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Mathematical Treatise on the Technique of Linkage

An Annotated English Translation of Takebe Katahiro's

Tetsujutsu Sankei

Preserved in the National Archives of Japan

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I Introduction

The *Tetsujutsu Sankei* (*Mathematical Treatise on the Technique of Linkage*)[3] is a classic Japanese mathematical text written by Takebe Katahiro[4] (1664–1739) in 1722. In this treatise, Takebe presents his most notable mathematical achievements, including, for example, an efficient calculation of π up to 42 digits and three expansion formulas for circular arc length in terms of the sagitta (maximum separation between the arc and its chord).

Although Takebe's book contains outstanding results of other early 18th century Japanese mathematicians, the main purpose of the *Tetsujutsu Sankei* is to present the author's personal view on mathematics and mathematical research. According to Takebe, there are three aims in mathematical research, i.e., rules, procedures and numbers, and two methods to reach these aims, i.e., by reasonable evidence and by numerical evidence. To illustrate his idea he employs twelve examples, including the above-mentioned calculation of π and of arc length. Since it was a rare occasion for a mathematician of the *Edo* period to express his philosophy on mathematical

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[3] At the first appearance, names of Japanese texts are followed by their English translations in parentheses.

[4] The names of Japanese mathematicians are written in vernacular order: family name first, followed by the given name.

research, the *Tetsujutsu Sankei* has for generations attracted the interest of many Japanese mathematicians.

Numerous attempts have been made by scholars on Takebe Katahiro's achievements. The early studies are mainly made by Hayashi Tsuruichi (1873–1935), Shibata Kwan (1886–1983), Fujiwara Matsusaburo (1881–1946), and so on. See [Hayashi1911], [Hayashi1915], [Shibata1935], [Shibata1935b], [Fujiwara1941], [Fujiwara1945], and Volume 2 of the *History of Japanese Mathematics before the Meiji Restoration* [Fujiwara1954]. Hayashi's almost all works can be found in his collected works [Hayashi1937]. Recently, Fujiwara's collected works on the history of Asian mathematics was published as [Fujiwara2007].

As for the life and works of Takebe Katahiro in general we refer the reader to our monograph [OgawaEa2008]. Recently published book [Horiuchi2010] (the French original edition [Horiuchi1994]) describes in English the history of Japanese mathematics in the *Edo* period, especially Takebe and his teacher Seki Takakazu.

Takebe's calculation of π and arc length was considered in [Sugiura1982] and [Murata1982]. The reconstruction of his calculations with a computer began from 1980s downward. The first attempt was most probably [Wada1983] in 1983 and followed by [Morimoto1990], [Morimoto1990b], [Ogawa1997], [Ogawa2000], [Morimoto2003], and so on. Some studies have since increased ([Horiuchi1994b], [MorimotoEa2004], [Morimoto2006], [Morimoto2007], [Nonaka2010], and [Morimoto2011]). In particular, detailed studies on some chapters of the *Tetsujutsu Sankei*, that include the translation of the text into modern Japanese, can be found in [Ogawa1998], [Ogawa1998b], [Ogawa2002], [Ogawa2007] and so on. The collected works of Seki Takakazu [HirayamaEa1974] contains a commentary on Seki's mathematics in English. There are two recently published monograph on Seki's life and works [Sato2005] and [UenoEa2008], which put together much research on Seki [Ogawa1996], [Sato1996], for example.

The general history of Japanese traditional mathematics was compiled first by Endō Toshihide (1843 – 1915) [Endo1896], which later has been corrected and augmented. The *History of Japanese Mathematics before the Meiji Restoration* [Fujiwara1954] in five volumes was prepared by Fujiwara during the World War II but published by the name of the publishing committee in 1954. The history of Japanese mathematics was also written in English as early as in 1910s ([Mikami1913] and [SmithEa1914]).[5] The bibliography on the Japanese traditional history written in European language are listed in [Ogawa2001]. The following books also are concerned with history of mathematics in Japan: ([Mikami1921], [Murata1981], [OgawaEa2003].) For the history of Chinese traditional mathematics see [Li1984], [Martzloff1987], and [Qian1990].

[5] As these two books are already obsolete, a new general history of Japanese mathematics in English is urgently needed.

There are few English translations of Takebe's main works, while [TakenouchiEa2004] is the only one today, but it is a preliminary edition. In this connection, the most popular mathematical book in Japan, the *Jinkōki*, was translated into English [WasanInst2000], in which we can learn how to use the abacus. Furthermore, [Kojima1963] is an introduction to the Japanese abacus.

In this English translation of the *Tetsujutsu Sankei*, we have made an effort to capture the original manner in which Takebe discusses mathematics. At the same time, to make his work more understandable for the reader, we have included additional historical background and commentary, which interpret his ideas in light of the more familiar mathematical terminology and methods that we employ today.

II The author

At the age of thirteen, Takebe Katahiro became a student of Seki Takakazu (ca.1642 – 1708)[6] , an illustrious master of mathematics. Under the guidance of his master, he learned, among others, mathematics of the *Yuan* dynasty from the *Suanxue Qimeng* (*Sangaku Keimō* in Japanese, *Introduction to Mathematics*) [7] written in 1299 by Zhu Shijie (Shu Seiketsu in Japanese). By his mid-thirties, Takebe had already published three books: the *Kenki Sanpō* (*Mathematical Methods to Investigate the Minute*) in 1683 the *Hatsubi Sanpō Endan Genkai* (*Colloquial Commentary on Series of Operations in the Hatsubi Sanpō*) in 1685; and the *Sangaku Keimō Genkai Taisei* (*Great Colloquial Commentary on the Suanxue Qimeng*) in 1690. See [Morimoto2004] and [Ogawa2005].

The first book, *Kenki Sanpō*, contains answers to the problems raised in the *Sūgaku Jōjo Ōrai* (*Text on Multiplication and Division in Mathematics*) written in 1674. See [Sato1996b], [Fujii2002], [Takenouchi2004], and [Takenouchi2006].

The second book, the *Hatsubi Sanpō Endan Genkai*, is an annotation to Seki Takakazu's *Hatsubi Sanpō* (*Mathematical Methods to Explore Subtle Points*). The latter book was difficult to understand, prompting need for an annotation. See [Ogawa1994], [Ogawa1996] and [Sato1996].

The third book, the *Sangaku Keimō Genkai Taisei*, is a detailed annotation to the important Chinese work *Suanxue Qimeng*. Together with the *Suanfa Tongzong* (*Sanpō Tōsō* in Japanese, *Systematic Treatise on Mathematical Methods*) by Cheng Dawei (Tei Daii in Japanese, 1533–1593) of the *Ming* dynasty, the *Suanxue Qimeng* most influenced early 17th century Japanese mathematics.

Takebe Katahiro also began in 1683 an encyclopedic work, the *Taisei Sankei* (*Great Accomplished Mathematical Treatise*), in collaboration with his master Seki

[6] Seki's birth year is estimated between 1640 and 1645.

[7] At the first appearance, names of Chinese texts are followed by their Japanese reading and their English translation in parentheses.

Takakazu and his brother Takebe Kata'akira (1661–1716). See [Komatsu2007] and [Ogawa2006]. Their intent was to reveal the entirety of mathematics of their day. By the mid-1690's, they had completed a preliminary version in twelve volumes. After that, Takebe Katahiro took leave of mathematics as an appointed government official, and Seki Takakazu a respite due to illness. It was not until 1711 that the entire twenty volumes of the *Taisei Sankei* were completed, mainly due to the individual effort of Takebe Kata'akira. This evolution is recorded in the *Takaebe-shi Denki* (*Biography of the Takebe*). See [Fujiwara1954].

Between 1704 and 1715, Takebe Katahiro served as an officer of the Shogunate and completed no mathematical works. In 1716 Tokugawa Yoshimune became the eighth *shōgun*. The new *shōgun* had a keen interest in the science of calendars and mathematics, and could appreciate Takebe Katahiro's mathematical ability. He surveyed the land in 1720 and edited the *Kuni Ezu* (*Illustrated Atlas of Japan*) in 1725. Being encouraged by the *shōgun*, in addition to writing about the science of calendars, Takebe resumed writing books on mathematics. This was the context in which he wrote in 1722 the book under our consideration, the *Tetsujutsu Sankei.* The same year he wrote the *Fukyū Tetsujutsu* (*Master Fukyū's Technique of Linkage*), and the *Shinkoku Gukō* (*A Humble Consideration on the Time*). A prolific author, Takebe later wrote the *Saishū Kō* (*A Consideration on the Period of Years*) in 1725; the *Ruiyaku Jutsu* (*Methods of Repeated Division*) in 1728.

He wrote several other books whose dates are unknown: the *Koritsu* (*Arc Rate*) (see [Fujiwara1941]), the *Sanreki Zakkō* (*Various Considerations on Mathematics and the Calendar*) (see [Fujiwara1945] and [SatoS1995]), the *Hōjin Shinjutsu* (*A New Method of Magic Squares*), the *Kyokusei Sokusan Gukō (Humble Considerations of the Observation and the Calculation of the Polestar)*, the *Chūhi Ron* (*Imprecision in Measurement*), and the *Jujireki Gi Kai* (*Commentary on the Time Granting Calendar*).

Fujiwara [Fujiwara1954] claimed that Takebe Katahiro also wrote the *Enri Kohai Jutsu* (*Studies on the Circle — Methods to Calculate the Length of Circular Arc*), which is sometimes called the *Enri Tetsujutsu* (*Technique of Linkage in Studies on the Circle*). Recently many scholars raised questions about Fujiwara's claim.

In 2005, a copy of a book entitled the *Kohai Setsuyaku Shū* (*Method of Pulvelizing Back Arc*) was discovered. It describes Takebe's discovery of infinite expansion formula of the square of arc length in terms of sagitta, and was recognized as a book of Takebe Katahiro (see [Yokotsuka2004] and [Yokotsuka2006]).

Takebe retired in 1733, when he was seventy years old, and he died six years later in 1739, at the age of seventy five.

III Organization of the *Tetsujutsu Sankei*

The *Tetsujutsu Sankei* begins with a Preface, followed by a Catalogue of twelve examples of mathematical investigation presented in the book:

Part 1. Four Examples on Investigation of the Rule and Law

1. Investigating Multiplication and Division (Investigation of rules by reasonable evidence)
2. Investigating the Rule of Element Placement (Investigation of rules by reasonable evidence)
3. Investigating the Rule of Reduction (Investigation of rules by numerical evidence)
4. Investigating the Rule of Finding Differences (Investigation of rules by numerical evidence)

Part 2. Four Examples on Investigation of the Reason of Procedure

5. Investigating the Procedure of Repeated Exchange of Weavers (Investigation of procedures by reasonable evidence)
6. Investigating the Procedure for Finding the Extreme Value of a Parallelepiped (Investigation of procedures by reasonable evidence)
7. Investigating the Procedure of Arithmetic Removal (Investigation of procedures by numerical evidence)
8. Investigating the Procedures for Finding the Surface Area of Sphere (Investigation of procedures by numerical evidence)

Part 3. Four Examples on Investigation of the Numerical Quantity

9. Investigating Numbers Stemming from Pulverization (Investigation of numbers by reasonable evidence)
10. Investigating Numbers Related to Square Root Extraction (Investigation of numbers by reasonable evidence)
11. Investigating Numbers Related to the Circle (Investigation of numbers by numerical evidence)
12. Investigating Numbers Related to the Arc (Investigation of numbers by numerical evidence)

The author claims there are three aims in mathematical research; the rule and law, the reason of procedure, and the numerical quantity. Three aims are sometimes called, in short, the rule, the procedure and the number, respectively. He also claims there are two means of investigation; one by reasonable evidence and other by numerical evidence. The organization of twelve examples reflects the author's three aims and the two means in mathematical investigation. After each example, the author explains why this example is classified to the aim and the method. After presenting these twelve examples, there is a single chapter on Takebe's philosophy of mathematics, in which the author describes the psychology of mathematicians and the characteristics of mathematical research. The book ends with an appendix that Takebe added in 1725.

IV Editions of the *Tetsujutsu Sankei*

The version of the *Tetsujutsu Sankei* which serves as the source of our English translation is preserved in the National Archives of Japan. Since this text is said to be dedicated to the *shōgun* Tokugawa Yoshimune, it was carefully preserved and may be regarded as authoritative.

The *Fukyū Tetsujutsu* is in some way very similar to the *Tetsujutsu Sankei*. *Fukyū* is a pseudonym of Takebe Katahiro. An English translation of the *Fukyū Tetsujutsu* is included in [TakenouchiEa2004]. Although the *Fukyū Tetsujutsu* and the *Tetsujutsu Sankei* have nearly identical introductions and appendices, the organization of the *Fukyū Tetsujutsu* is quite different, and has distinctive content:

1. Searching for the rule of multiplication (the first half of Chapter 1 of the *Tetsujutsu Sankei*)
2. Searching for the rule of division (the second half of Chapter 1 of the *Tetsujutsu Sankei*)
3. Searching for the procedure of permutation (Chapter 5 of the *Tetsujutsu Sankei*)
4. Searching for the square root (Chapter 10 of the *Tetsujutsu Sankei*)
5. Searching for the rule for placing the element (Chapter 2 of the *Tetsujutsu Sankei*)
6. Searching for the procedure of preparing medical prescriptions (No corresponding chapter in the *Tetsujutsu Sankei*)
7. Searching for and understanding the rule of finding differences repeatedly in the research of the procedure of the quadrangular pile (Chapter 4 of the *Tetsujutsu Sankei*)
8. Searching for the procedure to find the surface area of a sphere (Chapter 8 of the *Tetsujutsu Sankei*)
9. Searching for the rule of arithmetic removal (Chapter 7 of the *Tetsujutsu Sankei*)
10. Searching for the circle constant (Chapter 11 of the *Tetsujutsu Sankei*)
11. Searching for the arc constant (Chapter 12 of *Tetsujutsu Sankei*)
12. Searching for the procedure of decomposition (Chapter 9 of the *Tetsujutsu Sankei*)

Note that Chapters 3 and 6 the *Tetsujutsu Sankei* have no corresponding chapters in the *Fukyū Tetsujutsu*, while only the latter discusses medical prescriptions. We adopt the viewpoint that the *Fukyū Tetsujutsu* is a different work rather than revised version of the *Tetsujutsu Sankei*. The relation between these two books is a subject for further research.

The most reliable manuscript of the *Fukyū Tetsujutsu* is preserved in the University of Tokyo library. Another interesting manuscript of the *Fukyū Tetsujutsu* is held in the Kanō collection of Tōhoku University, in which the calculation of π is carried out to 70 digits. It is beyond the scope of this study to compare these manuscripts. See [Komatsu2000], [Komatsu2004], [Ogawa2004], and [Suzuki2005].

We remark that many manuscripts of the *Fukyū Tetsujutsu* have the title *Tetsujutsu Sankei* but maintain the particular table of contents for the *Fukyū Tetsujutsu* which we have described above. The *Tetsujutsu Sankei* which we have translated here is not the *Fukyū Tetsujutsu* that sometimes bears the same name, but rather, the distinct work which we feel merits consideration in its own right.

V Translation

1r

Preface to the Mathematical Treatise on the Technique of Linkage

[1]With the technique of linkage we can understand the reason of procedure investigating and linking [evidence]. [2]Generally speaking, there are two methods of investigation, one by reasonable evidence, [3]another by numerical evidence. [4]If investigating a [single] case is not sufficient for finding out the reason of procedure, investigate two cases. [5]If two cases are not enough, investigate three cases. [6]Even though the reason of procedure is deeply buried, if one keeps investigating enough times, a point of maturation will be reached where it is impossible not to find it. [7]But it happens that what is hidden can be found out immediately in one step; [8]also it happens that what is simple can be found out gradually in several steps. [9]Certainly, nobody is purely straight in man's character. [10]In nature some people are fast and others are slow [in understanding]; all these cannot be certain.
1v [11]By this, sometimes there are bending and stretching: if one stretches, he gains knowledge; if one bends, he stagnates. [12]Therefore, there are indeed differences in understanding; some people are slow and dull, while others are fast and sharp.

[13]Mathematics consists of the establishment of the rule and law, the clarification of the reason of procedure, and the calculation of the numerical quantity. [14]These are arranged in direct [order] if the reason is discerned, procedures are applied and numbers are obtained by the procedures, [15]and in inverse [order] if procedures are tested according to numbers and a reasons is sought by the procedures. [16]The direct and the inverse [orders] are all unified in the technique of linkage. [17]Therefore, establish the rule and law by investigation, clarify the reason of procedure by investigation, and determine the numerical quantity by investigation. [18]Accordingly,

recognizing three [aims], the rule, the procedure and the number, distinguishing numerical and reasonable evidence and citing twelve examples of procedures, we describe an outline of investigation and proclaim the technique of linkage. [19]In addition, I explain that my distorted and inconsistent native character cannot really
2r be changed and state the reason why this book is written.

[20]According to the History of the *Sui* dynasty, Zu Chongzhi **"wrote a book called the *Zhuishu* (Technique of Linkage). [21]There were neither scholars nor officers who could understand the deep contents of the book. [22]Therefore, they abandoned it [as curriculum] and no longer cared it."** [23]Having been led to use the word *zhui* (Linkage) and reflecting deeply, we cannot help thinking that Zu Chongzhi was a genius of antiquity. [24]Certainly, this marvelous truth cannot be recognized through education nor can it be reached through contemplation.

[25][Lunar] January 7, *Mizunoe Tora*, the seventh year of *Kyōhō*.

[26]Written by Fukyū, a humble aged *samurai* at the city of *Edo* in *Musashi* Province.

Catalogue

3r

Four Examples on the Investigation of the Rule and Law

I. Multiplication and Division (Investigation of rules by reasonable evidence)

II. Element Placement (Investigation of rules by reasonable evidence)

III. Reduction (Investigation of rules by numerical evidence)

IV. Finding Differences (Investigation of rules by numerical evidence)

Four Examples on the Investigation of the Reason of Procedure

V. Weavers (Investigation of procedures by reasonable evidence)

VI. Parallelepiped (Investigation of procedures by reasonable evidence)

VII. Arithmetic Removal (Investigation of procedures by numerical evidence)

3v VIII. Sphere (Investigation of procedures by numerical evidence)

Four Examples on the Investigation of the Numerical Quantity

IX. Decomposition (Investigation of numbers by reasonable evidence)

X. Root Extraction (Investigation of numbers by reasonable evidence)

XI. Numbers Related to the Circle (Investigation of numbers by numerical evidence)

XII. Numbers Related to the Arc (Investigation of numbers by numerical evidence)

A theory of proper character

The Technique of Linkage

4r

Four Examples on the Rule and Law

I. Investigating Multiplication and division

[1]Multiplication. [2]Suppose there are 12 *koku* of [unhulled] rice. [3]The price is 27 *sen* in silver per *koku*. [4]Question: How much is the total price?
[5]Answer: 324 *sen* in silver.

[6]Because the price of 1 *koku* is 27 *sen* in silver, for 2 *koku*, two prices added together, the price is 54 *sen* in silver. [7]For 3 *koku*, three prices added together, the price is 81 *sen* in silver. [8]For 4 *koku*, four prices added together, the price is 108 *sen* in silver. [9]For 5 *koku*, five prices added together, the price is 135 *sen* in silver.
4v [10]For 6 *koku*, by addition it is 162 *sen*. [11]For 7 *koku*, by addition it is 189 *sen*. [12]For 8 *koku*, by addition it is 216 *sen*. [13]For 9 *koku*, by addition it is 243 *sen*. [14]For 10 *koku*, by addition it is 270 *sen*. [15]For 11 *koku*, by addition it is 297 *sen*. [16]For 12 *koku* of rice, by addition we obtain 324 *sen*, [17]which is the corresponding price.

[18]After we obtain the true number decomposing repeatedly in this way ([19]this is, the so-called calculation at sight), we search for a simplified procedure. First, we take pairs of one-digit numbers between 1 and 9, form 45 products from "one times one makes one" till "nine times nine makes eighty one," and write the
5r multiplication chant. [20]Secondly, recite and memorize this table. Then place 12 *koku* of rice. By 27 *sen*, the price of a *koku*, first we multiply 10 [*koku*] and get 200 *sen* saying "one times two makes two," and 70 *sen* saying "one times seven makes seven." Secondly, [by 27] we multiply 2 *sen* and get 40 *sen* saying "two times two makes four," and 14 *sen* saying "two times seven makes fourteen." By adding these values we obtain the corresponding price 324 *sen* in silver in one step. Understanding this procedure, we establish the rule of multiplication.

[21] **Main procedure to solve the problem** **[22]Place the *koku* of rice. [23]Multiply this by the price in silver per *koku* and we obtain the corresponding price in silver.**

[24]Although we seem to obtain the price immediately without decomposing repeatedly, in fact we do not obtain [it] immediately; [25]the repetition is done in one step. [26]Generally speaking, the decomposition is the basis of number determination and the establishment of rules is the basis of procedure application.

[27]Therefore, in Mathematics, it is most important to establish rules.

5v **[28]Division. [29]Suppose there are 15 *koku* 6 *to* of rice. [30]Let it be divided by 6 people. [31]Question: How much is the share per person? [32]Answer: 2 *koku* 6 *to* per person.**

[33]If each person is given 1 *koku* of rice, we need 6 *koku* for 6 people. [34]This is less than what we have. [35]If each person is given 2 *koku* of rice, we need 12 *koku* for 6 people. [36]This is again less than what we have. [37]If each person is given 3 *koku* of rice, we need 18 *koku* for 6 people. [38]We know that this is, instead, more than what we have at first. [39]Therefore, we know each person's share is 2 *koku* and something. [40]We remove 12 *koku*, which we need if we distribute 2 *koku* to 6 people, from what we have at first. The remainder is 3 *koku* 6 *to*. [41]If each person is given 1 *to*, we need 6 *to* for 6 people. [42]This is less than the remainder. [43]If
6r each person is given 2 *to*, we need 1 *koku* 2 *to* for 6 people. [44]This is less than the remainder. [45]If each person is given 3 *to*, we need 1 *koku* 8 *to* for 6 people. [46]This is less than the remainder. [47]If each person is given 4 *to*, we need 2 *koku* 4 *to* for 6 people. [48]This is less than the remainder. [49]If each person is given 5 *to*, we need 3 *koku* for 6 people. [50]This is still less than the remainder. [51]If each person is given 6 *to*, we need 3 *koku* 6 *to* for 6 people. [52]This is exactly equal to the remainder. [53]Therefore, we know that each person's share is exactly equal to 6 *to*. [54] Because the remainder is exhausted completely by 3 *koku* 6 *to* that is, 6 *to* times 6 persons, each person's share is found to be 2 *koku* 6 *to* of rice.

[55]After we obtain the true number decomposing and investigating in this way,
6v we search for a simplified procedure. We place the original *koku* of rice in the Reality row and the number of people in the Norm row. First, we guess the first quotient is 2 *koku* and, reciting the multiplication chant, multiply the Norm row by it saying "two times six makes twelve" and subtract it from the Reality row. Then we guess the second quotient is 6 *to*, multiply the Norm row by it saying "six times six makes thirty six" and see the Reality row is completely exhausted by it. Understanding this [operation], we establish the rule of division by quotient.

[56]Also, we take two numbers from 1 to 9, one for the Reality row and other for the Norm row, ([57]where 1 shall not be taken for the Norm row, and the number in the Reality row shall not be greater than the number in the Norm row,) we calculate the quotient and the remainder dividing the Reality row by the Norm row and write the nine-division chant. [58] Recite and memorize this chant. Then divide, from the higher digit, the Reality row by the Norm row; saying "let six divide one and get fourteen" and "let six meet six and get
7r ten" to get the first quotient 2 *koku*, and then saying "let six divide three and get heavenly five" and "let six meet six and get ten" to get the second quotient

6 *to*, we establish the rule of nine-division.

[59] **Main procedure to solve the problem** [60]**Place the *koku* of rice in the Reality row and the number of persons in the Norm row.** [61]**Apply the division to this [configuration] and** [62]**we obtain the *koku* of rice per person.**

[63]Although, relying on the division by quotient or on the nine-division [chant], we seem to obtain the solution immediately without investigation, but it is not the case. [64]We investigate just in one step. [65]It is true that we cannot understand this rule from the beginning. [66]After employing the decomposition we investigate and understand how to organize [the results], and compose the rule's chant and employ it.

[67]The above two rules of multiplication and of division are to determine numbers by decomposition and to investigate and understand the rules relying upon rea-
7v sonable evidence. [68]As they are very simple, there are no secrets hidden in the determination of numbers according to the rule; the reasons are clearly manifested.

