothesis in a subsequent publication.

4.3 An interesting parallel between *Mathematical procedures* and the *Mathematical Classic by Xiahou Yang* 夏侯陽算經

The last point that we would like to discuss with respect to the excerpt that we have analyzed in this paper relates to a parallel that can be found in mathematical documents that have been handed down through the written tradition. This brings us back to the 8th century mathematical book which we mentioned above: the *Mathematical Classic by Xiahou Yang*. Let us recall that to begin with, this book did not belong to the *Ten Classics of Mathematics*, but that it was kept thanks to confusion with another classic (see footnote 41).

It turns out that the opening chapter of the *Mathematical Classic by Xiahou Yang* has a passage directly parallel to sentences (I) to (III) of the excerpt from *Mathematical procedures* in a context in which the concern clearly relates to converting quantities expressed with respect to certain units into quantities expressed with respect to smaller units. Let us translate the passage before we draw some conclusions from this similarity:

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[57] The context either relates to a multiplication by a single digit or by a combination of 10 and a single digit. Earlier in footnote 40, we noted an occurrence of a similar phenomenon in the *Book of mathematical procedures*. In bamboo strip 55, to carry out a multiplication by 50, one reads "有(又)十而五之以為實 One further decuples (these numbers) and quintuples them to make the dividends." It is interesting to note that the multiplication by 50 is precisely decomposed into two multiplications, which may be another reflection of the same concern.

[58] Compare, for instance, [Chen Yifu, 2007]. The specific use of 因 *yin* is described clearly by Yang Hui in his *Foundations and consequences of variations for multiplication and division* (*Chengchu tongbian benmo* 乘除通變本末), see [Guo Shuchun 郭書春, 1993, 1.1050].

[59] The only exception to the rule is the expression "one takes half of this 因而半之 *yin'er ban zhi*," which the 7th century commentator Li Chunfeng uses, in his commentary on problem 3 of Chapter 2 in *The Nine Chapters* [Chemla and Guo Shuchun, 2004, 226—227].

> “(丈之)[60]求尺，尺之求寸，皆上十之。斤之求兩，二而八之。兩之求銖，三而八之。銖之求絫、黍，皆上十之。斗之求升、合、（勺)[61]、抄、撮，皆上十之。里之求步，三百之。步之求尺，六之。氂、毫、絲、忽，可以意知。
>
> If having *zhang*, one looks for *chi*, having *chi*, one looks for *cun*, in each case, by moving upwards,[62] one decuples them. If having *jin*, one looks for *liang*, one doubles and then octuples them. If having *liang*, one looks for *zhu*, one triples and then octuples them. If having *zhu*, one looks for *lei*, *shu*, in all these cases, by moving upwards, one decuples them. If having *dou*, one looks for *sheng*, *ge*, (*sha*), *chao*, or *cuo*, in all these cases, by moving upwards, one decuples them. If having *li*, one looks for *bu*, one three-hundred-folds them. If having *bu*, one looks for *chi*, one sextuples them. As for *li*, *hao*, *si*, *hu*, it is possible to know according to the meaning.”

We see that the middle section of this passage in the *Mathematical Classic by Xiahou Yang* suggests decomposing the multiplications through which one converts the original quantities into quantities expressed with respect to smaller units in exactly the same way as we find in *Mathematical procedures*. Two differences can be noted. Firstly, the expression of the shift leftwards is not mentioned in the passage of *Mathematical procedures* on which we focused in this article. What this detail means for a history of place-value decimal systems remains to be understood. Secondly, the *Mathematical Classic by Xiahou Yang* systematically prescribes the operations to carry out for any type of conversion according to the same format. By contrast, it seems that the text from tomb

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[60] We follow here the critical edition published in [Guo Shuchun 郭書春 and Liu Dun 劉鈍, 1998], with some modifications. Note that in this edition, the page numbers start from 1 for each of the books included in the *Ten Classics of Mathematics*. The passage in question starts by “其物殘分,” which do not match the next characters. Moreover, clearly before the next characters “求尺”， the two characters “丈之” are missing. This can be deduced from three facts. First, the context makes it clear. Secondly, the next sentence is followed by a “in all these cases *jie* 皆” which indicates that there must have been at least two cases before. Lastly, below one reads the parallel statement “如斛中求斗、斗中求升、升中求合、合中求勺、勺中求抄及丈中求尺、尺中求寸、寸中求分, 皆言上十之、上百之。”［Guo Shuchun 郭書春 and Liu Dun 劉鈍, 1998, 2］.

[61] We do not follow here the emendation of the text by either [Qian Baocong 錢寶琮, 1963, 558-559] (footnote 1) or [Guo Shuchun 郭書春 and Liu Dun 劉鈍, 1998, 1, 29-30] (footnote 3). Instead we consider that a character was lost by the copyist. We rely for this on the parallel statement “如斛中求斗、斗中求升、升中求合、合中求勺、勺中求抄及丈中求尺、尺中求寸、寸中求分, 皆言上十之、上百之。” [Guo Shuchun 郭書春 and Liu Dun 劉鈍, 1998, 2］.

[62] Here the *shang* “moving upwards” refers to moves on the table on which computations are made with the counting rods. Note that, by contrast to the excerpt examined in this article, the statement includes the conversion of units into smaller ones, whether the system of measuring units is decimal or not.

M77 only focuses on weight measurement units.

Several conclusions can be derived from the parallel between the excavated manuscript and the text that has been handed down. First of all, computation practices reveal a relative stability between early imperial and Tang times, even though the classics handed down do not seem to reflect them.[63] Moreover, the *Mathematical Classic by Xiahou Yang* appears to have kept pieces of mathematical knowledge that were quite old. How far does this phenomenon hold true? What other ancient mathematical knowledge could be retrieved from the book, this is a question to which we shall return in another article.

More generally other conclusions can be derived from our analysis of the seven bamboo strips examined here.

First, the text examined contains the earliest known table that was included in a mathematical writing. Why did the scribes opt for this lay-out and why in other cases did they choose a linear exposition? This question also awaits further study.

Second, the document analyzed casts light on the history of number systems in early China and on practices of converting between measurement units.

Third, we suggested that the use of certain terms like *sheng* "move up" or "place as *sheng*" reflected practices with the system of counting rods in ancient China. In this respect, bringing together in this way all the clues that our sources contain to this computational practice would be an important achievement in better understanding an instrument for which nothing else remains.

Fourthly, analysis of the phonological features of the text from tomb M77 provides evidence that the various components of the text were rhymed.[64] This fact implies that rhymed texts were used not only for multiplications but also for conversions. This remark calls for further research on the composition and use of specific kinds of texts to carry out basic operations.

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[63] In footnote 8, [PENG Hao 彭浩, 2001, 127-128] suggests a similar conclusion. [Qian Baocong 錢寶琮, 1964, 125—126] discusses the numerous conversions of multiplications into sequences of multiplications and divisions by one digit, evidenced by the *Xiahou Yang suanjing*, which in his view were carried out on only one row. He suggests the idea that the system of multiplications on three positions using counting rods was not practical and was simplified into the practice of sequences of simple multiplications and divisions. The newly excavated manuscripts show that in fact the two practices coexisted as early as our evidence can tell. Why did the practice of sequences of simple multiplications and divisions become almost invisible in the classical texts brought together in the *Ten Classics of mathematics* collection? What can we conclude from that dissimilarity between the excavated manuscripts and the earliest extant mathematical documents handed down through the written tradition in this respect? What is the history of such computational practice? We shall come back to these questions in another article.

[64] The analysis relies on [Li Zhenhua, Zhou Changji 1999].

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