[69]Generally speaking, Mathematics culminates in the clarification of reasons and the determination of numbers. [70]It is required to rely on the decomposition in order to determine numbers, and to discern the reasons in order to apply a procedure. [71]The first and the latter, both jointly form the rule. [72]But if we try to scrutinize only relying upon reasons, we cannot always attain our objective; [73]inevitably we stagnate. [74]If we try to scrutinize only relying upon numbers, we cannot always attain our objective; [75]inevitably we are confused in reasons. [76]There are two kinds of reasoning: if, without distinguishing the direct and the inverse applications of the procedure, we simply apply thousands of procedures by decomposing [examples], we cannot profit from the advantages of mathematical rules and stagnate in direct application of procedures; [77]and if, without investigating [examples], we simply try to find [the solution] immedi-
8r ately only relying upon reasons, we can never attain the [proper] understanding in inverse application of procedures because there is no basis of reasonable evidence. [78]Therefore, if we distinguish the direct and the inverse applications of procedures, clarify numerical and reasonable evidence, discern according to form and character if the numbers and the rules are exhaustible or not, and investigate [examples] deeply, then there are no rules which cannot be understood and no numbers which cannot be determined.

II. Investigating the Rule of Element Placement

[1]We do not know yet in what age the rule of element placement started. [2]It was in the *Zhiyuan* period of the *Yuan* dynasty that Guo Shoujing used this rule when he completed the *Shoushili* (*Time Granting Calendar*). [3]In the *Dade* period of the same dynasty, this rule was explained in detail in the *Suanxue Qimeng* (*Introduction to Mathematics*) by Zhu Shijie. [4]This is a mysterious method to obtain the procedure to determine numbers. [5]Although it is dif-
8v ficult to explain how marvelous it is to understand this rule, we try to state an example of my understanding relying upon evidence and present here the meaning of investigation.

[6]Suppose there is a rectangle of area 180 [squared] *bu*. [7]Given: The sum of the length and width is 27 *bu*. [8]Question: How much are the length and width respectively?
[9]Answer: width 12 *bu*, length 15 *bu*.

[10]When we have an area and extract the square root from it, we place the area in the Reality row, make the Square row empty and place one rod in the Corner row and extract the square root using three rows. [11]When we have a volume and extract the cubic root from it, we place the volume in the Reality row, make the Square and the Side rows empty and place one rod in the Corner row and extract the cubic root using four rows. [12]When we have an [4 dimensional] accumulation and extract the 3-root from it, we place the accumulation in the Reality row, make the Square and the two
9r Side rows empty and place one rod in the Corner row, and extract the 4-root using five rows.[13] In this way, it will be taken as evidence that according to the number of multiplications we make use of lower and lower rows. [14]Generally speaking, if we can exhaust the Reality row by extraction, this is not because we are subtracting numbers with the same sign. [15]We must understand that, only by "adding numbers if they have the same sign and subtracting numbers if they have different signs," do we attain the solution. [16]Therefore, if the number in the Reality row is negative, then the Corner row is always positive. [17]Because the positive and the negative numbers appear simultaneously, the root can be extracted naturally. [18]Also, the obtained side goes back to the square accumulation [i.e., area] if we multiply it by itself, to the cubic accumulation [i.e., volume] if we multiply it by itself twice, and to the 3-multiplicational accumulation [i.e., 4 dimensional accumulation] if we multiply it by itself thrice. This is an ordinary manipulation. [19]Considering that the true number is always placed in the Reality row, we make the Reality row empty and
9v place a counting rod in the Square row and call it a virtual side. If we multiply the virtual side by itself, the Reality and Square rows become empty and the rod goes down to the Corner row; we use three rows in total. [20]This is the virtual square accumulation. [21]If we multiply the virtual side by itself twice, the Reality, the

Square and the Side rows become empty and the rod goes down to the Corner row; we use 4 rows in total. [22]This is the virtual cubic accumulation. [23]If we multiply the virtual side by itself thrice, the Reality, the Square, the [first] Side, and the [second] Side rows become empty and the rod goes down to the Corner row; we use 5 rows in total. [24]This is the virtual 3-multiplicational accumulation. [25]From the preceding argument, we understand that the rod which was first placed in the Square row goes down to the Corner row if the multiplication is repeatedly operated. [26]In this situation, if we cancel out the true value of accumulation with the virtual value of accumulation, it seems at first reasonable that the total cancellation happens,
10r but because the true and the virtual values are of different kinds, the result cannot be empty in number; the [true] value of accumulation stays in the Reality row as a negative number and the one rod stays in the Corner row as a positive number after several empty rows. ([27]If the value at the Reality row is positive, then [the value in] the Corner row is negative.) [28]In this way, we can establish naturally the complete equation to be extracted. [29]After that, we set the quotient, multiply up from the Corner row to the Reality row, "adding numbers if they have the same sign and subtracting numbers if they have different signs;" finally we can extract the root from this. [30]The obtained quotient is the side which we were looking for.

> [31]In the preceding, making the Reality row empty and placing a rod in the Square row, we name what we are to seek, find the same kind of true and virtual
> 10v numbers by the ordinary manipulation, and cancel out. It is a mysterious marvel to investigate and understand that we can thus establish the complete equation to be extracted using corresponding rows.

[32]Suppose there are a few articles and distribute them to a few persons. To solve the problem to find the number of articles per person, we place the number of articles in the Reality row and the number of persons in the Norm row and execute the division to determine each person's share. This is an ordinary procedure. [33]By the new procedure, making the Reality row empty and placing one rod in the Square row, we represent the virtual share per person, which, multiplied by the number of persons, represents the virtual total number of articles. [34]If we cancel it with the true total number, we find naturally the equation to be extracted with two rows, Reality and Norm. At once, we obtain the number of articles per person by the rule of division by quotient.

> [35]It is routine reasoning to divide the number of articles by the number of
> 11r persons and to get the share per person. [36]Although it seems not so easy, if we employ the rule of element placement, we can find naturally the equation to be divided without discerning the reason that division should be employed. How splendid it is!

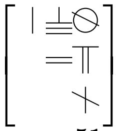
[37]Also, suppose there is a rectangle with known area. The problem is to find how long the length and width are when their difference is given. In an old method, we first multiply the area of the rectangle by 4 and add the square of the difference of the length and width to it, and extract the square root from it. Thus, we obtain the sum of the length and width, from which we subtract the difference, halve the result, and obtain the width. Adding the difference to it, we obtain the length.

[38]In another method, we place the area of the rectangle in the Reality row, the difference of the length and width in the Square row, and one rod in the Side row. We obtain the width by extracting the root from this. Because there is a number in the Square row, we call this the square root extraction with subordinate.

11v [39]In the new procedure, we make the Reality row empty and place a rod in the Square row. We call this the virtual width. We form the virtual length by adding the difference to the virtual width, and the virtual area by multiplying the virtual length and the virtual width. We cancel out the virtual area with the true area and find naturally the equation with subordinate, from which we extract the root.

> [40]Like in the old method, we arrange the area of the rectangle at the four corners and place the square, the side of which is the difference of length and width, in the center. Considering this figure of the square of the sum of length and width, we can find easily the sum of the length and width. This reason works quite fast with this problem. However, if we try to elaborate the procedure always in this manner, even with not so difficult problems we cannot formulate the reason after deliberation and cannot find the procedure to determine
> 12r numbers. [41]Now the rule of element placement is a mysterious method to find the procedure quickly, although its reason is hidden deeply. [42]It is not, however, to find [the solution] immediately without investigation. [43]Investigating repeatedly mainly by the reason of multiplication, addition, and subtraction, we obtain this equation.

[44]**Series of operations to solve the problem** [45]**Place the celestial element unit as the width** $\begin{bmatrix} \bigcirc \\ \text{𝍠} \end{bmatrix}$. [46]**Subtract this from the sum and make the remainder the long side** $\begin{bmatrix} \text{𝍪𝍦} \\ \text{𝍠̸} \end{bmatrix}$. [47]**Multiply the short side by it and make this the area of the rectangle** $\begin{bmatrix} \bigcirc \\ \text{𝍪𝍦} \\ \text{𝍠̸} \end{bmatrix}$. [48]**Move this to the left.** [49]**Place the area, which is canceled out by the number in the left, and**

obtain the equation  **.** [50]**Extract the square root from this and obtain the width 12** ***bu.*** [51]**Subtract this form the sum and obtain the length 15** ***bu.*** **(**[52]**We omit the main procedure.)**

[53]The investigation of procedure by reasonable evidence is sometimes visible and easy, and sometimes hidden and difficult. [54]If we use the rule of element placement, we will always be able to find its subtlety. [55]Although its rules and reasons are investigated in thousands of ways, it consists only of addi-
12v tion, subtraction and multiplication. [56]It should be called the greatest rule, [57]which we admire stating its meaning.

[58]Master Seki Takakazu was my teacher. [59]Once he invented further true and virtual numbers relying on the evidence of the rule of element placement and formulated the *Kai Fukudai no Hō* (*Method for Solving Concealed Problems*); [60]this should also be called a mysterious feat.

[61]If the above rule of element placement is to be investigated and understood with reasonable evidence, it can be explained almost as in the preceding. But we cannot say that it can be understood only by reasonable evidence, [62]nor can we say that it can be understood only by numerical evidence. [63]There is not necessarily the reasonable or numerical evidence; but it is marvelous that we understand it without
13r expectation and obtain it without noticing. [64]This understanding is completely the same as that of those who understand by evidence; it is attained by one's own native straight character when the time of the truth becomes mature. [65]There are many marvels besides the rule of element placement. [66]Without regards to shallow or deep, easy or difficult, all the understanding is attained in the same way. [67]If one is not given this straight character, even if he studies thoroughly all mathematics, he cannot attain to perception of the truth.

III. Investigating the Rule of Reduction

[1]**Suppose [we are] given 105 [parts] out of 168 parts.** [2]**Question: How much is it if reduced?**

[3]**Answer: 5 [parts] out of 8 parts.**

[4]Place [on the counting board] the denominator 168 and the numerator 105. Take 2 as a divisor. ([5]Although the divisor 1 is the starting number, we do not employ 1 because the denominator and the numerator do not change if they are
13v divided by 1.)

Incrementing the divisor stepwise by 1 until [we reach] the numerator, we examine divisibility taking these [incremental numbers] as divisors. It happens that neither the denominator nor the numerator are settled: ([6]the denominator and the numerator are called not settled if they have decimal places of *bu* and *ri* after division.) It may happen that the denominator is settled but the numerator is not settled or that the numerator is settled but the denominator is not settled. In these cases, we do not employ the divisor. [7]If both the denominator and the numerator are settled, the divisor is kept. [8]In this problem, we keep the settling divisors 3, 7 and 21; [9]3 and 7 are prime numbers and these two numbers multiplied give the main number 21. Therefore, we take 21 as the [greatest common] divisor, by which we divide the denominator and the numerator.

[10]In this way, starting from the divisor 2 and incrementing it stepwise by 1, we examine to find the cases where both the denominator and the numerator
14r are settled. After that, we investigate a simplified procedure. First, we remove completely the denominator by the numerator, then we remove completely the numerator by the remainder of the denominator, then we remove completely the remainder of the denominator by the remainder of the numerator, and then we remove completely the remainder of the numerator by the [second] remainder of the denominator. In this way, we repeatedly remove completely the remainders [of the denominator and of the numerator] by each other. If we find the remainders of the denominator and of the numerator coincide, we understand that it is the divisor [of reduction] and thus establish the rule of reduction and the procedure of mutual removal. ([11]If, at the last stage of removal, the remainder becomes empty, we stop at one step before to make the remainders of the denominator and of the numerator equal, which divide completely both the denominator and the numerator.)

[12] **Main procedure to solve the problem** **[13]Place the denominator 168 and the numerator 105. [14]We remove completely the denominator by the numerator; the remainder of the denominator is 63. [15]We remove completely the numerator by the remainder of the denominator; the re-**
14v **mainder of the numerator is 42. [16]We remove completely the remainder of the denominator by the remainder of the numerator; the second remainder is 21. [17]We remove completely the remainder of the numerator by the second remainder of the denominator; the second remainder of the numerator is 21. [18]At this stage, the remainders of the denominator and of the numerator coincide. [19]We take 21 as the divisor of reduction, [20]by which we reduce the denominator and the numerator to determine the reduced fraction.**

[21]The reduction of fraction controls cumbersome fractions. [22]By this procedure, which removes completely the denominator and the numerator mutually,

we can investigate and determine the reduced factor in one step. [23]Generally speaking, in all problems or in all procedures related to numbers, we cannot escape from the reduction of fraction. [24]That is, in order to extend the procedure [of reduction], although a variety of fractions are produced, according to the meaning of a problem, all are based on the reduction and can be handled
15r by the procedure of mutual removal. [25]This [rule] looks very elementary but is indeed very profound. [26]Therefore, by examples we explained its meaning.

[27]The rule of reduction and the procedure of mutual removal are very simple. Although we rely on some bases if we try to understand the reason behind them, the rule of reduction can be established thoroughly by numerical evidence, as reduction is independent of articles' names in the problem. Therefore, we regard it as an investigation of rules by numerical evidence.

IV. Investigating the Rule of Finding Differences

[1]Suppose there is a quadrangular pile [2]with a base length of 19. [3]Question: How much is the sum?
[4] Answer: 2470.

[5]When the base length of the quadrangular pile is 1, the sum is counted to be 1. [6]This is case 1. [7]Next, when the base length is 2, we count the sum and obtain
15v 5. ([8]That is, we add 1 and 4.) [9]This is case 2. [10]Next, when the base length is 3, we count the sum and obtain 14. ([11]That is, we add 1, 4 and 9.) [12]This is case 3. [13]Next, when the base length is 4, we count the sum and obtain 30. ([14]That is, we add 1, 4, 9 and 16. [15]Similar calculations for case 5 and onwards.) [16]This is case 4. [17]Next, when the base length is 5, we count the sum and obtain 55. [18]This is case 5. [19]Next, when the base length is 6, we obtain the sum 91. [20]This is case 6. [21]Next, when the base length is 7, we obtain the sum 140. [22]This is case 7. ([23]The calculations for case 8 and onwards are similar.)

[24]The value of a sum is, originally, a kind of cubic accumulation. [25]Therefore, if we take the differences of terms three times according to the base length, all terms become equal to each other. This indicates that we should determine the sum using the number of 2-multiplication accumulation of the base length.
16r [26]Thus, based upon this evidence we understand the rule of finding differences.

[27]At each case, we divide the sum by the base length. We call this the first definite sum. [28] We obtain 1 for case 1, $2\frac{1}{2}$ for case 2, $4\frac{2}{3}$ for case 3, $7\frac{1}{2}$ for case 4, 11 for case 5, $15\frac{1}{6}$ for case 6 and 20 for case 7. [29]We subtract the definite sum of each case from that of the subsequent case and call it the definite sum difference of each

case. [30]We obtain $1\frac{1}{2}$ for case 1, $2\frac{1}{6}$ for case 2, $2\frac{5}{6}$ for case 3, $3\frac{1}{2}$ for case 4, $4\frac{1}{6}$ for case 5 and $4\frac{5}{6}$ for case 6.

> [31]It seems that we should divide the definite sum difference by the base length. But if we divide it by the base length, the number becomes uneven and not equal to each other. [32]Therefore, we search and understand that the division should be done by the difference between the base lengths of the case and the subsequent case.

[33]In each case, we subtract the base length from that of the subsequent case and call it the "square case difference divisor." [34]For each case, we obtain 1, [35]by which we divide the definite sum difference of each case and call this the square sum for the case. [36]We obtain $1\frac{1}{2}$ for case 1, $2\frac{1}{6}$ for case 2, $2\frac{5}{6}$ for case 3, $3\frac{1}{2}$ for case 4, $4\frac{1}{6}$ for case 5 and $4\frac{5}{6}$ for case 6. [37]We subtract the square sum from that of the subsequent case and call this the square sum difference. [38]We obtain $\frac{2}{3}$ for case 1, $\frac{2}{3}$ for case 2, $\frac{2}{3}$ for case 3, $\frac{2}{3}$ for case 4 and $\frac{2}{3}$ for case 5.

> [39]It seems that we should divide the square sum difference by the base length,
> 17r but if we divide it by the base length or by the difference between the base length and that of the subsequent case, we find the numbers uneven and not equal to each other. [40]Therefore, we search and understand that the division should be done by the difference between the base length and that of the 2 cases before. [41]Also, using this method, if we want to calculate the 3-multiplication sum difference, we take the difference between the base length and that of the 3 cases before as the 3-multiplication case difference divisor; if we want to calculate the 4-multiplication sum difference, we take the difference between the base length and that of the 4 cases before as the 4-multiplication case difference divisor. [42]Further cases can be treated similarly. We understand that the case difference divisor of higher order can be obtained step by step.

[43]In each case, we take the difference between the base length and that of the 2 cases before as the "cubic case difference divisor." [44]For each case we obtain 2,
17v [45]by which we divide the square sum difference of each case and call this the cubic sum of that case. [46]We obtain $\frac{1}{3}$ for case 1, $\frac{1}{3}$ for case 2, $\frac{1}{3}$ for case 3, $\frac{1}{3}$ for case 4 and $\frac{1}{3}$ for case 5. The numbers being equal to each other, [47]we take $\frac{1}{3}$ as the "cubic difference."

[48]We multiply the base length of each case by itself and multiply this by the "cubic difference," subtract this from the first definite sum, and call this the second definite sum. [49]We obtain $\frac{2}{3}$ for case 1, $1\frac{1}{6}$ for case 2, $1\frac{2}{3}$ for case 3, $2\frac{1}{6}$ for case 4, $2\frac{2}{3}$ for case 5, $3\frac{1}{6}$ for case 6 and $3\frac{2}{3}$ for case 7. [50]Consecutively we subtract the second definite sum from that of the subsequent case and call this the [second] definite sum

difference. [51] We obtain $\frac{1}{2}$ for case 1, $\frac{1}{2}$ for case 2, $\frac{1}{2}$ for case 3, $\frac{1}{2}$ for case 4, $\frac{1}{2}$ for case 5 and $\frac{1}{2}$ for case 6. [52]It each case, we divide them by the square case difference divisor and call them the [second] square sum of the case. [53]We obtain $\frac{1}{2}$ for case
18r 1, $\frac{1}{2}$ for case 2, $\frac{1}{2}$ for case 3, $\frac{1}{2}$ for case 5 and $\frac{1}{2}$ for case 6. The numbers being equal to each other, [54]we take $\frac{1}{2}$ as the square difference.

[55]We multiply the base length of each case by the square difference, subtract the second definite sum of the case by this, and call it the third definite sum. [56] We obtain $\frac{1}{6}$ for case 1, $\frac{1}{6}$ for case 2, $\frac{1}{6}$ for case 3, $\frac{1}{6}$ for case 4, $\frac{1}{6}$ for case 5, $\frac{1}{6}$ for case 6 and $\frac{1}{6}$ for case 7. The number being equal to each other, [57]we take $\frac{1}{6}$ as the "definite difference." ([58]When we determine the cubic sum, the numbers at each case become equal. Therefore, we only need to determine the three kinds of numbers, those of the first, the second and the third cases. But, for the moment we calculate seven kinds of numbers to show that they are equal in each case.)

[59]We reduce the three differences to a common denominator and obtain 2 for the cubic difference, 3 for the square difference and 1 for the definite difference, [60]the common denominator being 6.

18v

[61]It is difficult to search how to determine the square difference, cubic difference and furthermore if the differences of base lengths of each case are equal. [62]Therefore, we search and understand the situation making the base lengths uneven in different cases. [63]Also, it is hard to search how to determine the positive or negative signs of the three differences by the numbers of the quadrangular pile. [64]Therefore, as in the calendrical calculation of the difference of degrees in the movement of the sun and the moon, making the sum numbers larger or smaller we search and understand the rule of signature. [65]We mention no further details.

[66] **Main procedure to solve the problem** **[67]We double the base length, add 3 to this, multiply this by the base length, add 1 to this, also, multiply this by the base length, and divide this by 6; we obtain the sum.**

[68]Knowing that the sum number corresponds to the 2-multiplication sum of
19r the base length, by this evidence we investigate the solution, comparing it with the three kinds of numbers, namely, the base lengths, the square of the base lengths and the 2-multiplication sum of the base lengths, and understand the rule of [linear] equations. [69]This is equivalent to the following: we arrange the base lengths for each case in one line, the square of the base lengths in the line below, the 2-multiplication sum of the base lengths in the line below and the sum numbers in the last line. Contracting this arrangement, we can find the coefficients. ([70]Also, Master Seki created the general procedure of square piles. [71]The procedure for the self-multiplication pile coincides naturally with

the calculation of quadrangular piles.) [72]We omit these procedures.

[73]In the above [rule of] finding differences, we consider similar examples, calculate numbers by decomposition, and understand the rule by numerical evidence. [74]Generally speaking, we cannot obtain the rule or the procedure, which are understood by numerical evidence, by discerning the reason completely. [75-76]Therefore, we do not insist on seeking its reason and apply the procedure naturally with the
19v help of the rules: this is to conform ourselves to the Way of Mathematics.

[77]Generally speaking, among methods of investigation, some rely necessarily on reasonable evidence, some rely necessarily on numerical evidence, and also some rely on both. [78]He who investigates relying on reasonable evidence, even though he does not search for numerical evidence, as long as he truly endeavors with his whole heart, he will certainly attain understanding; [79]if he masters the rule of element placement and employ it, he can overcome a lot of difficulties and attains understanding with less effort. [80-81]He who investigates relying on numerical evidence, even though he does not insist on discerning the reason, as long as he determines numbers entirely and investigates them deeply,
20r he certainly will attain understanding. [82]As the methods of investigation, which will increase or decrease at the extreme point of saturation or exhaustion, consist of the determination of numerical examples by decomposing and by slicing, according to the variation of the examples, either reasonable evidence or numerical evidence can be investigated, and relying on these evidence the rules or the procedures can be established in thousands of manners. [83]Certainly, although it is possible to learn how to apply a procedure relying on rules, it is rare to discern how to understand the rules recognizing the character. [84]Therefore, it is taken easy to discern the reason from the heart and difficult to determine numbers with the strength. [85]But, without distinguishing the two ways of investigation, one relying on reasonable evidence and the other relying on numerical evidence, he who insists on attaining complete understanding by reasonable evidence in the investigation where he should rely on numerical
20v evidence, he encounters obstacles and cannot attain [such] understanding; [86]he who insists to investigate by numerical evidence in the investigation in which he should rely on reasonable evidence, he cannot exert himself fully and stagnate. [87]But if he does not seek numbers, he should understand that it is because of the [problem's] character. [88]In this case, it is no use to continue thinking in vain and paying further attention [to the problem]. [89]Therefore, now we explain that it is fundamental to distinguish two paths of investigation of numbers, one by reasonable evidence, the other by numerical evidence. The reader is advised not to stray from the path of investigation.

Four examples on the Reason of procedure

V. Investigating the Procedure of Repeated Exchanges of Weavers

21r **[1]Suppose there are weavers. [2]3 weavers weave 4 *tan* of tapestry in 21 days. [3]Now 7 weavers weave in 45 days. [4]Question: How many *tan* of tapestry are woven?**

[5]Answer: 20 *tan* of tapestry.

[6]Place [on the counting board] 4 *tan* of tapestry as given at first. Divide this by 3 weavers and we find that one weaver weaves 1 *tan* 33333 strong of tapestry in 21 days. [7]Divide this further by 21 days and we find that one weaver weaves 6 *ri* 34921 weak of tapestry [per day]. [8]Therefore, multiplying this by 45 days given later, we find that one weaver weaves 2 *tan* 857143 strong of tapestry in 45 days. [9]Multiplying this further by 7 weavers, we find that 7 weavers weave 20 *tan* of tapestry in 45 days.

> [10]Although the original procedure is as stated, after several repetition of divisions we do not always return to a correct number when some numbers are not
> 21v "settled." [11]Therefore, following the rule of multiplying first and dividing later, we simplify the procedure.

[12] **Main procedure to solve the problem** **[13]Place 4 *tan*, the first given length of tapestry, [14]multiply this by the later given 7 weavers, and also multiply this by 45 days. Place this in the Reality row. [15]Place the first given 3 weavers, [16]multiply this by 21 days. Place this in the Norm row. [17]Divide this [configuration] and we obtain the length of woven tapestry.**

> [18,19]In the beginning, it is hard to understand why only one division is sufficient if we multiply all multipliers to form the Reality row and if we multiply all divisors to form the Norm row. [20]Only after we investigate in depth, decomposing the reason of procedure for determining the number per unit, can we formulate this procedure by putting together all multipliers and all divisors.

[21]In the above procedure of repeated exchanges of weavers, we decompose and
22r investigate the procedure by reasonable evidence; that is, by the procedure we establish the rule of exchange. [22]If we seek the reason according to the rule and procedure, it cannot be clarified immediately. But originally this procedure was established by reasonable evidence which we are searching for. Therefore, we classify this [example] as the investigation of procedure by reasonable evidence.

VI. Investigating the Procedure for Finding the Extreme Volume of a Parallelepiped

[1]Suppose there is a parallelepiped. [2]The difference of the length and width is 7 *shaku* and the sum of the width and height is 8 *shaku*. [3]We want to make the volume as large as possible. [4]Question: How much are the length, the width, the height and the extreme volume respectively? [5]Answer: Width 4 and 2/3 *shaku*; length 11 and 2/3 *shaku*; height 3 and 1/3 *shaku*; volume 181 and 13/27 [cubic] *shaku*.

[6]We do not investigate by numerical evidence. [7]Immediately relying on reason
22v we investigate by the rule of element placement.

[8]Place the celestial element unit as the width $\begin{bmatrix} \bigcirc \\ | \end{bmatrix}$. [9]Add the difference to this and make this the length $\begin{bmatrix} |\,D \\ | \end{bmatrix}$. [10]And by the width we subtract the sum and make this the height $\begin{bmatrix} |\,S \\ \not| \end{bmatrix}$. [11]Multiply the length, the width, and the height, and make this the volume

$\bigcirc$	**Reality**
$|\,DS$	**Square**
$\not|\,D\,|\,S$	**Side**
$\not|$	**Corner**

.

[12]We take this as the original formula and search for its meaning in [solution] procedures. If the volume is given numerically in the problem, we cancel the original formula by the value of the volume, which remains in the Reality row. [13]Because the Square row will be extracted completely when the Reality row becomes extremely large, we make the width, which we established first, as the quotient and applying the rule of extraction of the quotient number to the original formula we find the extreme case of the Square row and obtain the
23r equation by cancellation.

[14]We make the width the quotient $\begin{bmatrix} \bigcirc \\ | \end{bmatrix}$. [15]Place the original Corner row -1 and multiply it by the quotient and add the original Side row to it and make it the first number to extract the Side row: $\begin{bmatrix} \not|\,D\,|\,S \\ \not| \end{bmatrix}$. (We obtain a negative.) [16]Further multiply this by the quotient and make this the first number which ought to extract the Square row (neg-

ative) $\begin{bmatrix} \bigcirc \\ \not{|}D \mid S \\ \not{|} \end{bmatrix}$**.** **[17]Also, we place the original Corner row and multiply this by the quotient, add this to the first number to extract the Side row and make it the second number to extract the Side row (negative)** $\begin{bmatrix} \not{|}D \mid S \\ \not{||} \end{bmatrix}$**.** **[18]Further, we multiply this by the quotient and make this the second number which ought to extract the Square row (negative)** $\begin{bmatrix} \bigcirc \\ \not{|}D \mid S \\ \not{||} \end{bmatrix}$**.** **[19]Add this to the first number which ought to extract the Square row and make it the extreme case of the Square row (negative number)** $\begin{bmatrix} \bigcirc \\ \not{||}D \parallel S \\ \not{|||} \end{bmatrix}$**.** **[20]Move it to the left.** **[21]Place the original Side row (positive number), and cancel it by what was moved to the left.** ([22]In the rule of extraction of the quotient [number], we use addition for same signatures and subtraction for differing signatures. [23]Therefore, also in the cancellation, we combine by addition same signatures and by subtraction differing signatures.) **We obtain the equation**

$\mid DS$	**Reality**
$\not{||}D \parallel S$	**Square**
$\not{|||}$	**Side**

.

23v [24]In this problem, we should obtain the solution using numbers. But in order to describe this procedure, we employ the names given in the problem.

[25] **Main procedure to solve the problem** **[26]Place the sum. [27]By the difference we multiply this and make this the Reality row (positive). [28]Also, place the sum. By the difference we subtract it. Double the remainder and make this the Square row (positive). [29]Make 3 the Side row (negative). [30]Extracting the square root from it, we obtain the width. [31]Adding the difference to it we obtain the length. [32]By the width we subtract the sum and obtain the height. [33]Multiplying the length, the width, and the height we obtain the volume. ([34]The obtained width has the inexhaustible digit under the *shaku*. [35]Therefore, in the original formula, we multiply the Reality row by 3, leave the Side row unchanged, and divide the Side row by 3, and extracting the square root from it we obtain 14. [36]Dividing it by 3 we obtain 4 and 2/3 *shaku*.)**

[37]The above procedure for a parallelepiped is an example of an investigation of a procedure by reasonable evidence. [38]If we seek the reason according to this procedure, it is hidden and cannot be observed. But because we established this

[procedure] discerning the reason by the rule of element placement, we recognize this
24r as an investigation of a procedure by reasonable evidence. [39]Generally speaking,
the rule and procedure are not always established by numerical evidence. [40]Even
if they are established with reasonable evidence, if we seek the reason according to
the rule and procedure, it may be hidden and cannot be observed. [41-42]In this
case, we do not try to discern the reason by force, entrusting the reason to the rule
and procedure, we simply follow the rule and procedure and employ them; this is to
conform ourselves to the Way of Mathematics.

> [43]Once, someone asked me the procedure to find the extreme number of in-
> crease and decrease in the calculation of the delay of lunar movement in the
> *Shoushili* (Time Granting Calendar) using three differences, cubic, square, and
> definite. [44]I did not discern the reason, [45]but decomposed example numbers
> and could immediately search out the evidence to put 1 for the Reality row,
> 2 for the Square row, and 3 for the Side row and understood this procedure.
> ([46]We omit the obtained numbers.) [47]But later, when I changed the example
> 24v problem and asked for this extreme volume of the parallelepiped, I did not rec-
> ognize the similarity of these problems. [48]Then relying on the rule of element
> placement to discern the reason, I searched out the procedure immediately.
> [49]Depending on the time and the problem, we choose following our intuition
> reasonable evidence or numerical evidence [for our investigation]. [50]By this we
> should realize that we attain the same understanding either through investiga-
> tion by numerical evidence or through investigation by reasonable evidence.

VII. Investigating the Procedure of Arithmetic Removal

[1]We arrange 30 pebbles, ([2]half of which are black and the other half of which are
white,) alternately, and remove every tenth pebble repeatedly in the conforming or-
der. [3]We reach an arrangement, where there remains only one black pebble, having
25r removed the rest of the 14 black pebbles. [4]From here we start from the remain-
ing black pebble and remove every tenth pebble counting backward. ([5]Although
it amounts to the same thing removing [pebbles] in the conforming order, we tem-
porally follow the old tradition.) Finally, all white pebbles are removed and the
one black pebble remains. [6]For ages this game has been called the choice of the
step child. [7]Now we investigate and try to understand this procedure, looking for
different arrangements [that end with one black pebble.]

[8]We arrange pebbles ([9]all are white except for one black pebble) and examine if the arrangement is appropriate or not. If we remove every second pebble, 1, 3, 7, 15, 31, etc. are the appropriate cases where the black pebble remains. [10]If we remove every third pebble, 3, 5, 8, 30, etc. are the appropriate cases where [only] the black pebble remains [at the end].

[11]If we remove every fourth pebble, 1, 4, 8, 11, 15, etc. are the appropriate cases
where the black pebble remains. [12]If we remove every fifth pebble, 2, 5, 11, 14,
36, etc. are the appropriate cases where the black pebble remains. [13]If we remove
25v every sixth pebble, 1, 2, 7, 13, etc. are the appropriate cases. [14]After repeating several trials, we search and find that there are necessarily inappropriate cases [15]and appropriate cases and [16]that an appropriate case may also be appropriate or inappropriate [with a different removal number]. [17]Relying upon this evidence, we understand the main procedure.

[18]**Main procedure to find the cases** **[19]Place one rod ([20]representing the black pebble) in the Norm row, [21]making the Reality row empty. [22]Add 1 consecutively to the Norm row and the removal number to the Reality row. If the Reality row becomes larger than the Norm row, subtract the former by the latter. If the Reality row is exhausted, we remove one rod ([23]representing the black pebble) from the Norm row and find an appropriate case.**

[24]The procedure of arithmetic removal was investigated and understood by
my elder brother Kata'akira. [25]Kata'akira's native intelligence was close to
Takakazu [26]but his state of mind was so weak that he was sick for many days.
[27]Once he tried to apply the simplified procedure of the fifth side and found it
very complicated. [28]He said that, even though the solution involved numbers
with ten thousands digits, it would require only a hundred days if he calculated
26r one hundred digits in one day. Indeed, he finished all the calculation in about
one month. [29]After Kata'akira passed away, I remembered this episode and
admired his great achievement. [30]After less than ten days I calculated the
seed numbers for the table of the ecliptic and gave it to Nakane Jōemon. [31]I
was then fifty seven years old. [32]Also, when I was young, by a given mandate
I performed several steps of calculation to find the accumulated years from the
original date of the universe using the four astronomical data of the *Xuanmingli*.
After I completed the calculation, I thought that it required numbers with many
digits and was very difficult. [33]Now I am old and have lost half of my vigor but
with effort I can calculate numbers two times larger than what I could in my
26v earlier days. [34]Moreover I find no difficulty [in obtaining such results] because
Mathematics truly follows my heart. [35]Generally speaking, if one experiences
difficulty in determining numbers, in applying procedures, or in investigating
rules, it is because mathematics does not follow one's heart and so one is not
attaining the truth. [36]Was it only Kata'akira who truly recognized the reality
that mathematics may or may not follow one's heart? [37]It is the power by
which the soft smashes the hard and the small controls the large to stay calm
and to continue calculation without interruption, neither relying on one's own
intelligence nor using one's own physical power.

[38]In the above [procedure of] arithmetic removal, making examples and investigating them by decomposition, we attain the understanding of the rule and procedure
27r by numerical evidence. [39]Although there is a reason in the basis, we dare not discern it to obtain [the procedure]. [40]Only by looking at what numbers are [obtained in examples], can we orient our heart to this understanding by the numbers.

VIII. Investigating the Procedures for Finding the Surface Area of a Sphere

[1]Suppose there is a sphere [2]with diameter 1 *shaku*. [3]Question: How much is the surface area?

[4]Answer: The surface area is 314 [squared] *sun* 159265359 weak.

[5]We employ the procedure of whittling. ([6]We do not slice because slicing is not conformable to the character [of the sphere].) First, we determine the volume of the sphere of diameter 1 *shaku* 001 *ri*, remove the volume of the sphere of diameter 1 *shaku* from this, and obtain the real volume of the shell (1 *sun* 57236764672 strong). [7]Divide this by the width of the shell (5 *mō*) and so obtain the surface area of the shell (314 *sun* 473529344 strong). [8]Second, we determine the volume of the sphere of diameter 1*shaku* 00001 *shi*, remove the volume of the sphere of diameter 1
27v *shaku* from this, and obtain the real volume of the shell (1 *ri* 57081203481 strong). [9]Divide this by the width of the shell (5 *kotsu*) and so obtain the area of the shell (314 *sun* 162406962 strong). [10]Thirdly, we determine the volume of the sphere of diameter 1 *shaku* 0000001 *bi*, remove the volume of the sphere of diameter 1 *shaku* from this, and obtain the real volume of the shell (1 *shi* 57079648387 strong). [11]Divide this by the width of the shell (5 *sen*) and so obtain the area of the shell (314 *sun* 159296775 weak). [12]Thus, as the width of the shell becomes smaller, the true number [for the surface area] appears gradually.

> [13]Observing the surface areas of the three shells, relying on the procedure of decremental divisor, we can obtain the true surface area of the sphere 314 *sun* 159265359 weak. Investigating this, we find the number for the circular ratio appearing in the number for the [surface] area. [14]Therefore, we understand
> 28r that the circular ratio should be multiplied. Dividing the surface area by the circular ratio, we find the quotient is exactly equal to the whole number 100. [15]Investigating and understanding that it is the square of the diameter, we establish the main procedure.

[16]Also, regarding the center of the sphere as the apex of a cone, the radius of the sphere as the height of the cone and the volume of the sphere as the volume of the cone, we multiply the volume by the conic divisor 3, and divide this by the height of the cone to find the [base] area of the cone, which corresponds with the surface

area of the sphere.

[17]Multiply twice the diameter of the sphere by itself, multiply this by the circular ratio, and divide this by 6 to obtain the volume of the sphere. [18] Multiply this by the conic divisor 3, divide this by the radius of the sphere, and find the surface area of the sphere. Therefore, to simplify this procedure, first omit one [multiplication by the] diameter in the procedure for determining the volume
28v of the sphere, and also the division by 6. Finally, by this [simplified] procedure, multiplying the diameter by itself and multiplying this by the circular ratio, we obtain the surface area of the sphere immediately.

[19]**Main procedure to solve the problem** **[20] Place the diameter of the sphere, multiply it by itself, [21] multiply this by the rate of the circular circumference, divide this by the rate of the diameter, and obtain the surface area.**

[22]Master Seki said that, in order to understand thousands of rules, it is most essential to observe the form and to establish the path [of reasoning]. [23]His hidden purpose was to understand the true procedure from the beginning without any investigation. [24]Thus, in the latter procedure, he observed the form of a sphere and considered it as a cone and its center as the apex. In this way, observing the form and establishing the path [of reasoning], he understood the true procedure immediately without any investigation. [25]Therefore, he con-
29r sidered the former procedure second-rate. [26]Because originally I am of foolish character, if I want to understand, by reasonable evidence, the true rule only by observation, although it may be very easy if we encounter a procedure like this, which has a simple reason, I cannot always attain a solution when a given procedure is not based on a simple reason. [27]In such a case, we investigate repeatedly, relying exclusively on numbers, to understand there is some evidence, on which we can establish the true rule. [28]For this reason, I do not dare to consider the former procedure second-rate. [29]Certainly, is it because of my distorted character that it is difficult for me to understand without any investigation? [30]If I were straight in mind, without distinguishing numerical and reasonable evidence, I would be able to understand everything immediately without any investigation. [31]But because I am of distorted character, even
29v though I study deeply, I will not be able to attain such a state. [32]Generally speaking, in the numerical quantity, in the reason of procedure, and in the rule and law, everything is originally natural. [33]He who understands this does not tread on a new path; [34]his path merges with the natural path to attain understanding. [35]If this is the case, it is also appropriate to attain understanding after investigation. [36]I strongly recognize that Master Seki's natural intelli-

> gence is without parallel in the world. [37]He always said that problems on the circular area were very difficult to solve. [38]Alas, this is because he [chose to] operate in a relaxed manner, [39]but I dare say that even problems on the circular area can certainly be solved by tenacity. [40]This is only because I work in a painstaking manner. [41]The reason why Master Seki said that he could not solve this type of problem was that he operated in a relaxed manner to find a quick and easy solution, endeavoring to solve problems immediately without
> 30r any investigation. [42]It was not because he could not solve them. [43]Perhaps, he did not like to go into the matters thoroughly. [44]Because natively I am of foolish character, I cannot reach a quick and easy solution operating in a relaxed manner. [45]I am confident in a way to be peaceful even operating always in a painstaking manner. [46]Therefore, if I investigate [in this way], I know I will surely obtain the solution. [47]Reflecting on this, I know that my native character is one [part] out of ten less than that of Takakazu.

[48]In the above procedures for determining the [surface] area of a sphere, the former consisted in the determination of numbers by whittling and in the investigation of the procedure with this numerical evidence; [49]in the latter, without the determina-
30v tion of numbers and the investigation of procedure, [50]the reason was immediately discerned and the procedure was also immediately obtained.

[51]Certainly, these procedures being compared, the investigation by numerical evidence is complicated to apply but immediate to introduce; [52]the investigation by reasonable evidence indicates the reason very easily but is subtle and difficult to introduce. [53]Having proposed these two procedures I discussed their meaning and proved that both turned out to be the same understanding.

Four Examples on the Numerical Quantity

IX. Investigating Numbers Stemming from Decomposition

> [1]If we want to investigate by reasonable evidence, there is the rule of element placement, which unifies all the procedures. [2]If we want to investigate by numerical evidence, there is no way other than the procedure of decomposition. Furthermore, there is no definite rule, and processes to the solution differ
> 31r according to thousands of rules. [3]This means, the [procedure of] decomposition is the basis of determining numbers and discerning reasons, the way of investigation, and the method to find rules and procedures. [4]Therefore, if we decompose according to the form and character and investigate deeply to determine numbers, we surely understand the rule and procedure. [5]In this manner, we state its meaning and witness its importance.

[6]If he who decomposes the circumference of a circle cuts the diameter equally and

horizontally into thin slices, seeks the [length of the] right and left oblique chords cut by the horizontal lines and adds the oblique chords to seek the [approximate] circular circumference, then the parts of the circumference are not equal even if he cuts the diameter equally. [7]Therefore, if he seeks the circumference doubling
31v the sections of the diameter, these numbers being disobedient to the character, he stagnates in determining the extreme number and never obtain the evidence to understand the circle's character. [8]Therefore, when he cuts the circumference into the four angular form [i.e., by an inscribed square] and further doubling angles [i.e., forming an inscribed octagon, etc.], the circumference is cut into equal length and the numbers are obedient to the character of the circumference. Therefore, doubling the number of angles and seeking the angular circumferences at each step, by the repeated application of the procedure of incremental divisor he can determine the extreme number rapidly and obtain an evidence to understand the character of a circle.

[9]He who decomposes the volume of a ball, slices the diameter of the ball equally and makes each slice into the shape of a circular platform. Because the sum of the widths of these slices is the sagitta of an arc, we can calculate the chord of the arc, which we take as the diameters of the upper and the lower ends of the platform; the width of the slice is the height of the platform. By the procedure to seek the volume
32r of a circular platform, one finds the volume of each slice and summing these slices forms the cut out volume. ([10]If he omits the circle rates in seeking the volume of a platform, he can obtain the volume of a square platform.) [11]Further, doubling the number of slices and seeking the cut out volume at each step, investigating the obtained numbers to determine the incremental divisors, according to the procedure, we find the extreme number of the true volume. [12]Because this does not disobey the reason of volume seeking, he does not stagnate in determining the extreme number. But further investigating deeply, we find that the procedure to find the volume of a platform seems good as a reason but the numbers do not converge well. [13]Therefore, multiplying the sum of the square of the upper radius and the square of the lower radius by the height, and halving this to form the volume of the tubular slices and adding them up, we form the cut out volume of accumulated tubes. If, doubling the number of slices and seeking the polyhedral volumes, we apply the procedure of incremental divisor to determine the extreme number, we can find the
32v extreme number rapidly even with a very small number of slices. [14]Certainly, it is not indeed the procedure to find the volume of a platform to find the volume of a tabular slice. [15]This is a miraculous procedure in the decomposition of the volume of a ball and follows the character of the decomposition of the volume of ball.

> [16]In the decomposition of the circle and related objects, we seek total conformance with the form and character [17]and never venture outside conformity with them. [18]If we cut into slices what should be whittled into shells, we

are disobedient. [19]When we cut according to the diameter what should be cut according the circular circumference, we are disobedient. [20]When we cut horizontally what we should cut vertically, we are disobedient. [21]When we do not obey the form and character, even when we can find the true number, we
33r are slow in searching the extreme number and have difficulty in understanding the reason of procedure. [22]In order to understand how to obey its form and character, we first discern the reason, determine numbers, and then, relying on the numbers, we investigate deeply and so attain understanding. [23]Therefore, if we want to employ the [procedure of] decomposition, we should neither concentrate only in seeking the true number nor lose sight of the reason which distinguishes obedience and disobedience.

[24]The above decomposition is the investigation of numbers by reasonable evidence. [25]But once we start to investigate according to its form and character, we should recognize that numbers are to be investigated by numerical evidence.

X. Investigating Numbers which are Square Roots

[1]Suppose there is a regular square of area 1166 [squared] *bu*. [2]Question:
33v **How much is its square root?**
[3]Answer: One side is 34 *bu* with remainder 10 [squared] *bu*.
[4] **Main procedure to solve the problem** **[5]Place the area [of the regular square] in the Reality [row] [6]and 1 in the Side row. [7]Apply the [generalized] division to this [configuration] to extract a square root and obtain the side of the square.**

[8]We place the area in the Reality row and 1 in the Side row and moving over orders we observe the first quotient is on the order of 10. ([9]We omit the manipulation of moving over orders.) [10]If its first quotient is 10, then because of “one times one makes one hundred” it is smaller than the Reality [row]. [11]If it is 20, then because of “two times two makes four hundred” it is also smaller than the Reality [row]. [12]If it is 30, then because of “three times three makes nine hundred” it is again smaller than the Reality [row]. [13]If it is 40, then because of “four times four makes thirteen hundred” [14]it is instead larger than the Reality [row]. [15]Therefore,
34r it is known to be 30 *bu* and something. We take 30 as the first quotient, multiply the Side row by it, and place the product in the Square row. We multiply the Square row by the first quotient and subtract 900 from the Reality row. The remainder 266 [squared] *bu* [is now in the Reality row]. [16]Also, we multiply the Side row by the first quotient, add it to the Square row and obtain 60 *bu* in the Square row. [17]Now seek the second quotient. If it is 1 *bu*, then 61 [squared] *bu* being subtracted from the Reality [row], [we find] it is too small. [18]If it is 2 *bu*, then 124 [squared] *bu* being subtracted from the Reality [row], it is also too small. [19]If it is 3 *bu*, then 189

[squared] *bu* being subtracted from the Reality [row], it is also too small. [20]If it is 4 *bu*, then 256 [squared] *bu* being subtracted from the Reality [row], it is still too small. [21]If it is 5 *bu*, then 325 [squared] *bu* being subtracted from the Reality row,
34v it is too large. [22]Therefore, it is known to be 4 *bu* and something. We take 4 *bu* as the second quotient. Multiply the Side row by it, add the product to the Square row, multiply the Square row by the second quotient 4 *bu* and subtract 256 [squared] *bu* from the Reality row. The remainder 10 [squared] *bu* is now [in the Reality row]. [23]Repeating this investigation, we find the third and the fourth quotients and so on.

> [24]Although we start from the first quotient 10, make it larger and larger, examine whether the root is smaller or larger, and finally know the definite quotient to be 30, once we master the manipulation, we can observe immediately the quotient to be 30 neither searching several cases nor relying on any rule. [25]Also, as for the second quotient, although we start from 1 *bu*, make it larger and larger, examine whether it is smaller or larger, and finally know the definite quotient to be 4 *bu*, we can observe immediately the second quotient to be 4 *bu*,
> 35r establishing the rule of dividing the remainder in the Reality row by the Square row. [26]Although we seem to know immediately without any investigation, in truth, we do not know it immediately; [27]we investigate it in a single step. [28]A novice cannot obtain the definite quotient from the beginning without several cases of investigation. [29]Once he obtains the definite quotient by repeated investigation, with matured manipulation, he understands how to know the definite quotient at once.

[30]In the above [procedure of the] extraction of a square root, we establish the procedure by reasonable evidence and then determine numbers by the procedure. [31]Although it is hard to clarify the reason relying on the procedure of extraction of a square root, because we establish the procedure by discerning the reason, we classify the [procedure of the] extraction of a square root as the determination of
35v numbers by reasonable evidence.

XI. Investigating Numbers Related to the Circle

[1]Cutting a circle of diameter 1 *shaku* we form the quadrangle [inscribed square] and determine the square of the cut out [i.e., inscribed polygon's] perimeter. [2]Also, cutting it again we form the octagonal and determine the square of the cut out perimeter. [3]Also, cutting it again we form the 16-angle [regular polygon] and determine the square of the cut out perimeter. [4]Also, cutting it further we form the 32-angle, also the 64-angle, and also the 128-angle. [5]Doubling the number of angles, we determine the square of the cut out perimeters successively. Observing these

numbers, we find, although the numbers are coming closer and closer to the true number as the number of angles are doubled, they do not attain it. [6]Therefore, subtracting the consecutive squares of cut out perimeters from each other, investigating the value attained by the successive quotients, we can elaborate the true number by the procedure of incremental divisor. ([7]The procedures to determine the square of the cut out perimeters and the numbers determined were described in the *Enritsu* (*Circle Rates*) [8]and are omitted here.)

36r

> [9]At the beginning Master Seki extracted the root from the square of the angular [i.e. polygon's] side to determine the angular side and employed the cut out [polygon's] perimeter [to approximate the perimeter of the circle]. [10]Now we determine the square of the cut out perimeter by means of the square of the angular sides, thus skipping the task of root extraction. [11]It is not from the beginning that we discern we have only to employ the squared numbers. [12]First we employed the cut out perimeter and then with deep investigation we understood we could employ the squared numbers.

[13]Starting from the quadrangle, we subtract the square of the cut out perimeter from the following one, and call the remainder the first difference. [14]Dividing the difference by the preceding one, we investigate and understand that the ratios of consecutive discrepancies tend to 1/4. [15]Therefore, by the procedure of incremental divisor, we divide the first difference by 3, which is the denominator minus 1, and add it to the square of the cut out perimeter, to make the square of the first approximate
36v circumference.

[16]Starting from the [regular inscribed] octagon we subtract the square of the first approximate circumference from the following one, and call the remainder the second difference. [17]Dividing the difference by the preceding one, we investigate and understand that the ratios of consecutive discrepancies tend to 1/16. [18]Therefore, by the procedure of incremental divisor, we divide the second difference by 15, which is the denominator minus 1, and add it to the square of the first approximate circumference, to make the square of the second approximate circumference.

[19]Starting from the 16-angle [regular inscribed polygon] we subtract the square of the second approximate circumference from the following one, and call the remainder the third difference. [20]Dividing the difference by the preceding one we investigate and understand that the ratios of consecutive discrepancies tend to 1/64. [21]Therefore, by the procedure of incremental divisor we determine the square of the third approximate circumference. [22]When we determine the square of the fourth
37r approximate circumference, the incremental divisor is 1/256. For the fifth approximation, the divisor is 1/1024. [23]In this way, we investigate and understand that the denominator of the incremental divisors are of the repeated power of 4. By applying repeatedly the procedure of incremental divisor to the square of the approximate

circumference we determine the square of the definite circumference. ([24]The numbers [in this procedure] of incremental divisor are recorded in the *Enritsu* [25]and are omitted here.)

> [26]At the beginning, Master Seki recognized how to determine the definite circumference by the procedure of incremental divisor, but applied it only once. [27]Therefore, by determining the cut out perimeter of up to a 131072-angle [regular inscribed polygon] he could elaborate the true number to fifteen or sixteen digits. [28]Now we investigate and understand that by repeated application of the procedure of incremental divisor, determining the square of the cut out perimeter of up to the 1024-angle [regular inscribed polygon], we elaborate the true number by a little more than 40 digits. [29]Also in this case, we could not discern from the beginning that we should apply repeatedly the [procedure of]
> 37v incremental divisors. [30]After employing the [procedure of] incremental divisor one time, with deep investigation, we understood that we should repeat the application.

[31]By the procedures of decomposition and of incremental divisor we can determine the definite circumference:

3 *shaku* 1 *sun* 41592653589793238462643383279502884197l2 strong

By the procedure of residual division we form the rates of the circumference and of the diameter.

[32]Now put the original number 1 *shaku*, [33]by which we divide the definite circumference to get the first quotient and the first inexhaustible. ([34]Always divide the large number by the small.) [35]Divide the original number 1 by the first inexhaustible to get the second quotient and the second inexhaustible. [36]Divide the
38r first inexhaustible by the second inexhaustible to get the third quotient and the third inexhaustible. [37]Divide the second inexhaustible by the third inexhaustible to get the fourth quotient and the fourth inexhaustible. [38]Divide the third inexhaustible by the fourth inexhaustible to get the fifth quotient and the fifth inexhaustible. [39]In this way, dividing the inexhaustible of the preceding step by the inexhaustible of the present step, we determine the quotients consecutively.

[40]Let the original number 1 be the rate of the diameter and let the first quotient be the rate of the circumference. [41]These rates are called the first weak rates. [42]By the second quotient multiply the first rates of the diameter and of the circumference respectively, and add the original number 1 to the rate of the circumference, to make the second strong rates. [43]By the third quotient multiply the second rates of the diameter and of the circumference respectively, and add the first rates of the diameter and of the circumference to the said rates respectively, to make the third weak rates. [44]By the fourth quotient multiply the third rates of the diameter and of

the circumference respectively, and add the second rates of the diameter and of the
38v circumference to the said rates respectively, to make the fourth strong rates. [45]In this way, multiplying the rates of the diameter and of the circumference of the said step by the quotients of the following step and adding the rates of the diameter and of the circumference of the preceding step to the rates of the said step, we determine the rates of the following step. They become strong and weak alternatively and are convergent. ([46]The rates [obtained by the procedure] of residual division are recorded in the *Enritsu* [47]and are thus omitted here.)

[48]At the beginning when Master Seki employed the procedure of residual division, he added 1 repeatedly to the diameter and 3 to the circumference respectively to form the rates of the diameter and of the circumference, and at every step divided the rate of the circumference by that of the diameter. If the obtained number is smaller than the definite circumference, he added 1 to the diameter and 4 to the circumference respectively. [49]Kata'akira, having found this procedure too complicated, investigated and established this procedure. [50]It is also not from the beginning that he discerned this procedure. [51]After using the procedure to determine [numbers] one by one, with deep investigation
39r he understood the true rule.

[52]Although the true procedure of residual division is in this way, we do not look for the exhaustive elaboration of the exact number in a case such as calculation of the denominator of the fractional part of a day's length from the fractional part of a lunar month in calendar making; [53]we only need to decide the values in the order of *byō*.[54]Therefore, we only use the first strong ratio and the second weak ratio, or the second weak ratio and the third strong ratio, or the third strong ratio and the fourth weak ratio. We add them successively to determine the several ratios with not so big numbers and use them conveniently.

[55]Generally speaking, in calendar calculation there is another set of rules. [56]We should know it. [57]For example, when we establish a procedure, if we set up a rule according to the truth, it may be so difficult that we cannot apply it for calculation. [58]Therefore, we consider the necessary accuracy of the true number to be determined and we investigate and set up a simple provisory procedure and employ it. [59]In the determination of numbers, we do not need to elaborate
39v the exact number of great accuracy by the true procedure. [60]Considering the order of accuracy we set up the provisory procedure, by which we determine numbers of necessary accuracy and use them. [61]It is the same for numbers [obtained by the procedure] of residual division. [62]Sometimes we use intermediate ratios instead of true rates.

[63]In the *Sui Shu* (History of the *Sui* dynasty), [there is the following statements]: **In the *Jiu Shu* (*Nine Numbers*) from antiquity, the rate of the circular circumference was 3 and the rate of the diameter was 1;**

[64]the procedure is crude. [65]People like Liu Xin, Zhang Heng, Liu Hui, Wang Fan, and Pi Yanzong, each proposed new rates, [66]which had not yet reached conformance. [67]In the *Song* Kingdom, Zu Chongzhi, an officer at South *Xuzhou*, started a more exact rule: [68]He supposes the diameter of a circle one hundred million to be 1 *jō*. The upper bound of the circular circumference is 3 *jō* 1 *shaku* 4
40r **sun 1 bu 5 ri 9 mō 2 byō 7 kotsu; the lower bound is 3 jō 1 shaku**
4 *sun* 1 *bu* 5 *ri* 9 *mō* 2 *byō* 6 *kotsu*. [69]The right number is between the upper and the lower bounds. [70]The exact rates are 113 for the circular diameter and 355 for the circular circumference. The reduced rates are 7 for the circular diameter and 22 for the circular circumference.

[71]In old days, Master Seki determined the definite circumference by decomposing the circle and formed the rates of the circumference and of the diameter by the procedure of residual division. [72]After more than twenty years, when I first looked at the *Sui Zhi* (*Monograph on Calendar in the book of the Sui dynasty*) and found that the number of the circumference and the [two] rates happened to coincide. [73]Alas, how great were Master Zu and Master Seki! Although living in different countries and in different ages, they attained the same truth looking for the true numbers. How marvelous it is!

[74]In the above investigation of the circular numbers, the determination of the square of the cut out perimeter by the procedure of decomposition is the investigation of
40v numbers by reasonable evidence; [75]the determination of the limit number by repeat-
ing the procedure of incremental divisor is the investigation of numbers by numerical evidence; [76]the determination of the rates by the rule of residual division is also the investigation of numbers by numerical evidence. [77]Although the procedure of incremental divisor and the rule of residual division are a procedure and a rule respectively, which were originally established by investigation by numerical evidence, we regard all [of this chapter] as the investigation of numbers by numerical evidence.

XII. Investigating Numbers Related to the Arc

[1]In the search of the form and character of the back arc, the true number is hidden if it is close to the half circle and the true number appears if it is close to the side. [2]If it is close to the half circle, it belongs to the latitude and its curve is rapid; [3]if it is close to the side, it belongs to the longitude and its curve is slow. [4]Therefore,
41r taking the sagitta to be extremely small, we should search for the number and seek
the procedure.

[5]At the beginning, assuming the diameter to be 1 *shaku* and the sagitta to be 1

sun, 2 *sun*, 3 *sun*, or 4 *sun* we searched for the definite back arc by procedures of decomposition and of incremental divisor. Further, we continued to determine the definite back arc for the sagitta of 4 *sun* 5 *bu*, 4 *sun* 9 *bu*, etc. We examined these numbers but could not find any evidence when the back arc is close to the half circle. [6]Therefore, although Master Seki formed and revised the rate of the back arc twice and I [myself] also formed and revised it once, we abandoned these procedures because all the formulas were not accurate. [7]Relying on the fact that the square of the half back arc for a 1 *sun* sagitta is 10 *sun* 3 strong and that that for a 1 *bu* sagitta is 1 *sun* 0033 strong, discerning in advance that the true number will appear if the sagitta is extremely small, we determined the definite number of the square of the half back arc taking the sagitta to be
41v 1 *kotsu* and searched and understood its character.

[8]Cutting an arc with sagitta 1 *kotsu* we form two sides. Next cutting them again we form 4 sides, cutting them again we form 8 sides, and cutting them again we form 16 sides. [9]In this way, doubling the number of sides, we determine each of the squares of the cut out half back arcs and then by the procedure of repeated incremental divisor we obtain the square of the definite half back arc

1 *shi* 0000003333335111112253969066667282347769479595875 strong

([10]The rules of decomposition and of incremental divisor are the same as for the determination of the square of the circular circumference. [11]Hereafter we determine the squares of the bisected half back arcs up to 64 sides, and we elaborate the true number of about 50 orders by the procedure of incremental divisor. [12]We omit these numbers for the bisected half back arc.)

[13]If the sagitta is 1 *sun*, the square of the half back arc is of order 10 *sun*;
42r if the sagitta is 1 *bu*, then the square of the half back arc is of order 1 *sun*; if the sagitta is 1 *kotsu*, then the square of the half back arc is of order 1 *shi*. Therefore, we search and understand that the number of the base is the product of the sagitta and the diameter. [14]This coincides with the squares of the bisected chords.

[15]Multiply the sagitta and the diameter. The number obtained is called the square of the approximate half back arc, [16]which we subtract from the square of the definite half back arc; call the remainder the first definite difference.

[17]Observing that the order of the first definite difference is 7 less than that of the square of the half back arc, we find that we should determine the number of the order of the square of the sagitta. [18]Now we divide the first definite

difference by the square of the sagitta and obtain 3 *bu* 33333511111. [19]By the
42v procedure of residual division we search and obtain the extreme value 1/3.

[20]It corresponds with the old method where the square of the sagitta multiplied by the rate 5.8696 strong is added to the square of the chord to find the square of the back arc. [21]From old time, people did not observe the divisor 1/3. [22]Because they were looking for the formula which is exact for the half circle, they employed the multiplicative rate of the square of the sagitta.

[23]Self-multiply the sagitta and divide it by 3. The number obtained is called the first approximate difference, [24]which we subtract from the first definite difference; call the remainder the second definite difference.

[25]Observing that the order of the second definite difference is 6 less than that of the first approximate difference, we find that we should determine the second difference multiplying the second approximate difference by the sagitta and dividing it by the diameter. [26]Now we divide the second definite difference by the first approximate difference multiplied by the sagitta and divided by the
43r diameter, and obtain 5 *bu* 33333676191 weak. [27]By the procedure of residual division we search and obtain the extreme value 8/15.

[28]Place the first approximate difference [at the position], multiply it by the sagitta, divide it by the diameter, also multiply it by 8, and divide it by 15. The number obtained is called the second approximate difference, [29]which we subtract from the second definite difference; call the remainder the third definite difference.

[30]Observing that the order of the third definite difference is 6 less than that of the second approximate difference, we find that we should determine the third difference multiplying the second approximate difference by the sagitta and dividing it by the diameter. [31]Now we divide the third definite difference by the second approximate difference multiplied by the sagitta and divided by the diameter and obtain 6 *bu* 428576 slightly strong. [32]By the procedure of
43v residual division we search and obtain the extreme value 9/14.

[33]Place the second approximate difference [at the position], multiply it by the sagitta, divide it by the diameter, also multiply it by 9, and divide it by 14. The number obtained is called the third approximate difference, [34]which we subtract from the third definite difference; call the remainder the fourth definite difference.

[35]Observing that the order of the fourth definite difference is 7 less than that of the third approximate difference, we find that we should determine the fourth

difference multiplying the third approximate difference by the sagitta and dividing it by the diameter. [36]Now we divide the fourth definite difference by the third approximate difference multiplied by the sagitta and divided by the diameter, and obtain 7 *bu* 11111649832 strong. [37]By the procedure of residual division we search and obtain the extreme value 32/45.

44r
[38]Place the third approximate difference [at the position], multiply it by the sagitta, divide it by the diameter, also multiply it by 32, and divide it by 45. The number obtained is called the fourth approximate difference, [39]which we subtract from the fourth definite difference; call the remainder the fifth definite difference.

[40]Observing the order of the fifth definite difference is 6 less than that of the fourth approximate difference, we find that we should determine the fifth difference multiplying the fourth approximate difference by the sagitta and dividing it by the diameter. [41]Now we divide the fifth definite difference by the fourth approximate difference multiplied by the sagitta and divided by the diameter, and obtain 7 *bu* 57576356977 weak. [42]By the procedure of residual division we search and obtain the extreme value 25/33.

[43]Place the fourth approximate difference [at the position], multiply it by the
44v sagitta, divide it by the diameter, also multiply it by 25, and divide it by 33. The number obtained is called the fifth approximate difference, [44] which we subtract from the fifth definite difference; call the remainder the sixth definite difference.

[45]Observing the order of the sixth definite difference is 6 less than that of the fifth approximate difference, we find that we should determine the sixth difference multiplying the fifth approximate difference by the sagitta and dividing it by the diameter. [46]Now we divide the sixth definite difference by the fifth approximate difference multiplied by the sagitta and divided by the diameter, and obtain 7 *bu* 91209437363 strong. [47]By the procedure of residual division we search and obtain the extreme value 72/91.

[48]Place the fifth approximate difference [at the position], multiply it by the sagitta, divide it by the diameter, also multiply it by 72, and divide it by 91. The number obtained is called the sixth approximate difference. ([49]We omit how to determine the seventh and further differences.)

[50]

Square of the definite half back	1 *shi* 0000 00333 33351 11112 25396 90666 67282 34776 94795 95875 strong
Square of the approximate half back	1 *shi*
First definite difference	0 *shi* 0000 00333 33351 11112 25396 90666 67282 34776 94795 95875 strong
First approximate difference	0 *shi* 0000 00333 33333 33333 33333 33333 33333 33333 33333 33333 strong
Second definite difference	0 *shi* 0000 00000 00017 77778 92063 57333 33949 01443 61462 62542 strong
Second approximate difference	0 *shi* 0000 00000 00017 77777 77777 77777 77777 77777 77777 77778 weak
Third definite difference	0 *shi* 0000 00000 00000 00001 14285 79555 56171 23665 83684 84764 strong
45r Third approximate difference	0 *shi* 0000 00000 00000 00001 14285 71428 57142 85714 28571 42857 strong
Fourth definite difference	0 *shi* 0000 00000 00000 00000 00000 08126 99028 37951 55113 41907 strong
Fourth approximate difference	0 *shi* 0000 00000 00000 00000 00000 08126 98412 69841 26984 12698 strong
Fifth definite difference	0 *shi* 0000 00000 00000 00000 00000 00000 00615 68110 28129 29209 weak
Fifth approximate difference	0 *shi* 0000 00000 00000 00000 00000 00000 00615 68061 56806 15681 weak
Sixth definite difference	0 *shi* 0000 00000 00000 00000 00000 00000 00000 00048 71323 13528 strong
Sixth approximate difference	0 *shi* 0000 00000 00000 00000 00000 00000 00000 00048 71319 15703 strong

45v **[51]This original procedure runs as follows: Multiply the sagitta and the diameter to make the square of the approximate half back arc. [52]Self-multiply the sagitta, divide it by 3, to make the first difference. [53]Place the first difference, multiply it by the sagitta, divide it by the diameter, also multiply it by 8, and divide it by 15, to make the second difference. [54]Place the second difference, multiply it by the sagitta, divide it by the diameter, also multiply it by 9, and divide it by 14, to make the third difference. [55]Place the third difference, multiply it by the sagitta, divide it by the diameter, also, multiply it by 32, and divide it by 45, to make the fourth difference. [56]Place the fourth difference, multiply it by the sagitta, divide it by the diameter, also multiply it by 24, and divide it by 33, to make the fifth difference. [57]Place the fifth difference,**

multiply it by the sagitta, divide it by the diameter, also multiply it by 72, and divide it by 91, to make the sixth difference. ([58]The seventh
46r **and further differences can be determined similarly.) [59]We add the differences repeatedly to the square of the approximate half back arc, to make the square of the definite half back arc.**

[60]Applying this procedure to the half circle, the sagitta being large, we find two orders by using two differences, three orders by using three differences, four orders by using four differences, [61]and one more order by one more difference. [62]That is, this coincides with Master Seki's 4-multiplication procedure of finding the back arc. [63]But he did not understand that the natural number should be searched and sought by means of the back arc close to the side. [64]Only requiring it to be exact for the half circle, he formed the rate and abandoned it because it was not precise, without knowing only 4 orders could be obtained with 4-multiplication.

46v [65]In the procedure, observing the multipliers and divisors to determine successive differences, we search and understand that the multipliers at each steps are the square of the seed, which is 1 for the first step and incremented by 1 at each later step ([66]2 for the second, 3 for the third, 4 for the fourth), directly for the odd steps ([67]which are the first, third, fifth and further differences), and doubled for the even steps ([68]which are the second, fourth, sixth, and further differences).

[69]Similarly, we search and understand that the divisors are the products of the left seed, which is 3 for the first difference, and incremented by 2 at each later step ([70]5 for the second, 7 for the third, 9 for the fifth, and so on), and the right seed, which is 1 for the first difference, and incremented by 1 for the odd steps ([71]2 for the third, 3 for the fifth, 4 for the seventh and so on), and 3 for the second difference and incremented by 2 for the even steps ([72]5 for the fourth, 7 for the sixth, 9 for the eighth and so on).

47r

	difference	step	multiplier		divisor		
	1	odd	1	seed 1 square	3	left seed 3 right seed 1	multiplied
	2	even	8	seed 2 2 square	15	left seed 5 right seed 3	multiplied
	3	odd	9	seed 3 square	14	left seed 7 right seed 2	multiplied
	4	even	32	seed 4 2 square	45	left seed 9 right seed 5	multiplied
[73]	5	odd	25	seed 5 square	33	left seed 11 right seed 3	multiplied
	6	even	72	seed 6 2 square	91	left seed 13 right seed 7	multiplied
	7	odd	49	seed 7 square	60	left seed 15 right seed 4	multiplied
	8	even	128	seed 8 2 square	153	left seed 17 right seed 9	multiplied
	9	odd	81	seed 9 square	95	left seed 19 right seed 5	multiplied
	10	even	200	seed 10 2 square	231	left seed 21 right seed 11	multiplied

47v [74]When, using the multipliers and divisors given in the previous paragraph, we determine by adding successively differences as in the original procedure, we obtain directly the true number without employing the decomposition, the square of the half back arc. [75]Therefore, it [the procedure] exhausts the natural character of the back arc. [76]We should understand [the theory] [77]that the character of an arc and circle is inexhaustible, [78]and that, consequently, the corresponding procedure must be also determined inexhaustibly. [79]Certainly, some numbers are exhaustible and others are inexhaustible; [80]some procedures are exhaustible and others are inexhaustible; [81]some characters are exhaustible and others are inexhaustible. [82]Numbers like 1/4 and 1/5 are exhaustible; [83]numbers like 1/3 and 1/7 are inexhaustible. [84]Procedures like addition, subtraction, and multiplication are exhaustible; [85]procedures like division and root extraction are inexhaustible. [86]The character of the circumference of a square and of the area of a rectangle is exhaustible; [87]the character of the circumference of a circle and of the area of
48r an arc [sector] is inexhaustible. [88]That is, the circle and arc are of inexhaustible character, the procedure to handle them is also inexhaustible; the procedures being inexhaustible, the numbers are also inexhaustible. [89]But many people do not recognize the character, supposing it is exhaustible, they investigate with exhaustible procedures similar to finding the hypotenuse of a right triangle and the volume of a

cone. [90]How can they [expect to] obtain the answer?

[91]The original procedure is a natural method which follows the character of the arc. [92]If we seek the square of the half back arc for an extremely small sagitta, the successive differences decrease more rapidly and the truer number can be achieved quickly. But if the sagitta is getting larger in the case of a half circle, the successive differences decrease slowly and more and more differences must
48v be calculated. [93]In this case, many multipliers are required and the procedure is not easy. [94]It cannot be considered as the definite rate. [95]Therefore, we search and seek a simplified procedure by arranging divisors. Multiplying the first difference by the sagitta repeatedly and dividing it by the difference of the diameter and the sagitta we seek the second difference above. But the decrease [of differences] is not yet rapid. [96]We must also investigate more deeply. By the remainder of subtraction of the sagitta multiplied by a rate from the diameter we tried and divided differences repeatedly; we find the decrease is abruptly rapid. [97]Therefore, we take this as the fundamental procedure of the definite rate. [98]Although the procedure which uses repeated division by the difference of the sagitta and the diameter would not be employed, we describe it as one step of the ladder of investigation.

49r

[99]To search the differences by the division of the difference of the diameter and the sagitta.

[100]The beginning of the procedure. Multiply the sagitta and the diameter, to make the square of the approximate half back arc. [101]Self-multiply the sagitta and divide it by 3, to make the additive first difference. [102]Place the first difference, multiply it by the sagitta, divide it by the difference of the sagitta and the diameter, also multiply it by 8, and divide it by 15, to make the additive second difference. [103]Place the second difference, multiply it by the sagitta, divide it by the difference of the sagitta and the diameter, also multiply it by 5, and divide it by 14, to make the subtractive third difference. [104]Place the third difference, multiply it by the sagitta, divide it by the difference of the sagitta and the diameter, also multiply it by 12, and divide it by 25, to make the additive fourth difference. [105]Place the fourth difference, multiply it by the sagitta, divide it by the difference of the sagitta and the diameter, also multiply it by 223, and divide it by 398, to make the subtractive fifth difference. ([106]The sixth and further differences can be determined similarly.) [107]Place the square of the approximate half back arc, add or subtract the differences accordingly, to make the square of the definite half back arc. ([108] We omit the numbers thus sought.)

[109]Applying this procedure to the half circle, the sagitta being large, we find 3 orders by using two differences, 4 orders by using 3 differences, 4 orders by
49v using three differences, and 5 orders by using 4 differences. [110]We find one order more if we use one more difference. [111]That is, this coincides with the 6-multiplication original procedure of finding the back arc, which I [myself] established earlier. [112]Originally, expecting to find 7 orders using 6-multiplication we established the method, which turned out not to be accurate even using multi-multiplication. [113]Therefore, also we did not employ that procedure and abandoned it.

[114]In an old method, multiplying the sagitta by itself, multiply by the norm of the square of the sagitta, add it to the square of the approximate back arc. [115]Subtract the double of the sagitta from the diameter. Multiply the remainder by the square of the sagitta, divide it by the difference of the sagitta and the diameter and halve it. Subtract the obtained number from the square of the approximate back arc and call it the square of the definite back arc. This old procedure corresponds naturally to the previous main procedure with two differences.

50r [116]To search the use of the higher power of the sagitta in the division.

[117]Multiply the sagitta and the diameter and make it the square of the approximate half back arc, [118]which we subtract from the square of the definite half back arc; call the remainder the first definite difference.

[119]Self-multiply the sagitta and divide it by 3, to make the first approximate difference, [120]which we subtract from the first definite difference; call the positive remainder the second definite difference. ([121]The procedure of the preceding search is the same as before.)

[122]Observing the order of the second definite difference is 6 less than that of the first approximate difference, we find that we should determine the second difference multiplying the second approximate difference by the sagitta and dividing it by the diameter. [123]Now we divide the second definite difference by the first approximate difference multiplied by the sagitta and by the diameter. We obtain 5 *bu* 33333676191 weak. [124]By the procedure of residual division
50v we search and find the extreme value 8/15. [125]At this step it is not accurate if we seek the second approximate difference multiplying the first approximate difference by the sagitta and dividing it by the diameter. [126]Even if we divide it by the difference of the sagitta and the diameter, we cannot obtain the desired accuracy. [127]Therefore, we search whether it will be accurate if the division is by the difference of the sagitta multiplied by a coefficient and the diameter. Now we multiply the first approximate difference by the sagitta, also multiply

it by 8, divide it by 15, and divide it by the second definite difference. Subtract the obtained number from the diameter and find the negative remainder 6 *bi* 4285718673 strong. Divide it by the sagitta, and obtain the negative number 6 *bu* 4285718673 strong, [128]which is called the approximate coefficient (to be subtracted) of the sagitta. [129]By the procedure of residual division we search and find the extreme value 9/14.

[130]Place the first approximate difference [at the position], multiply it by the sagitta.
The result is placed in the Reality row. [131]Place the sagitta [at the position], mul-
51r tiply it by 9 and divide it by 14. Subtract the obtained number from the diameter.
The remainder is placed in the Square row. Divide the Reality row by the Square row. Also, multiply the quotient by 8 and divide it by 15. The result is called the second approximate difference of 2-multiplication, [132]which we subtract from the second definite difference; the obtained positive remainder is called the third definite difference.

[133]Observing that the order of the third definite difference is 14 less than that of the second approximate difference, we find that the third difference should be obtained by the second approximate difference multiplied by the square of the sagitta divided by the square of the diameter. Now we divide the third definite difference by the second approximate difference multiplied by the square of the sagitta and divided by the square of the diameter. We obtain 4 *ri* 38776034632 strong. [134]By the procedure of residual division we search and find the extreme value 43/980. [135]At this step it is not accurate if we
seek the third difference multiplying the second approximate difference by the
51v square of the sagitta divided by the square of the diameter. [136]Also, it is no
more accurate if we divide it by the square of the difference of the sagitta and the diameter [instead of the square of the diameter]. [137]Further investigation shows that it will be accurate if the division is by the square of the diameter adjusted by the product of the sagitta and the diameter and the square of the sagitta multiplied by some coefficients by addition or subtraction. [138]Now we multiply the second approximate difference by the square of the sagitta, also multiply it by 43 and divide it by 980. We divide the obtained number by the third definite difference and subtract the square of the diameter from it. The negative remainder is 1 *shi* 1952076352824992496 strong and is called the numerator of the product coefficient. [139]Because this is 6 digits lower from the top digit of the square of the diameter, we understand that we should use
the product of the sagitta and the diameter. [140]Now we divide the numerator
52r by the product of the sagitta and the diameter, obtain 1.1952076352840992496
strong, [141]which is called the approximate coefficient (to be subtracted) of the product of the sagitta and the diameter. [142]By the procedure of residual

division we search and find the extreme value 1696/1419. [143] Also, multiply the sagitta by the diameter, then multiply it by 1696 and divide it by 1419. Subtract the obtained number from the numerator of the product coefficient and obtain the positive remainder 2 *byō* 575998122373 strong. [144]Because this is 13 digits lower from the top digit of the square of the diameter, we understand that we should use the square of the sagitta. [145]We divide it by the square of the sagitta and obtain 2 *bu* 575998122373 strong, [146]which is called the approximate coefficient (to be added) of the square of the sagitta.
52v [147]By the procedure of residual division we search and find the extreme value 6743008/26176293. ([148]To search and seek the coefficient of the square of the sagitta, the sagitta still being large, the true number is hidden. [149]Therefore, with the sagitta 1 *jin* seeking the square of the half back arc in 90 digits I succeeded to search out the coefficient in detail. [150]The obtained numbers are so complicated that we omit them.)

[151]Place the second approximate difference [at the position], multiply it by the square of the sagitta and place it in the Reality row [of a counting board]. [152]Self-multiply the sagitta, multiply it by 6743008, divide it by 26176293, add the square of the diameter to it, subtract from it the product of the sagitta and the diameter, multiply it by 1696 and divide it by 1419. Place the remainder in the Normal row, by which divide the Reality row, also multiply it by 43, divide it by 980, and make it the third approximate difference of 4-multiplication.

53r

[153]If we want to obtain the fourth difference, we subtract the third approximate difference from the third definite difference and call the negative remainder the fourth definite difference. [154]Then looking at its top digit we evaluate how much lower it is from the top digit of the third approximate difference and then find that we divide the product of the third approximate difference and the cube of the sagitta by the cube of the diameter. First seek the coefficient of multiplication and division, by the procedure, the coefficient of the product of the sagitta and the square of the diameter, then the coefficient of the product of the square of the sagitta and the diameter, and at last the coefficient of the cube of the sagitta. Then we search and seek the extreme values of the coefficients and obtain the fourth difference (to be subtracted) of 7-multiplication. [155]After the difference of 7-multiplication, we seek the difference of 11-multiplication, next that of 16-multiplication, next that of 22-multiplication and more. [156]These are very complicated and will be omitted.

[157]

53v

Square of the definite half back	1 *shi* 0000 00333 33351 11112 25396 90666 67282 34776 94795 95875 strong
Square of the approximate half back	1 *shi*
First definite difference	0 *shi* 0000 00333 33351 11112 25396 90666 67282 34776 94795 95875 strong
First approximate difference	0 *shi* 0000 00333 33333 33333 33333 33333 33333 33333 33333 33333 strong
Second definite difference	0 *shi* 0000 00000 00017 77778 92063 57333 33949 01443 61462 62542 strong
Second approximate difference	0 *shi* 0000 00000 00017 77778 92063 56553 29270 48945 16638 53847 strong
Third definite difference	0 *shi* 0000 00000 00000 00000 00000 00780 04678 52498 44824 08695 weak
Third approximate difference	0 *shi* 0000 00000 00000 00000 00000 00780 04678 52498 44824 09177 weak

[158]This main procedure runs as follows: Multiply the sagitta and the diameter and call [the product] the square of the approximate half back arc. [159]Place the sagitta, self-multiply it and divide it by 3; call [the result] the first difference. [160]Place the first difference, multiply it by the sagitta. [The result is] now in the Reality row. [161] Place the sagitta, multiply it by 9 and divide it by 14. We subtract the product from the
54r **diameter. [The result is] now in the Norm row. [162]Divide the Reality row by the Norm row, also multiply it by 8 and divide it by 15. Call [the result] the second difference. [163]Place the second difference and multiply it by the square of the sagitta. [The result is] now in the Reality row. [164]Place the sagitta, self-multiply it, multiply it by 6743008, divide it by 26176293, add the square of the diameter to it and subtract from it the product of the sagitta and the diameter multiplied by 1696 and divided by 1419. The remainder is placed in the Norm row. [165]Divide the Reality row by the Norm row, also multiply it by 43 and divide it by 980. Call [the result] the third difference. [166]Place the square of the approximate half back arc and add to it the first, second, and third differences. The sum is found to be the square of the definite half back arc. [167]We subtract the square root from it and obtain the half back arc. ([168]We omit the differences of more than 7-multiplication of the main procedure.)**

[169]In this procedure, if we use two differences the original number is exhausted to the order 5, and if we use three differences the original number exhausted
54v to the order 8. [170]Therefore, the procedure of three differences is applied for

the half circle, the sagitta being large, the accuracy of order about 10. Taking 6 cases, we seek by the procedure the arc rates and regard them the general procedures. [171]All details are recorded in the *Koritsu*.

[172]In the above investigation of numbers of the arc, the determination of the coefficients of multipliers and divisors at each difference is the investigation of numbers by numerical evidence. [173]The procedure to determine the back arc is the investigation of rules by numerical evidence. [174]Certainly, in [the investigation of] the circular circumference and the back arcs, neither numbers nor procedures can be obtained by investigation by reasonable evidence. [175]We can obtain them only through investigation by numerical evidence. [176]This is because of the character of the arc and the circle.

[177]This ends the examples of procedures.

One Chapter on a Theory of Proper Character

55r

[1]We are at peace when we follow the spirit of mathematics. [2]We are in trouble when we do not follow it. [3]To follow the spirit is to follow its character. [4]If we follow it, acknowledging that we will obtain a solution even before we understand [the problem], we are at peace without any doubt. [5]Because we are at peace, we always proceed and do not stagnate. [6]Because we always proceed and do not stagnate, there is nothing which cannot be accomplished. [7]If we do not follow it, then without knowing if we will be able to obtain [a solution] or not before we understand [the problem], we are in doubt. [8]Because we are in doubt, we suffer and are daunted. [9]Because we suffer and are daunted, it is difficult to obtain [a solution]. [10]After I [myself] started to learn mathematics, looking for the easy way I was suffering from mathematical rules for a long time. [11]Certainly, this was
55v because I did not exhaust my own character. [12]Gradually after 60 days' struggle, I could realize my born character was distorted and became convinced that I should follow the spirit of mathematics.

[13]Alas, our own born character, straight or distorted, is native, we cannot change it. Even if we study hard, it cannot be improved; even if we forget and abandon it, it cannot be damaged in the least. [14]That is, we should speculate about its distortion [15]but we should not speculate about its straightness. [16]If we do not exhaust our own character, we cannot understand the truth which follows the character of mathematics. [17]But many people do not understand the it is natural that the native character may be straight or distorted. [18]Instead, they think that everything becomes clear after complete study and that it is not necessary to use force. [19]How misled they are! [20]These people think that one can obtain the straight character
56r by study. [21]How can such study change the [person's] character [into one which is] purely straight?

[22]Certainly, even if, exhausting our own character, we embody the Way [of Mathematics], the native character is the native character; it does not move, does not change. Also, there is nothing to be puzzled and nothing to be clarified. At any time when we are given a problem, following its difficulty, we cannot be away from using force.

[23]Also, once I heard that one person swallowed his art. [24]Does this refer to the person whose character is purely straight? [25]Deliberating about him, when I make the art follow me and enter into my heart, although what can be planned follows me, what cannot be planned may not follow me; this is because there is a difference between what can be planned and what cannot be planned. [26]I declare that, when
56v I immerse myself completely in mathematics without any resistance, I [myself] and the Way [of Mathematics] become mixed together, what can be planned follows me as what can be planned and what cannot be planned also follows me as what cannot be planned. [27]This is one outcome of the embodiment of the Way. [28]If one knows the Way of Mathematics in heart and explain it in words, he is dishonest. [29]If one embodies the Way and proceeds [in mathematics], he is [honest] in the truth. [30]We cannot speculate about the truth of the embodiment of the Way. [31]But in training myself in this truth which should not be speculated, I [myself] am sure there is one rule which concerns the native character. [32]But I [myself] am not yet mature in the Way. [33]Therefore, I dare not explain it. [34]When I become confident about its meaning, I will explain it. [35]This is indeed my distorted character.

[36]Certainly, if I were of purely straight character, I would have no intention to
57r explain a single word about it. [37]Why should I explain? [38]What is to be explained is that the native character is distorted.

[39]Generally speaking, the character is not equal among people; it may be straight or distorted, warm or cold. [40]It is indeed in this way that I [myself] follow the character of mathematics. But it is not always like this that others also follow it. [41]Therefore, when a student of mathematics looks at this book, he should not take it [as being] right immediately; [42]he should not take it [as being] wrong without thinking. [43]I would like to explain the reason why one can recognize one's own
58r native character and that the truth of mathematics follows the character.

[44]End of the Treatise on *Tetsujutsu*.

Appendix

57v

[1]Regulated length of the middle line when sides of a triangle differ by 1.

[2]Suppose there is a triangle. [3]The difference of the large and the middle sides and that of the middle and the small sides are 1 respectively. [4]We want to make the middle line regulated. [5]We ask how long the three sides and the middle line are respectively.

[6]

small side	middle side	large side	middle line (to be sought)
1	2	3	empty
3	4	5	2 2/5
13	14	15	11 1/5
51	52	53	44 8/53
193	194	195	167 9/65

58v [7]First we take the small side to be 1, the middle side to be 2, and the large side to be 3; we call them the basic numbers. ([8]That is, the middle and the small sides form the large side.) [9]The middle line is empty. [10]This is the first case. [11]Adding 1 to the three sides we make the small side 2, the middle side 3, and the large side 4. [12]With these values we seek the middle line and find it is not regulated. [13]Also, adding 1 again to the three sides we make the small side 3, the middle side 4, and the large side 5. ([14]That is, these numbers form the regular triangle.) [15]With these values we seek the middle line and find it is regulated. [16]This is the second case. [17]Also, adding 1 again to the three sides we make the small side 4, the middle side 5, and the large side 6. [18]With these values we seek the middle line and find it is not regulated. [19]Also, adding 1 again to the three sides we make the small side 5, the middle side 6, and the large side 7. [20]With these values we seek the middle
59r line and find it not regulated, either. [21]Also, adding again 1 to the three sides we make the small side 6, the middle side 7, and the large side 8. [22]With these values we seek the middle line and also find it not regulated, either. [23]In this way adding 1 to the three sides repeatedly we obtain the numbers of three sides and seek the middle line for investigation; arriving at the small side 13, the middle side 14, and the large side 15, we find the number of middle line become regulated. [24]This is the third case. [25]Next, arriving at the small side 51, the middle side 52, and the large side 53, we find the middle line become regulated. [26]This is the fourth case. ([27]We omit the numbers for the fifth and further cases.)

[28]At this stage, we search the numbers of the three sides for which the middle line become regulated and find the following: multiply the middle side by 4, subtract the middle side of the previous case from it and we obtain the middle side of the following case. [29]Subtracting 1 from the middle side we obtain the small side;
59v adding 1 to it we obtain the large side. ([30]If we want to obtain the small side directly, multiply the small side by 4, add 2 rods to it, subtract the small side of the previous case from it and we obtain the small side of the following case. [31]Also, if we want to obtain the large side directly, multiply the large side by 4, subtract 2 rods and the large side of the previous case from it and we obtain the large side of the following case.)

[32]If we want to obtain the middle line directly, add 1 rod to the large side of the previous case and halve it. The obtained number is the numerator of the

inexhaustible part of the middle line. [33]Divide it by the large side of the present case and we obtain the inexhaustible part of the middle line. [34]By the numerator of the inexhaustible part we subtract the middle side of the present case to obtain the integer part of the middle line. [35]Adding the inexhaustible part to it we obtain the exact value of the middle line.

[36]The above procedure was understood by Nakane Jōemon. Thus he searched numbers and obtained numerical evidence [of this procedure]. [37]Because there
60r is a reason in the basis, we can obtain the solution. But the reason is hidden and is very hard to be discerned. [38]In such a case we do not seek the reason. Only employing the numerical examples we follow the Way of Mathematics. [39]But someone thinks there is no reason because it cannot be evaluated; he is not knowledgeable. [40]Someone is puzzled and wants to discern the reason by force; he is not clever.

[41]End of Appendix

[42]The thirteenth day of the summer solstice, *Kinoto mi.*

VI Commentary

The procedure of translation:

1. We italicize the Japanese and Chinese words (names of dynasties, periods, etc.) which are not translated.
2. Quotations in the text are surrounded by quotation marks, like "..."
3. The two lined parts of the original text, which is called *warichū*, are surrounded by parenthesizes, like (...).
4. Bold face indicates that the original text is in Chinese.
5. Items in brackets "[...]" have been added for the sake of clarity but are only implied in the original text.
6. Numbers in brackets "[...]" indicate verse numbers in the Japanese original text.

Comments on Preface

[1] *tetsujutsu* 綴術 (*zhuishu* in Chinese) is translated in this monograph as "technique of linkage." This word is one of the key words of this monograph. Literally it should be rendered the "Procedure of Linkage," The word *tetsu* means "to link," "to knit," "to intertwine" and the word *jutsu* means "procedure," "technique," "method," etc.. Zu Chongzhi 祖冲之 wrote a book called the *Zhuishu*, of which we only know the title.

jutsuri 術理 is translated in this monograph as "reason of procedure", where *ri* 理 (reason) is a philosophical term of Chinese scholar Zhu Ji 朱熹 and his followers, which the samurai of the *Edo* period learned at the school, while *jutsu* 術 (procedure) is a technical term employed in Chinese mathematicaltexts since the *Jiuzhang Suanshu* 九章算術, which are a collection of problems, answers, and procedures. Once the problem is given accompanied with the answer, then the procedure is to give steps to attain the answer. (See [ChmlaEa2004] and [ShenEa1999].) We can say the procedure's role in Chinese classics is a program in the modern computer language. Here the *jutsuri* (reason of procedure) indicates not only the program itself but also the algorithm behind it.

[9] *shitsu* 質 is translated in this monograph as "character". Note that *shitsu* was translated into "attribute" in [Horiuchi1994b]. At the last chapter on Proper Character, Takebe Katahiro 建部賢弘 discusses the character of mathematical objects as well as that of a mathematician.

[13] The three aims of mathematical research are *hōsoku* 法則, *jutsuri* 術理, and *insū* (or *ensū*) 員数, which are rendered into "rule and law", "reason of procedure", and "numerical quantity". The author sometimes abbreviate *hōsoku* as *hō* 法, *jutsuri* as *jutsu* 術, and *insū* as *sū* 数.

[20] *Sui* 隋 is a Chinese dynasty (581 – 618).
[25] *mizunoe tora* 壬寅 is a year in the sexagenarian cycle. *Kyōhō* 享保 is a Japanese period (1716–1736). The seventh year of *Kyōhō* corresponds with 1722 AD.
[26] *Edo* 江戸 is an old name of *Tokyo*. *Musashi* 武蔵 is a Province consisted of today's *Tokyo*, *Saitama* and a part of *Kanagawa* prefectures.

Comments on Catalogue

In this catalogue chapter titles are represented simply by two Chinese character. The full titles can be found at the beginning of each chapter.

Comments on Chapter 1

Chapter 1 of the *Tetsujutsu Sankei* 綴術算経 corresponds to Chapters 1 and 2 of the *Fukyū Tetsujutsu* 不休綴術 and deals with multiplication and division. The reasons of the usage of the multiplication chant, the rule of division, and the rule of division by one digit numbers are made clear. Takebe Katahiro had learned these rules and reasons in the summary 総括 of the *Suanxue Qimeng* 算学啓蒙 (Zhu Shijie 朱世傑, 1299), from which he cited examples on multiplication and division. He emphasizes that multiplication is fundamentally decomposable into repeated addition, and division into repeated subtraction. As for multiplication and division using an abacus, we refer the reader to the *Jinkōki* 塵劫記 in English [WasanInst2000] and [Kojima1963].

The structure of this Chapter is as follows: [1-27] multiplication, [28-66] division, [67-68] closing remark, and [69-78] comments on the closing remark.

[2] Each chapter starts generally with a problem written in Chinese of the format "Suppose 仮如 ···. Question 問 ···. Answer 答 ···."

The rice is unhulled according to the *Sangaku Keimō Genkai Taisei* 算学啓蒙諺解大成. *koku* 斛 is a unit for grain. See VII Comments on Units.
[3] *sen* 銭 is a unit for silver money.
[20] Here the author explains the operation on a counting board or on an abacus.

Comments on Chapter 2

Chapter 2 treats the "rule of element placement 立元", i.e., the so-called "procedure of celestial element 天元", which is a way to treat algebraic equations or formulas in traditional Chinese mathematics. Note that Takebe Katahiro does not use the terminology 'procedure of celestial element" in the *Tetsujutsu Sankei.*

The structure of Chapter 2 is as follows: [1-5] historical remarks; [6-9] problem and answer; [10-30] explanation on the counting board algebra; [31] comments; [32-34] division viewed from the counting board algebra; [35-36] comments; [37-39] old elementary method to solve the problem; [40-43] comparison of the old elementary method and the new algebraic method; [44-52] statement of procedure; [53-60] evaluation of the rule of element placement and comments on Seki Takakazu's achievement; and [61-67] closing remark.

[2] *Zhiyuan* 至元 is a Chinese period (1335–1340) in the Yuan 元 dynasty (1271 – 1368). Guo Shoujing 郭守敬 (1231 – 1316) is a Chinese astronomer, engineer, and mathematician.
[3] *Dade* 大徳 is a Chinese period (1297–1307) in the Yuan Dynasty.
[6] *bu* 歩 is a unit for length and for area. 1 *bu* is approximately equal to 1.8 m and 1 [squared] *bu* is approximately equal to 3.3m^2. See VII Comments on Units.
[10] The original sense of *seki* 積 is something accumulated, the accumulation. Here, *seki* is rendered the "area", the areal accumulation.

Here, the rows of a counting board are called Reality 実, Square 方, and Corner 隅. See VIII Comments on Counting Board.
[11] Here, the rows of a counting board are called Reality, Square, Side 廉, and Corner. See VIII Comments on Counting Board.
[12] Here, the rows of a counting board are called Reality, Square, [first] Side, [second] Side, and Corner. See VIII Comments on Counting Board.
[13] The n-th root of x, namely $\sqrt[n]{x}$, was called the $(n-1)$-root of x in traditional Japanese mathematics. The 1-root was called square root and the 2-root the cubic root.
[18-28] It is important to note that a configuration

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{1}$$

on the counting board is used to represent both a polynomial

$$a_0 + a_1x + a_2x^2 + a_3x^3 \tag{2}$$

and an algebraic equation

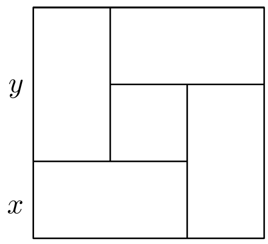
$$a_0 + a_1x + a_2x^2 + a_3x^3 = 0. \tag{3}$$

In ancient China, the configuration (1) on the computing board represented the equation (3). Because Takebe Katahiro found it difficult to overcome this ambiguity, he tried to give a lengthy rational interpretation why he regarded the configuration (1) as a polynomial (2).

[24] The n-th power of x, namely x^n, was called the $(n-1)$-multiplication accumulation of x, because the quantity is obtained by $n-1$ multiplications. The 1-multiplication accumulation is called the square and the 2-multiplication accumulation the cube.
[29] This passage explains the procedure of extraction. See VIII Comments on Counting Board.
[31] This passage explains the rule of the element placement according to the *Kai Indai no Hō* 解隠題之法 written by Seki Takakazu 関 孝和.
[32] The Square row was sometimes called the Norm row, especially when we are dealing with the division. Note that two Chinese characters *fa* (法, Norm) and *fang* (方, Square) have the same Japanese pronunciation *hō*.
[40] The problem considered here is to solve a simultaneous system of equations,

$$xy = 180, \quad x + y = 27.$$

Takebe employed the next figure of a square of the sum of the long and the short sides to illustrate an old method for solving such systems of equations.



If x and y represent, respectively, the short and long sides of the four congruent rectangles, then we see that $(x+y)^2 - 4xy = (y-x)^2$, or in other words, $y - x = \sqrt{(x+y)^2 - 4xy}$. In our case, $y - x = \sqrt{(27)^2 - 4(180)}$. The problem of finding x has been reduced to determining the square root of a natural number since $x = \{(x+y) - (y-x)\}/2$. (Techniques for determining such square roots were already known since the Jiuzhang Suanshu compiled in the Han 漢 dynasty.)
[44-52] The formal statement of the procedure to solve the problem [6-9]. The four symbols appeared in the text represent the configurations on the counting board by means of counting rods:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 27 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 27 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -180 \\ 27 \\ -1 \end{bmatrix}.$$

Note that, in the original text, positive numbers were printed in red and negative numbers in black. See VIII Comments on Counting Board for the manipulation.
[59] In 1683, Seki Takakazu wrote the *Kai Fukudai no Hō* 解伏題之法, in which Seki exposed the theory of resultants and determinants.

Both the *Kai Indai no Hō* and the *Kai Fukudai no Hō* are later complied in Volume 17 of the *Taisei Sankei* 大成算経.

Comments on Chapter 3

The rule of reduction 約分 is dealt with in Chapter 3. A main purpose of this chapter is to illustrate what we call Euclid's algorithm for finding the greatest common divisor.

The structure of the chapter is as follows: [1-3] the statement of problem and answer; [4-9] an elementary method of the reduction; [10-11] the reduction by means of Euclid's algorithm; [12-20] statement of procedure to solve the problem; [21-26] comments; and [27] closing remark.

[1-3] The problem is to simplify a fraction 105/168.
[4-9] First of all dividing the denominator and the numerator by 2, 3, 4, $\cdots$ consecutively, Takebe Katahiro obtains the common divisors 3, 7, and 21.
[10-11] The great common divisor (GCD), namely 21, is then used to determine the reduced fraction 5/8. The GCD is obtainable by Euclid's algorithm as follows:

$$\begin{aligned} 168 &= 105 \times 1 + 63, \\ 105 &= 63 \times 1 + 42, \\ 63 &= 42 \times 1 + 21, \\ 42 &= 21 \times 2. \end{aligned}$$

Comments on Chapter 4

This chapter deals with Chinese interpolation method using finite differences. The problem is to find the total volume of a stack of squares with unit thickness forming a "quadrangular pile" or pyramid. The volume is given by the series $S(n) = \sum_{k=1}^{n} k^2$ where the upper limit $n = 19$ is the base length of the particular pyramid given as an example.

The structure of Chapter 4 is as follows: [1-4] problem and answer; [5-23] numerical examples; [24-26] comments; [27-30] definition of definite product difference; [31-32] observations; [33-38] definition of planar product difference; [39-42] observation; [43-60] calculation of cubic, square, and definite differences; [61-65] observation; [66-67] statement of procedure; [68-72] comments; [73-76] closing remark; and [77-89] comments on the closing remark.

[1-4] The sum $S(n) = \sum_{k=1}^{n} k^2$ is called the accumulation 積 of the quadrangular pile 四角垜 with a base length 底面 n. The problem is to find a procedure to calculate $S(19) = 2470$.
[5-22] To have numerical examples, Takebe first calculates $S(1) = 1$, $S(2) = 5$, $S(3) = 14$, $S(4) = 30$, $S(5) = 55$, $S(6) = 91$, and $S(7) = 140$.

[24-26] The author claims $S(n)$ is a cubic polynomials and that the coefficients can be determined taking differences three times. The 2-multiplication accumulation of x is x^3.
[27-28] $S(n)/n$ is called the "first definite sum 第一定積". It must be a square polynomial. Assuming $S(n)/n = An^2 + Bn + C$, Takebe presents here an algorithm to calculate the "cubic difference 立差" A, the "square difference 平差" B, and the "definite difference 定差" C.

Let n_k, $k = 1, 2, 3, \cdots$ be an increasing sequence of natural numbers. (In the text it is assumed that $n_k = k$, $k = 1, 2, 3, \cdots$.) Denote the "first definite sum" by

$$q^{(1,1)}(k) = \{S(n_k)\}/n_k \, (= \, An_k^2 + Bn_k + C) \text{ for } k = 1,\ 2,\ 3,\ \cdots,\ 7.$$

If $n_k = k$, then $q^{(1,1)}(1) = 1$, $q^{(1,1)}(2) = 2\frac{1}{2}$, $q^{(1,1)}(3) = 4\frac{2}{3}$, $q^{(1,1)}(4) = 7\frac{1}{2}$, $q^{(1,1)}(5) = 11$, $q^{(1,1)}(6) = 15\frac{1}{6}$, $q^{(1,1)}(7) = 20$.
[29-30] Define the "[first] definite sum difference [第一] 定積差" by

$$d^{(1,1)}(k) = q^{(1,1)}(k+1) - q^{(1,1)}(k).$$

If $n_k = k$, then $d^{(1,1)}(1) = 1\frac{1}{2}$, $d^{(1,1)}(2) = 2\frac{1}{6}$, $d^{(1,1)}(3) = 2\frac{5}{6}$, $d^{(1,1)}(4) = 3\frac{1}{2}$, $d^{(1,1)}(5) = 4\frac{1}{6}$, $d^{(1,1)}(6) = 4\frac{5}{6}$.

[33-34] Define the "square case difference divisor 平限差法" by

$$\delta^{(1)}(k) = n_{k+1} - n_k.$$

If $n_k = k$, then $\delta^{(1)}(1) = 1$, $\delta^{(1)}(2) = 1$, $\delta^{(1)}(3) = 1$, $\delta^{(1)}(4) = 1$, $\delta^{(1)}(5) = 1$, $\delta^{(1)}(6) = 1$.
[35-36] Define "square sum 平積" by

$$q^{(1,2)}(k) = d^{(1,1)}(k)/\delta^{(1)}(k) \, (= \, A(n_{k+1} + n_k) + B).$$

If $n_k = k$, then $q^{(1,2)}(1) = 1\frac{1}{2}$, $q^{(1,2)}(2) = 2\frac{1}{6}$, $q^{(1,2)}(3) = 2\frac{5}{6}$, $q^{(1,2)}(4) = 3\frac{1}{2}$, $q^{(1,2)}(5) = 4\frac{1}{6}$, $q^{(1,2)}(6) = 4\frac{5}{6}$.
[37-38] Define the "square sum difference 平積差" by

$$d^{(1,2)}(k) = q^{(1,2)}(k+1) - q^{(1,2)}(k).$$

If $n_k = k$, then $d^{(1,2)}(1) = \frac{2}{3}$, $d^{(1,2)}(2) = \frac{2}{3}$, $d^{(1,2)}(3) = \frac{2}{3}$, $d^{(1,2)}(4) = \frac{2}{3}$, $d^{(1,2)}(5) = \frac{2}{3}$.
[43-44] Define "cubic case difference divisor 立限差法" by

$$\delta^{(2)}(k) = n_{k+2} - n_k.$$

If $n_k = k$, then $\delta^{(2)}(1) = 2$, $\delta^{(2)}(2) = 2$, $\delta^{(2)}(3) = 2$, $\delta^{(2)}(4) = 2$, $\delta^{(2)}(5) = 2$.
[45-46] Define the "cubic sum 立積" by

$$q^{(1,3)}(k) = d^{(1,2)}(k)/\delta^{(2)}(k) \, (= \, A).$$

If $n_k = k$, then $q^{(1,3)}(1) = \frac{1}{3}$, $q^{(1,3)}(2) = \frac{1}{3}$, $q^{(1,3)}(3) = \frac{1}{3}$, $q^{(1,3)}(4) = \frac{1}{3}$, $q^{(1,3)}(5) = \frac{1}{3}$.
[47] Thus, we have found the “cubic difference 立差” $A = q^{(1,3)}(n) = \frac{1}{3}$.
[48-49] Next, define the “second definite sum 第二定積” by

$$q^{(2,1)}(k) = q^{(1,1)}(k) - An_k^2 \,(= \,Bn_k + C).$$

If $n_k = k$, then $q^{(2,1)}(1) = \frac{2}{3}$, $q^{(2,1)}(2) = 1\frac{1}{6}$, $q^{(2,1)}(3) = 1\frac{2}{3}$, $q^{(2,1)}(4) = 2\frac{1}{6}$, $q^{(2,1)}(5) = 2\frac{2}{3}$, $q^{(2,1)}(6) = 3\frac{1}{6}$, $q^{(2,1)}(7) = 3\frac{2}{3}$.
[50-51] Define the “[second] definite sum difference [第二] 定積差” by

$$d^{(2,1)}(k) = q^{(2,1)}(k+1) - q^{(2,1)}(k).$$

If $n_k = k$, then $d^{(2,1)}(1) = \frac{1}{2}$, $d^{(2,1)}(2) = \frac{1}{2}$, $d^{(2,1)}(3) = \frac{1}{2}$, $d^{(2,1)}(4) = \frac{1}{2}$, $d^{(2,1)}(5) = \frac{1}{2}$, $d^{(2,1)}(6) = \frac{1}{2}$.
[52-53] Define the “[second] square sum [第二] 平積” by

$$q^{(2,2)}(k) = d^{(2,1)}(k)/\delta^{(1)}(k) \,(= \,B).$$

If $n_k = k$, then $q^{(2,2)}(1) = \frac{1}{2}$, $q^{(2,2)}(2) = \frac{1}{2}$, $q^{(2,2)}(3) = \frac{1}{2}$, $q^{(2,2)}(4) = \frac{1}{2}$, $q^{(2,2)}(5) = \frac{1}{2}$, $q^{(2,2)}(6) = \frac{1}{2}$.
[54] Thus, we have found the “square difference 平差” $B = q^{(2,2)}(n) = \frac{1}{2}$.
[55-56] Next, define the “third definite sum 第三定積” by

$$q^{(3,1)}(k) = q^{(2,1)}(k) - Bn_k \,(= \,C).$$

If $n_k = k$, then $q^{(3,1)}(1) = \frac{1}{6}$, $q^{(3,1)}(2) = \frac{1}{6}$, $q^{(3,1)}(3) = \frac{1}{6}$, $q^{(3,1)}(4) = \frac{1}{6}$, $q^{(3,1)}(5) = \frac{1}{6}$, $q^{(3,1)}(6) = \frac{1}{6}$, $q^{(3,1)}(7) = \frac{1}{6}$.
[57] Thus we have found the “definite difference 定差” $C = q^{(3,1)}(n) = \frac{1}{6}$.
[58] Takebe observes that we need only three values $k = 1, 2, 3$ to solve the given problem by the above method.
[62] In the text it is assumed that $n_k = k$ but the procedure can be understood better with the general case.
[63] As Takebe mentioned, this algorithm can be extended to the case of $q^{(1,1)}(k) = An_k^3 + Bn_k^2 + Cn_k + D$ or polynomials of higher degree.
[66-67] Formal statement of the Procedure: Let n be the base length. Then $S(n)/n = (An + B)n + C$ and we obtain the sum

$$S(n) = \{((2n+3)n+1)n\}/6.$$

Compare Takebe's solution $\sum_{k=1}^{n} k^2 = \{((2n+3)n+1)n\}/6$ with that typically given in today's calculus texts: $\sum_{k=1}^{n} k^2 = \{n(n+1)(2n+1)\}/6$. Note that Takebe's solution was well known among Japanese mathematicians of the 18th century.

[70-71] Seki found the formula for the sum of $S(n,p) = \sum_{k=1}^{n} n^k$ in the *Katsuyō Sanpō* 括要算法 (1712) for $p = 1,\ 2,\ \cdots,\ 10$. Seki's results can be compared with that of Jacques Bernoulli in 1713.
[82] "To increase or to decrease at the extreme point of saturation or exhaustion 満極干尽" refers Takebe's idea on the "three essentials 三要" of mathematics discussed in Volume 4 of the *Taisei Sankei*. (See [Xu2002] and [Ozaki2004].)

Comments on Chapter 5

The reason of multiplying first and dividing later had been considered as one of important methods of calculation since the *Jiuzhang Suanshu* and was explained in detail in the *Suanxue Qimeng*.

The structure of Chapter 5 is as follows: [1-5] problem and answer; [6-9] a common sense method; [10-11] observation; [12-17] statement of procedure; [18-20] comments; and [21-22] closing remark.

[2] *tan* 端 is a unit for length of cloth.
[6] 1.33333 strong 強 stands for a number x with $1.333331 \le x < 1.333335$. If $1.33333 < x < 1.333331$, x is called 1.33333 slightly strong 微強.
[7] 6.34921 weak 弱 stands for a number x with $6.349205 \le x \le 6.349209$. If $6.349209 < x < 6.34921$, x is called 1.34921 slightly weak 微弱.
[18-20] Takebe illustrates the reason of multiplying first and dividing later by means of a practical example. He obtains the solution by two different methods:

(1) Common sense method:

$$\left[\frac{\left(\dfrac{4\text{ tan}}{3\text{ weavers}}\right)}{21\text{ days}} \times 45\text{ days}\right] \times 7\text{ weavers} = \left[\frac{\left(\dfrac{4}{3}\right)}{21} \cdot 45\right] \cdot 7\text{ tan}$$

(2) Applying the reason of multiplying first and dividing later:

$$\frac{4 \cdot 7 \cdot 45}{3 \cdot 21}\,\frac{\text{tan weavers days}}{\text{weavers days}} = \frac{4 \cdot 7 \cdot 45}{3 \cdot 21}\text{ tan}$$

The common sense method employs understandable units (e.g., tan/weaver and (tan/weaver)/day) but introduces an infinite decimal in the first step of the computation ($4/3 = 1.\bar{3}$). Applying the reason of multiplying first and dividing later, one works entirely with whole numbers (desirable on an abacus) but the unit of both the numerator (tan weaver days) and denominator (weaver days) do not have practical meaning.

Comments on Chapter 6

Takebe Katahiro assumes that the readers are familiar with both the rule of element placement (Chapter 2) and the procedure of extraction (Chapter 10) . Readers are advised to read first our Comments on these two chapters.

The procedure for maximizing the volume of a parallelepiped subject to constraints on its dimensions is described in this chapter. (See [Ogawa1998b].)

> The structure of Chapter 6 is as follows: [1-5] statement of problem and answer. [6-7] comments; [8-11] calculation of the volume of a parallelpiped; [12-13] comments; [14-24] calculation of the counting board algebra; [25-36] statement of the procedure; [37-42] closing remark; and [43-50] comments on the closing remark.

[1-4] Suppose that the width x, length y, and height z of the parallelepiped satisfy the relations

$$x - y = 7, \quad y + z = 8.$$

The problem is to find the extreme value of the volume xyz.
[8-11] Let $D = 7$ and $S = 8$. (D stands for "Difference" and S stands for "Sum".) Knowing that the Square row vanishes if the value in the Reality row takes an extreme value (i.e., a maximal or minimal value), Takebe tries to find the equation satisfied by extreme values of the polynomial

$$V(y) = (D + y)y(S - y) = DSy + (S - D)y^2 - y^3.$$

[14-23] Takebe applies the procedure of extraction of the quotient number as follows:

Quotient	Reality	Square	Side	Corner
y			$S - D$	-1
		$(S - D)y - y^2$ (the first number which ought to extract the Square row)	$-y$	
		$(S - D)y - y^2$	$(S - D) - y$ (the first number to extract the Side row)	-1
		$(S - D)y - 2y^2$ (the second number which ought to extract the Square row)	$-y$	
		$2(S - D)y - 3y^2$ (the extreme case of the Square row)	$(S - D) - 2y$ (the second number to extract the Side row)	-1

He omits the Reality row which is unnecessary here. Moreover, in the Square row, he omits the original value DS.

In the first step of the procedure of element placement, Takebe declares that y is a new variable. See VIII Comments on Counting Board. His second step describes the above manipulations in y as operations on configurations. His third step is to cancel the original value DS in the Square row with the "extreme case of the Square row" to form the equation

$$DS + 2(S - D)y - 3y^2 = 0,$$

which is the equation $V'(y) = 0$.

Note that the notation in the text is an example of the side writing method invented by Seki Takakazu. Allowing algebraic combination of symbols as coefficients of a (one variable) polynomial, Seki inaugurated a method to handle polynomials of several variables.

Comments on Chapter 7

The arithmetic removal deals with a mathematical problem of congruence stemming from the problem known in the West as the Josephus problem.

> The structure of Chapter 7 is as follows: [1-7] presentation of problem; [8-17] numerical examples; [18-23] statement of procedure; [24-37] comments on Seki Takakazu, Takebe Kata'akira, and Nakane Genkei; and [38-40] closing remark.

[1-7] Let one black pebble and n white pebbles be arranged on a circle. Calling the black pebble the first pebble, we remove every m-th pebble repeatedly. The problem is to determine the number n for which the black pebble remains with all the white pebbles removed.

[8-17] We call m the removal number. By observation, Takebe Katahiro lists such numbers n for each removal number $m = 2,\ 3,\ 4,\ 5,\ 6$:

m	n
2	1, 3, 7, 15, 31
3	3, 5, 8, 30
4	1, 4, 8, 11, 15
5	2, 5, 11, 14, 36
6	1, 2, 7, 13

We now label on the circle the consecutive positions of the original $n+1$ pebbles as 0, 1, 2, 3, $\cdots$, n, with the black pebble at position 0. Let $N_{n+1,m}$ be the position of the last pebble remaining when we play the game with 1 black and n white pebbles and removal number m. Evidently, $N_{1,m} = 0$, because there is no white pebble at the beginning. When $m = 5$, $N_{n+1,m}$ are as follows:

$N_{n+1,m}$	0	1	0	1	1	0	5	2	7	2	7	0	5	10	0	5	10	15
$n+1$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

Note that our original problem is solved by finding n for which $N_{n+1,m} = 0$. From the table we see $n = 0,\ 2,\ 5,\ 11,\ 14$ satisfy this condition when $m = 5$. Based upon this numerical calculation, Takebe Katahiro recognized the following recursion relationship

$$N_{n+1,m} \equiv N_{n,m} + m \pmod{n+1} \quad (4)$$

With the initial condition $N_{1,m} = 0$ we can calculate $N_{n+1,m}$ recursively and find the n's for which $N_{n+1,m} = 0$.
[18-23] Takebe Katahiro was able to represent the above recursion formula by a sequence of operations involving two rows on the counting board. (The first row of the board he called Reality and the second row the Norm. See Comments on Chapter 2.):

> First leaving the Reality empty place 1 rod at the Norm.
> Then add m to the Reality and 1 to the Norm.
> If the number at the Reality exceeds that at the Norm, the number at the Reality shall be replaced by its remainder of the repeated subtraction by the number at the Norm.

This sequence of procedures on the counting board is equivalent to (4) in modern mathematical language.
[27] According to H. Komatsu, this is to eliminate variables of two algebraic equations using the determinant of the fifth order (resultant).
[30] Nakane Jōemon 中根上右衛門 is also known as Nakane Genkei 中根元圭 (1662-1733).
[32] The calculation of the accumulated years 積年 refers Problem 49 of the *Kenki Sanpō*.

Comments on Chapter 8

In this chapter the author describes two methods to calculate the surface area $S(d)$ of a sphere with diameter $d = 2r$.

> The structure of Chapter 8 is as follows: [1-4] problem and answer; [5-12] numerical examples; [13-15] observation; [16] a geometrical method; [17-18] comments; [19-21] statement of procedure; [22-47] comments on Seki's mathematics; and [48-53] closing remark.

[1-4] Here is the question and the answer. For the units see VII Comments on Units. In [4], *sun* means the squared *sun*.
[5-12] The first method amounts to numerical differentiation. Let $V(d)$ denote the volume of a ball with diameter $d = 2r$. Takebe assumes the formula $V(d) = \frac{\pi}{6}d^3 = \frac{4\pi}{3}r^3$ is known. (In Chapter 9 he describes two methods to find $V(d)$ and in Chapter

11 he mentions the calculation of π with 42 digit accuracy.) Considering shells, he calculates approximate values of the area of a sphere with diameter $d = 10$ as follows:

[5-7] $a_1 = \frac{V(10.01)-V(10)}{0.005} = \frac{1.57236764672\ (s)}{0.005} = 314.473529344\ (s)$,

[8-9] $a_2 = \frac{V(10.0001)-V(10)}{0.00005} = \frac{0.00157081203481\ (w)}{0.00005} = 314.162406962\ (s)$,

[10-11] $a_3 = \frac{V(10.000001)-V(10)}{0.0000005} = \frac{0.0000157079648387\ (s)}{0.0000005} = 314.159296775\ (w)$.

[13-15] Then by the procedure of the decremental divisor 損約の術 (see below) he calculates (6), which should give a better approximation of $S(d)$. He then finds $a = 314.159265359\ (w)$. By computer calculation, we find $a = 314.15926536944$ (s), which differs a little from Takebe's value. Takebe notices that this value is the same as πd^2 and claims $S(d) = \pi d^2$.

When the first three terms of a increasing series $a_1, a_2, a_3, \cdots, a_n, \cdots$ (i. e., $a_{n+1} - a_n > 0$) are given, Seki Takakazu claims that

$$a = a_2 + (a_2 - a_1)(a_3 - a_2)/\{(a_2 - a_1) - (a_3 - a_2)\} \tag{5}$$

gives a good approximation of $\lim_{n\to\infty} a_n$. (See Volume 2 of the *Katsuyō Sanpō*.) This is the procedure of the incremental divisor 増約の術. If the series a_n is decreasing (i. e., $a_n - a_{n+1} > 0$), (5) is rewritten as

$$a = a_2 + (a_1 - a_2)(a_2 - a_3)/\{(a_1 - a_2) - (a_2 - a_3)\} \tag{6}$$

and called the procedure of decremental divisor 損約の術. Because mathematicians of the Edo period preferred to have positive factors, the distinguished between (5) and (6).

Later, in his book the *Kigenkai* 起源解 Matsunaga Yoshisuke 松永良弼 (?–1744) explained this claim showing the right-hand side of (5) is the limit $\lim a_n$ when the first differences form a geometric sequence. In fact, suppose

$$(a_3 - a_2)/(a_2 - a_1) = (a_4 - a_3)/(a_3 - a_2) = (a_5 - a_4)/(a_4 - a_3) = \cdots = \rho,$$

then we have

$$\begin{aligned}\lim a_n &= a_2 + (a_3 - a_2) + (a_4 - a_3) + (a_5 - a_4) + \cdots \\ &= a_2 + (a_3 - a_2)\{1 + \frac{a_4 - a_3}{a_3 - a_2} + \frac{a_5 - a_4}{a_4 - a_3} \cdot \frac{a_4 - a_3}{a_3 - a_2} + \cdots\} \\ &= a_2 + (a_3 - a_2)\{1 + \rho + \rho^2 + \cdots\} \\ &= a_2 + (a_3 - a_2)/(1 - \rho) = a_2 + (a_3 - a_2)/(1 - (a_3 - a_2)/(a_2 - a_1)).\end{aligned}$$

In our case, taking $\epsilon = 0.001$ we have

$$\begin{aligned}a_1 &= \{V(d + \epsilon) - V(d)\}/(\epsilon/2) = (\pi/6)(6d^2 + 6d\epsilon + 2\epsilon^2) \\ a_2 &= \{V(d + \epsilon^2) - V(d)\}/(\epsilon^2/2) = (\pi/6)(6d^2 + 6d\epsilon^2 + 2\epsilon^4) \\ a_3 &= \{V(d + \epsilon^3) - V(d)\}/(\epsilon^3/2) = (\pi/6)(6d^2 + 6d\epsilon^3 + 2\epsilon^6)\end{aligned}$$

and

$$a_1 - a_2 = (\pi/6)(6d(\epsilon - \epsilon^2) + 2\epsilon^2 - 2\epsilon^4)$$
$$a_2 - a_3 = (\pi/6)(6d(\epsilon^2 - \epsilon^3) + 2\epsilon^4 - 2\epsilon^6),$$

which does not form a geometrical series but the major parts of which $\pi d\epsilon$, $\pi d\epsilon^2$, $\cdots$ form a geometric form. Therefore, we can expect the procedure of the decremental divisor (6) yields a good result.

[16] The second method to calculate $S(d)$ is more geometrical and ascribed to Master Seki Takakazu. Seki Takakazu considered intuitively the ball to be a cone with the center of the ball as the apex of the cone, the surface area $S(d)$ of the ball as the base B of the cone, and the radius $\frac{d}{2}$ of the ball as the height h of the cone.

[17-19] Because the volume V of the cone is given by $V = \frac{1}{3}Bh$, Seki found $V(d) = \frac{1}{3}S(d)\frac{d}{2}$. Because $V(d) = \frac{\pi}{6}d^3$ was known, he found $S(d) = \pi d^2$ without laborious calculation.

[19-21] Here stated the procedure in the final form: Let r be the radius of the ball. Then the diameter d is equal to $2r$. The surface area is given by $(2r)^2 \times (2\pi r)/(2r) = \pi d^2$.

[21] π is called the circular ratio 円周の法 in [17, 18]. If π is approximated by a fraction, the numerator is called the rate of circular circumference and the denominator is called the rate of the diameter

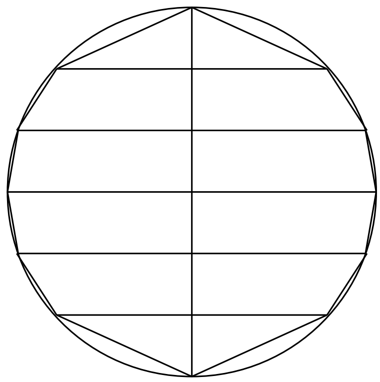
[22-47] Takebe Katahiro compares his method and his master's. Although his master's method is more elegant and works in this particular case, Takebe claims his method can be applied to more complicated cases, for example, in the study of numbers related to the circular arc (See Chapter 12.)

Comments on Chapter 9

This chapter deals with the procedure of incremental divisor, which we encountered in Chapter 8.

> The structure of Chapter 9 is as follows: [1-5] comments on the partitioning method; [6-8] two partitioning methods to find the circumference of a circle; [9-15] two partitioning methods to find the volume of a sphere; [16-23] comments on the merits and demerits of these methods in relationship to the natural attributes of the respective objects (circle or sphere.); and [24-25] closing remark.

[6-7] Mark $n-1$ points on a radius which divide it into n equal segments. Draw chords perpendicular to the radius through these $n-1$ points, and join consecutive points on the circle with chords.



The length of this piecewise linear curve Γ can be calculated by what Takebe calls the procedure of the right-angled triangle (i.e., Pythagoras' Theorem.) Because the length of the k-th half chord perpendicular to the radius is given by $rh_k = r\sqrt{1-(k/n)^2}$, the length of the k-th chord of Γ is equal to $r\sqrt{(1/n)^2+(h_k-h_{k-1})^2}$. The chords which approximate the semicircle come in pairs (left and right), so the n-th approximation of the full circumference is given by

$$S_n = 4r\sum_{k=1}^{n}\sqrt{(1/n)^2+(h_k-h_{k-1})^2}.$$

Doubling the partitioning number $n = 2, 4, 8, \cdots$, we obtain the following values with $r = 1/2$. (To apply recursive computation, Japanese mathematicians must have done the calculation in this way):

n	S_n	P.I.D.
2	3.03528	
4	3.1045	
8	3.12854	3.14134700
16	3.13699	3.14156089
32	3.13997	3.14158800
64	3.14102	3.14159191

(P.I.D. stands for the Procedure of Incremental Divisor.)

[8] Divide the circle equally into 4 parts and connect the dividing points to obtain the inscribed square. Then the length of a side is equal to $a_4 = \sqrt{2}r$ and the length of the perimeter of the inscribed regular square is equal to $4a_4 = 4\sqrt{2}r$. By the procedure of the right angled triangle, the length of a side of the inscribed regular octagon a_8 is given as follows: $a_8 = \sqrt{(r-\sqrt{r^2-(a_4/2)^2})^2+(a_4/2)^2}$. This relation holds in general. Let a_n be the length of a side of the inscribed regular n-gon. Then we have

$$a_{2n} = \sqrt{(r-\sqrt{r^2-(a_n/2)^2})^2+(a_n/2)^2}.$$

If we put $a_2 = 2r$, this holds even for $n = 2$. Therefore, if we know the length of the perimeter of the inscribed square, we can calculate, recursively, the length of the perimeters of the inscribed regular octagon, 16-gon, 32-gon, 64-gon $\cdots$. The numerical calculation with $r = 1/2$ gives us

n	na_n	P.I.D.
2	2.000000	
4	2.828443	
8	3.061467	3.15268277
16	3.121445	3.14223140
32	3.136548	3.14163181
64	3.134033	3.14159509

Takebe Katahiro considers the use of inscribed regular polygons to be more natural for a circle than the use of the piecewise linear curve Γ. However, numerical calculation by computer shows that there is no significant difference between these two approaches. We are not sure whether or not Takebe Katahiro really executed the former calculation.

[9] The formula for the volume of a circular platform is quoted here. A circular platform is a cone truncated by a plane perpendicular to the axis. Let r_1 be the radius of the bottom, r_2 that of the top, h the height. Then the platform's volume is given by

$$V = \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2).$$

If $r_2 = 0$, $V = \pi h r_1^2/3$ is the volume of a circular cone; if $r_1 = r_2$, $V = \pi h r_1^3$ is the volume of a cylinder.

Divide the radius of a sphere into n segments. Because the radius of the small circle perpendicular to the axis and passing through the k-th division point is given $r_k = r\sqrt{1-(k/n)^2}$, the volume of the k-th circular platform inscribed in the sphere is given by

$$V_k = \frac{\pi r}{3n}(r_{k-1}^2 + r_{k-1}r_k + r_k^2).$$

Therefore, the volume of the hemisphere is approximated by

$$V(n) = \frac{\pi r}{3n}\sum_{k=1}^{n}(r_{k-1}^2 + r_{k-1}r_k + r_k^2).$$

Because the (approximate) value of π is known, we calculate numerically $V(n)/\pi$ with $r = 1$.

[13] Along with $V(n)$, we also calculate numerically, with $r = 1$, the following

$$\bar{V}(n) = \frac{\pi r}{2n}\sum_{k=1}^{n}(r_{k-1}^2 + r_k^2).$$

The results are as follows:

n	$V(n)/\pi$	P.I.D.	$\bar{V}(n)/\pi$	P.I.D.
2	0.561004		0.625	
4	0.635799		0.65625	
8	0.657951	0.667271	0.664063	0.66667
16	0.664251	0.666754	0.666016	0.66667
32	0.666005	0.666682	0.666504	0.66667
64	0.666487	0.66667	0.666626	0.66667

As evident from this numerical calculation, $V(n)/\pi$ converges to the extreme value 2/3 sufficiently fast. (If we use the Procedure of Incremental Divisor, this convergence becomes faster.) But $\bar{V}(n)/\pi$ converges much faster to the extreme value (if we use the Procedure of Incremental Divisor the third term gives an accurate approximation.) Although a geometrical meaning cannot be given to $\bar{V}(n)$, this gives a very accurate approximation. Takebe Katahiro praised in [15] the latter approximation saying this was a "miraculous procedure".

Explanation of the "miraculous procedure." Approximating the hemisphere by circumscriptive circular cylinders, we obtain

$$U(n) = \frac{\pi r}{n} \sum_{k=1}^{n} r_{k-1}^2.$$

This gives an upper bound for the volume of the hemisphere. Approximating the hemisphere by inscribed circular cylinders, we obtain

$$W(n) = \frac{\pi r}{n} \sum_{k=1}^{n} r_k^2.$$

This gives a lower bound for the volume of the hemisphere. These quantities satisfy the inclusion relation $W(n) < V(n) < U(n)$. $\bar{V}(n)$ is nothing but the average of $W(n)$ and $U(n)$. By the Procedure of Piling 垛積術 (that is, the formulas $\sum_{k=1}^{n} k = n(n+1)/2$, $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$, etc.), we can calculate exactly $U(n)$, $W(n)$, and consequently $\bar{V}(n)$:

$$U(n) = (-1 + 3n + 4n^2)/6n^2, \quad W(n) = (-1 - 3n + 4n^2)/6n^2,$$

$$\bar{V}(n) = (-1 + 4n^2)/6n^2 = 2/3 - 1/6n^2.$$

Therefore, we have

$$\bar{V}(2^k) = \frac{2}{3} - \frac{1}{6}\left(\frac{1}{4}\right)^k.$$

In this case, the extreme value can be obtained exactly by the Procedure of Incremental Divisor. Only the first 3 terms of $\bar{V}(n)$ are necessary to obtain the extreme value 2/3.

Chapter 9 of the *Tetsujutsu Sankei* reveals that Takebe Katahiro recognized this phenomenon through numerical calculation. This passage does not suggest that both Seki Takakazu and Takebe Katahiro had some notion of upper and lower approximations, used in today's Riemann integration.

Instead, Takebe Katahiro tried to understand the fast or slow convergences by the character of the dividing method and that of the figure. If two characters are conformable, he said a good result could be expected. Takebe Katahiro also thought that if the character of a mathematician is conformable to the character of the method of investigation, he could produce a good result. This kind of reasoning is stated in the concluding chapter named One Chapter on a Theory of Proper Character.

Comments on Chapter 10

The procedure of root extraction is a method to calculate numerically the square root of 1166, digit by digit, using the counting board.

> The structure of Chapter 10 is as follows: [1-3] statement of problem and answer; [4-7] statement of procedure; [8-23] manipulation on the counting board to execute the procedure of root extraction; [24-29] comments on manipulation; and [30-31] closing remark.

[1] Problem is to solve numerically the quadratic equation

$$-1166 + x^2 = 0. \tag{7}$$

[5-6] The coefficients are represented by counting rods and placed on a counting board. See VIII Comments on Counting Board.
[8-23] Here is the series of operations to find the root of the equation (7). See VIII Comments on Counting Board.

Comments on Chapter 11

This chapter explains a method of calculation of the circular constant π up to more than 40 digits and a method of approximation of π by fractions. The method for calculating π is equivalent to the modern Romberg method which employs repeated Richardson extrapolation.

> The structure of Chapter 11 is as follows: [1-8] calculation of the square of the cut out inscribed 2^n-gon's perimeter; [9-12] comments on Seki Takakazu's calculation; [13-25] calculation of the square of π using repeatedly the procedure of the incremental divisor; [26-30] comparison with Seki's calculation; [31] π with 41 digits accuracy; [32-47] calculation of approximate fractions by means of residual devision; [48-51] explanation

of Seki's method to find approximate fractions; [52-62] desirable approximation for application; [63-73] Zu Chongzhi's result according to the Book of the *Sui* dynasty; and [74-77] closing remark.

[1-8] Let L_n be the perimeter of the regular 2^n-gon inscribed within a circle of diameter $d = 10$ and $A_n = L_n^2$ be the square of the cut out perimeter 截周冪. Takebe calculates first A_n for $n = 3,\ 4,\ \cdots,\ 9$ numerically by the repeated use of the regular triangle rule. A computer gives us the following values for $n = 3,\ 4,\ \cdots,\ 9$:

$$
\begin{aligned}
A_2 &= 800.000000000000000000000000000000000000000000,\\
A_3 &= 937.258300203047921917298041264483074288524999396,\\
A_4 &= 974.341983855529521559255175721109928673067188945,\\
A_5 &= 983.793643354601004739469509926961306934592978261,\\
A_6 &= 986.167977534077697057392003178512061342746884125,\\
A_7 &= 986.762276722775888059101381344711112127598833982,\\
A_8 &= 986.910896278011524659119173886945361719848934671,\\
A_9 &= 986.948053964673231795110590765801791730892115570.
\end{aligned}
$$

Takebe published these values in the *Enritsu* 円率, which is most probably Chapter 1 in the Volume 12 of the *Taisei Sankei.*
[9] In obtaining π, Takebe comments that it is better to calculate π^2 by means of $(L_n^{(i)})^2$ and to extract the square root of π^2 rather than calculate π directly. Though the text can be read as he first calculates π^2, the final value of π stated in this chapter is not $\sqrt{\pi^2}$, but π calculated directly. This can be inferred since the two approximations of π, one by means of $(L_n^{(i)})^2$ and the other directly, yield slightly different values.
[13-15] $D_n^{(1)} = A_n - A_{n-1}$ is called the first difference 一差.

$$
\begin{aligned}
D_3^{(1)} &= 137.258300203047921917298041264483074288524999939688,\\
D_4^{(1)} &= 37.0836836524815996419571344566268543845421189548493,\\
D_5^{(1)} &= 9.45165949907148318021433420585137826152578931581158,\\
D_6^{(1)} &= 2.37433417947669231792249325155075440815390586408 18,\\
D_7^{(1)} &= 0.594299188698191001709378166199050784851949856744 34,\\
D_8^{(1)} &= 0.148619555235636600017792542234249592250100689547 89,\\
D_9^{(1)} &= 0.037157686661707135991416878856430011043180898889459.
\end{aligned}
$$

By observing the values of $D_{n+1}^{(1)}/D_n^{(1)}$ for $n = 3,\ 4, \cdots,\ 8$, he states that this sequence tends to $1/4$. He erroneously writes in [14] that "dividing the difference by the proceeding one", he got the value $1/4$. He also makes similar mistakes in [17] and [20]. Based on the procedure of incremental divisor 增約術, he defines $A_n^{(1)} = A_n + D_n^{(1)}/(4-1)$ to be the squares of first approximate circumferences 一遍

約周冪 for $n = 3,\ 4,\ \cdots, 9$.

$$
\begin{aligned}
A_3^{(1)} &= 933.33333333333333333333333333333333333333333333333,\\
A_4^{(1)} &= 983.01106693739722922306405501931076571803333252918,\\
A_5^{(1)} &= 986.70321173969005477324088720665221346791458546154,\\
A_6^{(1)} &= 986.94419652095816579954095466224509968843490803313,\\
A_7^{(1)} &= 986.95942226056992783003283426236231281213151941330,\\
A_8^{(1)} &= 986.96037645234195172633784073344412905588281726760,\\
A_9^{(1)} &= 986.96043612975673685912510473435677825059896823475.
\end{aligned}
$$

[16-18] With the sequence $A_n^{(1)}$ he repeats the similar calculation. $D_n^{(2)} = A_n^{(1)} - A_{n-1}^{(1)}$ is called the second difference 二差. Observing the numerical values of $D_{n+1}^{(2)}/D_n^{(2)}$ for $n = 4,\ 5,\ \cdots,\ 8$, he finds that this sequence tends to $1/16$. Based upon the procedure of incremental divisor, he defines $A_n^{(2)} = A_n + D_n^{(2)}/(16-1)$ to be the squares of second approximate circumferences 二遍約周冪 for $n = 4,\ 5,\ \cdots,\ 9$.
[19-20] With the sequence $A_n^{(2)}$ he repeats the similar calculation. $D_n^{(3)} = A_n^{(2)} - A_{n-1}^{(2)}$ is called the third difference 三差. Observing numerical values $D_{n+1}^{(3)}/D_n^{(3)}$ for $n = 5,\ 6,\ \cdots,\ 8$, he finds this sequence tends to $1/64$. Based on the procedure of incremental divisor, he defines $A_n^{(3)} = A_n + D_n^{(3)}/(64-1)$ to be the squares of third approximate circumferences 三遍約周冪 for $n = 5,\ 6,\ \cdots,\ 9$.
[21-25] He continues the calculation repeatedly 5 times and finally obtains the good approximate value of π^2. Taking its square root, he finds the 42 digits approximate value of π (see commentary [9]):

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 68984.$$

[32-47] The procedure of residual division 零約術 refers to methods to approximate a real number by a fractional number. Seki invented his method of residual division but Takebe Kata'akira invented his own methods, which coincides with the so-called Euclid's algorithm. By the method of residual division due to Kata'akira, the calculation goes as follows:
[32-34] Let $b_0 = 1$ be the length of a diameter, which is called the original number 元数. First, divide π by b_0 and get the quotient a_1 and the remainder b_1; in this case, it amounts to that a_1 is the integral part of π and b_1 the fractional part: $\pi = a_1 \times b_0 + b_1$, where $a_1 = [\pi] = 3$ is called the first quotient 第一商 and b_1 the first inexhaustible 第一不尽.
[35] Second, decompose b_0 by b_1 and get the quotient a_2 and the remainder b_2: $b_0 = a_2 \times b_1 + b_2$, where $a_2 = [b_0/b_1] = 7$ is called the second quotient 第二商 and b_2 the second inexhaustible 第二不尽.
[36] Third, $b_1 = a_3 \times b_2 + b_3$, where $a_3 = [b_1/b_2] = 15$ is called the third quotient 第三商 and b_3 the third inexhaustible 第三不尽.

[37] Fourth, $b_2 = a_4 \times b_3 + b_4$, where $a_4 = [b_2/b_3] = 1$ is called the fourth quotient 第四商 and b_4 the fourth inexhaustible 第四不尽.
[38] Fifth, $b_3 = a_5 \times b_2 + b_5$, where $a_5 = [b_3/b_4] = 292$ is called the fifth quotient 第五商 and b_5 the fifth inexhaustible 第五不尽.
[39] In general, he puts $b_{n-1} = a_{n+1} \times b_n + b_{n+1}$, where $a_{n+1} = [b_{n-1}/b_n]$.
[40-41] Let k_1 be the first rate of the diameter 第一径率 and s_1 the first rate of the circumference 第一周率; that is, $k_1 = 1$, $s_1 = a_1 = 3$. Because $s_1/k_1 < \pi$, k_1 and s_1 are called the first weak rates 一等弱率.
[42] Let $k_2 = k_1 a_2 = 1 \cdot 7 = 7$ and $s_2 = s_1 a_2 + 1 = 3 \cdot 7 + 1 = 22$. Because $s_2/k_2 > \pi$, k_2 and s_2 are called the second strong rates 二等強率.
[43] Let $k_3 = k_2 a_3 + k_1 = 7 \cdot 15 + 1 = 106$ and $s_3 = s_2 a_3 + s_1 = 22 \cdot 15 + 3 = 333$. Because $s_3/k_3 < \pi$, k_3 and s_3 are called the third weak rates 三等弱率.
[44] $k_4 = k_3 a_4 + k_2 = 106 \cdot 1 + 7 = 113$ and $s_4 = s_3 a_4 + s_3 = 333 \cdot 1 + 2 = 335$. Because $s_4/k_4 > \pi$, k_4 and s_4 are called the fourth strong rates 四等強率.
[45] Let $k_{n+1} = k_n a_{n+1} + k_{n-1}$ and $s_{n+1} = s_n a_{n+1} + s_{n-1}$. Then we have

$$s_1/k_1 < s_3/k_3 < s_5/k_5 < \cdots < \pi < \cdots < s_6/k_6 < s_4/k_4 < s_2/k_2;$$

this fact is expressed by saying the rates are strong and weak alternatively.

[48] According to Seki's original procedure of residual division, the calculation of the rates goes as follows: $\frac{3}{1}(< \pi)$, $\frac{3+4}{1+1} = \frac{7}{2}(> \pi)$, $\frac{7+3}{2+1} = \frac{10}{3}(> \pi)$, $\frac{10+3}{3+1} = \frac{13}{4}(> \pi)$, $\frac{13+3}{4+1} = \frac{16}{5}(> \pi)$, $\frac{16+3}{5+1} = \frac{19}{6}(> \pi)$, $\frac{19+3}{6+1} = \frac{22}{7}(> \pi)$, $\frac{22+3}{7+1} = \frac{25}{8}(< \pi)$, $\frac{25+4}{8+1} = \frac{29}{9}(> \pi)$.
[53] Here *byō* 秒 means 1/60 minutes.
[63] The *Jiu Shu* 九数 refers to the names of nine chapters of the *Jiuzhang Suanshu*.

[65] Liu Xin (劉歆, ca. 50 – 23 BC), Zhang Heng (張衡, 78–139), Liu Hui (劉徽, ca. 3 c.), Wang Fan (王蕃, 228 – 266), and Pi Yanzong (皮延宗, ca. 5 c.) are Chinese mathematicians. In the original text, Wang Fan is erroneously written as Wang Shen/Ō Shin 王審.
[67] The *Song* Kingdom 宋 is a Chinese Kingdom (420 - 479).
[68] 1 *jō* 丈 is 10 *shaku*, i.e., 100 *sun*. 1 *oku* 億 $= 10^8$. This means he considered a number with 8 or 9 digits. The upper bound is 3.1415927 *jō*. Here *byō* means *shi*. 1 *byō* = 1 *shi* $= 10^{-4}$ *sun*. The lower bound is 3.1415926 *jō*.
[72] *Sui Zhi/Zui shi* 隋志 refers the monograph on calendar of *Sui Shu* 隋書, the Book of the *Sui* dynasty.

Comments on Chapter 12

In this chapter, Takebe Katahiro states three formulas for an inverse trigonometric function.

Let $t = c/d$. As we have $(s/2)^2 = d^2(\arcsin\sqrt{t})^2$, the formulas (18), (20), and (21) below give the following approximation formulas of $f(t) = (\arcsin\sqrt{t})^2$:

$$f(t) \approx t(1+\frac{1}{3}t(1+\frac{8}{15}t(1+\frac{9}{14}t(1+\frac{32}{45}t(1+\frac{25}{33}t(1+\frac{72}{91}t)))))), \tag{8}$$

$$f(t) \approx t(1+\frac{1}{3}t(1+\frac{8}{15}\frac{t}{1-t}(1-\frac{5}{14}\frac{t}{1-t}(1-\frac{12}{25}\frac{t}{1-t}(1+\frac{223}{398}\frac{t}{1-t}))))), \tag{9}$$

and

$$f(t) \approx t(1+\frac{1}{3}t(1+\frac{8}{15}\frac{t}{1-\frac{9}{14}t}(1+\frac{43}{980}\frac{t^2}{1-\frac{1696}{1419}t+\frac{6743008}{26176293}t^2}))). \tag{10}$$

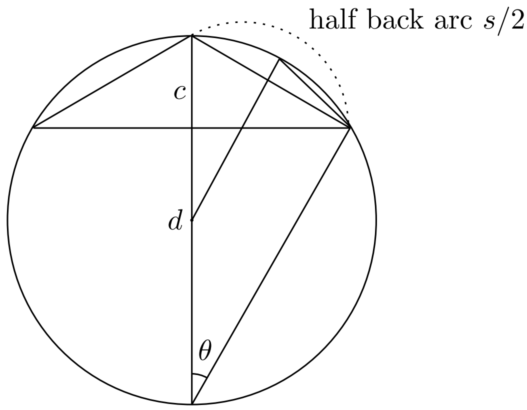
The above forms of mathematical expression were standard notation in the Japanese mathematics of the *Edo* period and they were convenient in the numerical calculation on the Japanese abacus (see e.g., [Ogawa2000]).

The structure of Chapter 12 is as follows: [1-98] the first formula (8); [1-7] philosophy of a back arc; [8-12] calculation of the definite half back arc; [13-19] calculation of the first definite difference and its approximations in a fractional expression; [20-22] comments on an old method; [23-27] calculation of the second definite difference and its approximations in a fractional expression; [28-32] calculation of the third definite difference and its approximations in a fractional expression; [33-37] calculation of the fourth definite difference and its approximations in a fractional expression; [38-42] calculation of the fifth definite difference and its approximations in a fractional expression; [43-47] calculations of the sixth definite difference and its approximations in a fractional expression; [48-49] calculations of the sixth approximate difference; [50] table of these numerical values; [51-59] statement of the original procedure; [60-64] comments on Seki's 4-multiplication procedure; [65-73] inductive inference of the coefficients and its results; [74-90] argument on inexhaustible numbers; [91-98] comments on the first formula (8);
[99-115] the second formula (9); [99-108] statement of the second formula; [109-115] evaluation of the formula and a comment on an old method;
[116-172] the third formula (10); [116-121] repetition of the calculation of the second definite difference; [122-129] calculation of the approximate coefficient of the sagitta; [130-132] calculation of the third definite difference; [133-150] calculation of the approximate coefficient of the square of the sagitta; [151-152] calculation of the third approximate difference of 4-multiplication; [153-156] general method for further calculations; [157] table of the numerical values obtained by the above method; [158-168] statement of the third formula (10); [169-171] comment on the formula; [172-176] closing remarks.

[1-98] *The first formula*

[5-7] Suppose we are given a circle of diameter $d = 10$. Let s be the length of the back arc with sagitta c.

As in the case of the calculation of the circumference, using the 5 operations (addition, subtraction, multiplication, division and the extraction of square roots) Japanese mathematicians of the 18c century could calculate the arc length s numerically once the sagitta c is given.



Seki Takakazu sought to find a "formula" which gives the approximate value of the arc length s when the sagitta c is given. Using the arc length s for $c = 1,\ 2,\ 3,\ 4,\ 4.5$. In the *Katsuyō Sanpō*, Seki obtained the formula:

$$\begin{aligned}113^2 \times 100^2(d-c)^5 s^2 = {} & 5107600cd^6 - 23835413c^2d^5 + 43470240c^3d^4 \\ & - 37997429c^4d^3 + 15047062c^5d^2 \\ & - 1501025c^6d - 281290c^7.\end{aligned} \tag{11}$$

If we approximate s by (11), the error is roughly of the order of 10^{-6}. In this sense Seki's approach was successful. But Takebe Katahiro was not satisfied by this result. [8-9] $(s/2)^2$ is called the square of the definite half back arc. In the modern notation, $(s/2)^2 = (d \arcsin(\sqrt{c/d}))^2$. The Japanese mathematicians could calculate $(s/2)^2$ numerically once c was given numerically. For example, $(s/2)^2 = 10.3523419254547$ for $c = 1$ and $(s/2)^2 = 1.003355122621573$ for $c = 0.1$. (We assume $d = 10$.)

Contrary to Seki's investigation, Takebe considered smaller values of c and calculated the corresponding arc length s. Observing the values carefully, he tried to approximate $(s/2)^2$. In doing so, he found that results improved with decreasing c. Finally, he took the sagitta as small as $c = 10^{-5}$. Takebe determines $(s/2)^2$ numerically for $c = 10^{-5}$ using repeatedly the procedure of incremental divisor:

$$(s/2)^2 = 1.0000003333335111112253969066667282347769479595875 \times 10^{-4},$$

which he calls the "definite half back arc 定半背冪".
[13-14] Its first approximation is 10^{-4}, which Takebe observes to equal $cd = 10^{-4}$.
[15-16] He calls cd the "approximate half back arc 汎半背冪" and $t_1 = (s/2)^2 - cd$ the "first definite difference 一定差".

[17-19] He finds $t_1 = 0.3333335111112253969066667282347769479595875 \times 10^{-10}$and observes the order of t_1 is equal to the order of $c^2 = 10^{-10}$. Then he calculates the ratio $t_1/c^2 = 0.333333511111$. By the procedure of residual division, he finds this decimal is approximated by the fraction 1/3.

The procedure of residual division or the "reiyaku" method is a method to convert a decimal to a fraction. Takebe Katahiro claims that his elder brother Takebe Kata'akira improved Seki Takakazu's "reiyaku" method given in the *Katsuyō Sanpō* because it was not efficient. Takebe Kata'akira's method was the same as Euclid's algorithm. The "reiyaku" method has been studies by many authors [Shibata1935], [Shibata1935b], and [Hayashi1915] (collected in [Hayashi1937]).

[23] Takebe defines the "first approximate difference 一汎差" h_1 by

$$h_1 := c^2 \times (1/3) \tag{12}$$

and finds $h_1 = 0.3333333333333333333333333333333333333333 \times 10^{-10}$.

[24] Next he defines the "second definite difference 二定差" $t_2 := t_1 - h_1$.

[25-27] He finds $t_2 = 0.177777892063573333394901443614626254 2 \times 10^{-16}$and observes that the order of t_2 is equal to the order of $h_1 \times (c/d)$ and calculates the ratio $t_2/(h_1 \times (c/d)) = 0.53333367619$. By the procedure of residual division, he finds this decimal is approximated by the fraction 8/15.

[28] Then he defines the "second approximate difference 二汎差" h_2 by

$$h_2 := h_1 \times (c/d) \times (8/15) \tag{13}$$

and finds $h_2 = 0.1777777777777777777777777777777777777777 \times 10^{-16}$.

[29] Next he defines the "third definite difference 三定差" $t_3 := t_2 - h_2$.

[30-32] He finds $t_3 = 0.11428579555561712366583684847 64 \times 10^{-22}$ and he observes that the order of t_3 is equal to the order of $h_2 \times (c/d)$ and calculates the ratio $t_3/(h_2 \times (c/d)) = 0.6428576$ slightly strong, where slightly strong means the number is between 0.6428576 and 0.64285761. By the procedure of residual division, he finds this decimal is approximated by the fraction 9/14.

[33] Then he defines the "third approximate difference 三汎差" h_3 by

$$h_3 := h_2 \times (c/d) \times (9/14) \tag{14}$$

and finds $h_3 = 0.1142857142857142857142857142857 \times 10^{-22}$.

[34] Next he defines the "fourth definite difference 四定差" $t_4 := t_3 - h_3$.

[35-37] He finds $t_4 = 0.8126990283795155113419 07 \times 10^{-29}$ and he observes that the order of t_4 is equal to the order of $h_3 \times (c/d)$ and calculates the ratio $t_4/(h_3 \times (c/d)) = 0.71111164983$. By the procedure of residual division, he finds this decimal is approximated by the fraction 32/45.

[38] Then he defines the "fourth approximate difference 四汎差" h_4 by

$$h_4 := h_3 \times (c/d) \times (32/45) \tag{15}$$

and finds $h_4 = 0.812698412698412698412698 \times 10^{-29}$.
[39] Next he defines the "fifth definite difference 五定差" $t_5 := t_4 - h_4$.
[40-42] He finds $t_5 = 0.615681102812929209 \times 10^{-35}$. He observes that the order of t_5 is equal to the order of $h_4 \times (c/d)$ and calculates the ratio $t_5/(h_4 \times (c/d)) = 0.75757635697$. By the procedure of residual division, he finds this decimal is approximated by the fraction $25/33$.
[43] Then he defines the "fifth approximate difference 五汎差" h_5 by

$$h_5 := h_4 \times (c/d) \times (25/33) \tag{16}$$

and finds $h_5 = 0.615680615680615681 \times 10^{-35}$.
[44] Next he defines the "sixth definite difference 六定差" $t_6 := t_5 - h_5$.
[45-47] He finds $t_6 = 0.487132313528 \times 10^{-41}$. He observes that the order of t_6 is equal to the order of $h_5 \times (c/d)$ and calculates the ratio $t_6/(h_5 \times (c/d)) = 0.79120943736$.

By the procedure of residual division, he finds this decimal is approximated by the fraction $72/91$.
[48] Then he defines the "sixth approximate difference 六汎差" h_6 by

$$h_6 = h_5 \times (c/d) \times (72/91) \tag{17}$$

and finds $h_6 = 0.487131915703 \times 10^{-41}$.
[49] He stops the calculation at this stage, then states the calculation can be continued similarly.
[51-59] In this formal statement of procedure, he repeats the definitions (12), (13), (14), (15), (16) and (17) and states the formula to represent the square of the back arc $(s/2)^2$ in terms of sagitta c and diameter d:

$$\begin{aligned}\left(\frac{s}{2}\right)^2 - cd &= t_1 = h_1 + t_2 = h_1 + h_2 + t_3 = \cdots \\ &= h_1 + h_2 + h_3 + h_4 + h_4 + h_5 + h_6 (+ t_7).\end{aligned}$$

Substituting the definitions in this formula we obtain

$$\begin{aligned}\left(\frac{s}{2}\right)^2 \approx cd &+ \frac{1}{3}c^2 + \frac{1}{3}\frac{8}{15}\frac{c^3}{d} + \frac{1}{3}\frac{8}{15}\frac{9}{14}\frac{c^4}{d^2} + \frac{1}{3}\frac{8}{15}\frac{9}{14}\frac{32}{45}\frac{c^5}{d^3} \\ &+ \frac{1}{3}\frac{8}{15}\frac{9}{14}\frac{32}{45}\frac{25}{33}\frac{c^6}{d^4} + \frac{1}{3}\frac{8}{15}\frac{9}{14}\frac{32}{45}\frac{25}{33}\frac{72}{91}\frac{c^7}{d^5}.\end{aligned} \tag{18}$$

[65-72] Takebe observes carefully the denominators and the numerators of the coefficients separately and deduces recursively that the fraction which should be multiplied to the $(i-1)$-th term to obtain the i-th term $(i \geq 2)$ is given by $\dfrac{2i^2}{(2i+1)(i+1)}$ when i is even, and by $\dfrac{i^2}{(2i+1)(i+1)/2}$ when i is odd. In this way, Takebe finds the calculation can be continued as many steps as one wishes using the following algorithm:

```
E := c^2/3;
S := cd + E;
for i := 2 to N do begin
    if i mod 2 = 0 then
        begin P := (2i+1)(i+1); Q := 2i^2 end
    else
        begin P := (2i+1)(i+1)/2; Q := i^2 end;
    E := E · (Q/P) · (c/d); S := S + E
end;
```

It can be said that formula (18) was the first infinite series expansion in the history of Japanese mathematics. In fact, it coincides with the Taylor expansion of the trigonometric function $(\arcsin x)^2$ in x at $x = 0$.

Note that (18) was later reformulated as

$$\left(\frac{s}{2}\right)^2 = cd\left\{1 + \frac{2^2}{3\cdot 4}\left(\frac{c}{d}\right) + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}\left(\frac{c}{d}\right)^2 + \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}\left(\frac{c}{d}\right)^3 + \cdots\right\} \tag{19}$$

in the *Enri Kohaijutsu*, where (19) was derived by a more algebraic method than the above.
[95] This sentence refers to the main procedure referred below.

[99-115] *The second formula*

[100] Let c be the sagitta and d the diameter. cd is called the square of the approximate half back arc.
[101] $+s_1 = c^2 \times 1/3$.
[102] $+s_2 = s_1 \times c/(c-d) \times 8/15$.
[103] $-s_3 = -s_2 \times c/(c-d) \times 5/14$.
[104] $+s_4 = +s_3 \times c/(c-d) \times 12/25$.
[105] $-s_5 = -s_4 \times c/(c-d) \times 223/396$. The last denominator was erroneously stated as 398 in the text.
[107] $(s/2)^2 = cd + s_1 + s_2 - s_3 + s_4 - s_5$.
[100-107] In sum, the second formula was as follows:

$$\begin{aligned}\left(\frac{s}{2}\right)^2 \approx cd &+ \frac{1}{3}c^2 + \frac{1}{3}\frac{8}{15}\frac{c^3}{d-c} - \frac{1}{3}\frac{8}{15}\frac{5}{14}\frac{c^4}{(d-c)^2} + \frac{1}{3}\frac{8}{15}\frac{5}{14}\frac{12}{25}\frac{c^5}{(d-c)^3} \\ &- \frac{1}{3}\frac{8}{15}\frac{5}{14}\frac{12}{25}\frac{223}{396}\frac{c^6}{(d-c)^4}\end{aligned} \tag{20}$$

[113] Takebe Katahiro abandoned the second formula saying its precision did not increase very much even with the increased number of multipliers.

[116-172] *The third formula*

[117] Let $d = 10$ be the diameter and $c = 10^{-5}$ the sagitta. The definite back arc [i.e., length of the back arc] is denoted by s. $(s/2)^2$ is called the square of the definite half back arc. $cd = 10^{-4}$ is called the square of the approximate half back arc.
[118] $t_1 = (s/2)^2 - cd$ is called the first definite difference.
[119] $h_1 = c^2/3$ is called the first approximate difference.
[120] $+t_2 = t_1 - h_1$ is called the second definite difference.
[122-124] $t_2/(h_1 \times c/d) = 5.333367619 \times 10^{-1} = 8/15$.
[130-131] $\tilde{h}_2 = (h_1 \times c)/(-c \times 9/14 + d) \times 8/15$ is called the cube of the second approximate difference.
[132] $\tilde{t}_3 = t_2\tilde{h}_2$ is called the third definite difference.
[133-134] $\tilde{t}_3/(\tilde{h}_2 \times c^2/d^2) = 4.387760346325 \times 10^{-2} \simeq 43/980$.
[138] $NPC = (\tilde{h}_2 \times c^2 \times 43/980)/\tilde{t}_3 - d^2 = -1.9520763527249924963 \times 10^{-4}$ is called the Numerator of the Product Coefficient 段積実.
[140-142] $-NPC/cd = 1.19520763528240024963 \simeq 1696/1419$.
[143] $A = NPC - cd \times 1696/1419 = 2.5759981223733 \times 10^{-12}$.
[145-147] $A/c^2 = 0.025759981223733 \simeq 6743008/2641762913$.
[151-152] $\tilde{h}_3 = (\tilde{h}_2 \times c^2)/(c^2 \times 6743008/26176293 + d^2 - cd \times 1696/1419) \times 43/980$ is called the third approximate difference of 4-multiplication.

[158-168] The third formula is written in Chinese.
[158] cd is called the square of the approximate half back arc.
[159] $h_1 = c^2/3$ is called the first difference.
[160-162] $\tilde{h}_2 = (h_1 \times c)/(-c \times 9/14 + d) \times 8/15$ is called the second difference.
[163-165] $\tilde{h}_3 = (\tilde{h}_2 \times c^2)/(c^2 \times 6743008/26176293 + d^2 - cd \times 1696/1419) \times 43/980$ is called the third difference.
[166] The final formula is as follows: $(s/2)^2 = cd + h_1\tilde{h}_2 + \tilde{h}_3$. We can rewrite it as follows:

$$\begin{aligned}\left(\frac{s}{2}\right)^2 \approx cd &+ \frac{1}{3}c^2 + \frac{1}{3}\cdot\frac{c^3}{d-\frac{9}{14}c}\cdot\frac{8}{15} \\ &+ \frac{1}{3}\cdot\frac{c^5}{d-\frac{9}{14}c}\cdot\frac{1}{d^2-\frac{1696}{1419}cd+\frac{6743008}{26176293}c^2}\cdot\frac{8}{15}\cdot\frac{43}{980}.\end{aligned} \tag{21}$$

If we calculate following Takebe Katahiro's instruction in the *Tetsujutsu Sankei*, we cannot obtain the fraction 6743008/26176293. But if we calculate without expanding into decimals by the procedure of residual division, we can get this value. In this sense, the fraction given here is right but it is unclear how Takebe Katahiro obtained this value. Mr. Yokotsuka Hiroyuki showed that this fraction can be obtained if we take $c = 10^{-13}$ instead of $c = 10^{-5}$. (See [Yokotsuka2004] and [Yokotsuka2006].) As the value $c = 10^{-13}$ for the sagitta was utilized in the *Sanreki Zakkō* ([SatoS1995]), we can conjecture Takebe Katahiro calculated with this value.

Takebe Katahiro said that he utilized the value $c = 10^{-9}$ for the calculation for the third approximation formula instead of $c = 10^{-5}$ in calculating 90 digits. But even with $c = 10^{-9}$ we cannot obtain this fraction. The *Sanreki Zakkō* had been studied for many years ([Fujiwara1941], [Fujiwara1945], etc.). But it was Yokotsuka who first obtained some meaningful results concerning its relation with the calculation of arc length in the *Tetsujutsu Sankei*.

Takebe Katahiro could not obtain these two formula algorithmically, but simply stated how to find the first few terms.

These two formulas were identified in the early stages of research on the history of Japanese mathematics (see, e.g., [Hayashi1911].) but their meaning was not clear. Recently one of the authors of this commentary proposed an interpretation [Morimoto2003], which we proceed to explain.

[71] It is most probably that the *Koritsu* is a chapter of the *Taisei Sankei*.

Comments on One Chapter on a Theory of Proper Character

The last Chapter is a summary of Takebe's philosophy on mathematics and mathematicians. He deliberates the psychological relationship between the character of mathematics and that of mathematicians and concludes that one can reach the solution of a mathematical problem if both correspond with each other but that one cannot if not. He also insists that the character of a mathematicians can never be changed even if one studies mathematics hard. It is essential for him to be in the Way of Mathematics.

We don't comment on this Chapter anymore because it does not contain any mathematical problems specifically. We refer the reader to [Murata1982], [Horiuchi1994b], and [Ogawa2007].

Comments on Appendix

[3] Three sides of a triangle are called large, middle, and small.

[4] The middle line is a line perpendicular to the large side passing through the opposite vertex. A regulated number means a rational number.

[42] The year *kinoto mi* 乙巳 is one in the sexagenarian cycle. This year corresponds with 1725 AD.

VII Comments on Units

- *sen* 銭 is a unit for silver money. (Chapter 1)
- *koku* 斛 and *to* 斗 are units for grain.
  1 *koku* = 10 *to*. (Chapter 1)
- *bu* 歩 is a unit for length and for area. 1 *bu* is approximately equal to 1.8 m.
  1 [squared] *bu* is approximately equal to 3.3 m^2. (Chapter 2 and 10)

- *sun* 寸 is the basic unit for length and is approximately equal to 3 cm.
  1 *jō* 丈 $= 10^2$ *sun* 寸.
  1 *shaku* 尺 $= 10$ *sun* 寸.
  1 *sun* 寸.
  1 *bu* 分 $= 10^{-1}$ *sun* 寸. Distinguish 1 *bu* 分 from 1 *bu* 歩.
  1 *ri* 厘 $= 10^{-2}$ *sun* 寸. (Chapter 3)
  1 *mō* 毛 $= 10^{-3}$ *sun* 寸.
  1 *shi* 糸 $= 10^{-4}$ *sun* 寸.
  1 *kotsu* 忽 $= 10^{-5}$ *sun* 寸.
  1 *bi* 微 $= 10^{-6}$ *sun* 寸.
  1 *sha* 沙 $= 10^{-8}$ *sun* 寸.
  1 *jin* 塵 $= 10^{-9}$ *sun* 寸.(Chapter 8, 11, 12)
  1 *byō* 渺 $= 10^{-12}$ *sun* 寸. (Chapter 12)
- *tan* 端 and *ri* 厘 are units for length of cloth.
  1 *tan* $= 10$ *ri*. (Chapter 5)

VIII Comments on Counting Board

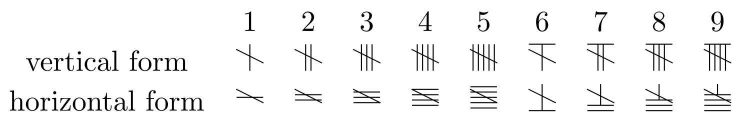
VIII.1 Counting Rods

In traditional Japanese mathematics, as in traditional Chinese mathematics, numbers were mostly integers or finite decimals. Positive numbers were represented by red counting rods and negative numbers by black counting rods. (See pages 273 and 267.) Counting rods representing the single digits 1 through 9 were placed in an appropriate box on the counting board and arranged as follows:

	1	2	3	4	5	6	7	8	9
vertical form	𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
horizontal form	𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

The vertical forms were used for every other non-zero digit as in 1, 100, 10000, etc., while the horizontal forms were used for the remaining non-zero digits as in 10, 1000, 100000, etc. In this way, in representing numbers with more than a single digit on a counting board, neighboring digits could be easily distinguished. (0 digits were represented by recognizably empty spaces.)

If red ink were not available, negative numbers were denoted with a slash:



In the text, the numbers were mostly written by Chinese characters, which transcribed in corresponding arabic numbers, but the numbers on the counting board were written using counting rods (Chapters 2 and 6).

VIII.2 Counting Board

A counting board was originated from Ancient China and still one of the most important calculating tools in Takebe's day. The following is an example of a counting board.

10^3	10^2	10	1	
				Quotient
				Reality
				Square
				Side
				Corner

Each row was named by a single Chinese character *shang* 商, *shi* 実, *fang* 方, *lian* 廉, and *yu* 隅 (see, for example, [Martzloff1987]) and was called *shō*, *jitsu*, *hō*, *ren* and *gū* in Japanese. In our translation, we employ English names, the Quotient, the Reality, the Square, the Side and the Corner, translating literally the respective Chinese characters.

VIII.3 Procedure of Root Extraction

The counting board was used, in traditional Chinese mathematics, to calculate the square root or the cube root of a number and more generally to solve numerically an algebraic equation with integral coefficients.

The procedure of root extraction had been well known in China since the *Jiuzhang Suanshu*, which is one of first Chinese mathematics books. At first it was applied to extract a square root or a cubic root. Later in the Song dynasty in China it was elaborated to handle with algebraic equations of higher degree. The procedure of root extraction is sometimes called Horner's method (introduced in the 19th century, in England) but its discovery was much earlier in China. Japanese mathematicians of the *Edo* period mastered this procedure completely and convinced that they could solve numerically any algebraic equation once it was given. See the *Sangaku Keimō Genkai Taisei* (Great Colloquial Commentary on the *Suanxue Qimeng*).

Note that, in the *Fukyū Tetsujutsu* the chapter on square root extraction is placed before the chapter on the rule of element placement. This order is more coherent logically than that in the *Tetsujutsu Sankei*.

An algebraic equation with numerical coefficients

$$a_0 + a_1(x-q) + a_2(x-q)^2 + a_3(x-q)^3 = 0 \tag{22}$$

had a particular representation on the counting board. Beginning at the top, the numbers q, a_0, a_1, a_2, and a_3 are placed in the Quotient row, in the Reality row, in the Square row, in the Side row, and in the Corner row, respectively. We can say that the cubic algebraic equation (22) was represented by the following column vector (which we call a *configuration* on the counting board).

q	Quotient
a_0	Reality
a_1	Square
a_2	Side
a_3	Corner

(23)

For example, the quadratic equation (7) in Chapter 10 is represented as follows:

	10^3	10^2	10	1	
					Quotient
black	—	|	⊥	⊤	Reality
					Square
red				|	Side
					Corner

For simplicity, we replace counting rods by corresponding Arabic numbers. For example, the above configuration is represented as follows:

	Quotient
−1166	Reality
0	Square
1	Side
0	Corner

or

	Quotient
−1166	Reality
0	Square
1	Corner

(24)

As the highest coefficient was placed on the Corner row in the *Edo* period, the quadratic equation like (7) was placed using the Reality, the Square and the Corner rows. Mathematically speaking, a quadratic equation is nothing but a cubic equation with null highest coefficients and we shall use both of configurations in (24).

Suppose we are given a cubic equation

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0,$$

represented by the configuration

	Quotient
a_0	Reality
a_1	Square
a_2	Side
a_3	Corner

(25)

Whenever the Quotient row was increased by an amount q, the counting board was manipulated from bottom to top three times (in the following from right to left three times);

Quotient	Reality	Square	Side	Corner
0	a_0	a_1	a_2	a_3
q	$((a_3q+a_2)q+a_1)q$	$(a_3q+a_2)q$	a_3q	0
	$((a_3q+a_2)q+a_1)q+a_0$	$(a_3q+a_2)q+a_1$	a_3q+a_2	a_3
		$(a_3q+(a_3q+a_2))q$	a_3q	0
		$(a_3q+(a_3q+a_2))q$ $+(a_3q+a_2)q+a_1$	$a_3q+(a_3q+a_2)$	a_3
			a_3q	0
			$a_3q+a_3q+(a_3q+a_2)$	a_3
q	$((a_3q+a_2)q+a_1)q+a_0$	$(a_3q+(a_3q+a_2))q$ $+(a_3q+a_2)q+a_1$	$a_3q+a_3q+(a_3q+a_2)$	a_3

The purpose of this manipulation is to calculate a'_0, a'_1, a'_2, a'_3 in (26) from a_0, a_1, a_2, a_3:

$$a_0+a_1x+a_2x^2+a_3x^3 = a'_0+a'_1(x-q)+a'_2(x-q)^2+a'_3(x-q)^3 \qquad (26)$$

Now suppose we want to solve the equation

$$a_0+a_1x+a_2x^2+a_3x^3=0 \qquad (27)$$

and that there is a solution between 10 and 100.

First step: Choose a natural number q among 10, 20, 30, $\cdots$, 90 appropriately so that $|a'_0|$ becomes smallest.

Second step: Choose q' among 1, 2, 3, $\cdots$, 9 so that $|a''_0|$ in (28) becomes smallest:

$$\begin{aligned} &a'_0+a'_1(x-q)+a'_2(x-q)^2+a'_3(x-q)^3 \\ &= a''_0+a''_1(x-q-q')+a''_2(x-q-q')^2+a''_3(x-q-q')^3 \end{aligned} \qquad (28)$$

The calculation of a''_0, a''_1, a''_2, a''_3 from a'_0, a'_1, a'_2, a'_3 is same as that of a'_0, a'_1, a'_2, a'_3 from a_0, a_1, a_2, a_3.

When the Reality row became empty (i.e. 0) after several steps, the number in the Quotient row gives a root of the cubic equation (27). It was then said that "the root was extracted" or "the counting board was divided to extract the root". In this way, the algebraic equation could be solved numerically digit by digit.

This procedure looks very complicated but it consists of one simple calculation which can be seen if we write this procedure in a computer language. The Reality, Square, Side, Corner rows on the counting board may well be considered as being registers of a computer. It is fundamental in Japanese mathematics to consider the cubic equation (22) as the column vector (23), the component of which are registers.

Let us illustrate this procedure using equation (7) in Chapter 10, which is represented by configuration (24). First add $q = 30$ to the Quotient row and calculate according to the following program:

$$\begin{aligned}&a_2 := a_3 \times q + a_2, a_1 := a_2 \times q + a_1, a_0 := a_1 \times q + a_0\\&a_2 := a_3 \times q + a_2, a_1 := a_2 \times q + a_1\\&a_2 := a_3 \times q + a_2\end{aligned} \tag{29}$$

Then the counting board looks as follows:

30	Quotient
−266	Reality
60	Square
1	Corner

(30)

Since $a'_0 = -266$, $a'_1 = 60$, $a'_2 = 1$, configuration (30) tells us that

$$-1166 + x^2 = -266 + 60(x - 30) + (x - 30)^2.$$

Next add $q' = 4$ to the Quotient row.

Now put $a_0 = a'_0$, $a_1 = a'_1$, $a_2 = a'_2$, $a_3 = a'_3$, and $q = q' = 4$ and apply the program (29). We will obtain the coefficients a''_0, a''_1, a''_2, and a''_3 such that:

$$\begin{aligned}&a''_0 + a''_1(x - q - q') + a''_2(x - q - q')^2 + a''_3(x - q - q')^3\\&= a'_0 + a'_1(x - q) + a'_2(x - q)^2 + a'_3(x - q)^3\\&= a_0 + a_1 x + a_2 x^2 + a_3 x^3.\end{aligned}$$

At this stage, the counting board becomes:

34	Quotient
−10	Reality
68	Square
1	Corner

(31)

Configurations (31) on the counting board tells us that

$$-1166 + x^2 = -10 + 68(x - 34) + (x - 34)^2.$$

Therefore, $-1166+34^2 = -10$, that is, $34^2+10 = 1166$. In other words, $\sqrt{1166} \doteqdot 34$.

The program (29) can be applied to the numerical calculation of the cube root, or to the numerical solution of a cubic equation. Japanese mathematicians could solve numerically any algebraic equation of any order generalizing the program (29). Note that, if $a_0 = -N$, $a_1 = D$, $a_2 = 0$, and $a_3 = 0$, repeated applications of the program (29) calculate the decimal expansion of the quotient N/D. In this sense, the extraction of root was considered as a generalization of the division operation and was called generalized division.

VIII.4 Counting Board Algebra

In the Song dynasty, the procedure of celestial element was invented and transmitted to Japan by the *Suanxue Qimeng* (1299) of Zhu Shijie. Takebe Katahiro called this procedure the rule of element placement, which is, in modern terminology, a method to represent a polynomial of one variable by means of the counting board. In this sense, this rule is sometimes called the counting board algebra.

Suppose we are given a polynomial

$$a_0 + a_1x + a_2x^2 + a_3x^3, \tag{32}$$

where a_0, a_1, a_2, a_3 are integers and x is an unknown variable. In the rule of element placement, a_0 is placed in the Reality row, a_1 in the Square row, a_2 in the Side row, and a_3 in the Corner row. This means that the cubic polynomial (32) is represented by the following configuration on the counting board:

	Quotient
a_0	Reality
a_1	Square
a_2	Side
a_3	Corner

(33)

When the Quotient row is empty, we abbreviate (33) as a column vector: $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$. If $a_3 = 0$, we omit a_3; if $a_2 = 0$ and $a_3 = 0$, we omit a_2 and a_3; if $a_1 = 0$, $a_2 = 0$ and $a_3 = 0$, then the vector is considered as a scalar:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \qquad \begin{bmatrix} a_0 \\ a_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \qquad \begin{bmatrix} a_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [a_0] = a_0.$$

Note that an actual number would be represented on the counting board by placing counting rods in a single row (the Reality row), using alternating vertical and horizontal forms of counting rods as mentioned earlier. On the other hand, *virtual quantities* (that is, those represented by polynomials in which an unknown variable, x, appears explicitly) were represented by column vectors with at least two entries.

In the *Kai Indai no Hō* (ca.1685), Seki Takakazu described the addition, subtraction and multiplication for column configurations. Namely, addition, subtraction and scalar multiplication are defined in the usual way for column vectors:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \pm \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_0 \pm b_0 \\ a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}, \qquad c \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ca_0 \\ ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}.$$

Multiplication of a configuration by x, that is, by the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is defined as a downward shift operator:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Multiplication of other column configurations can be computed using known rules, such as distributivity, associativity, commutativity and bi-linearity. For example

$$\begin{aligned} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \times \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} &= (a_0 + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \times (b_0 + b_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + a_1 b_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a_0 b_0 \\ a_0 b_1 + a_1 b_0 \\ a_1 b_1 \end{bmatrix}. \end{aligned}$$

Let us illustrate four steps of the rule of element placement using the example given in Chapter 2. Suppose a rectangle be given. The sum of the short side and the ling side is equal to 27 and the area is equal to 180. The problem is to find the short side.

The first step is to place one rod in the Square row and consider the configuration $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as the virtual short side. (In modern terminology, let x be the short side.)

Note that in ancient China, the Reality row was called the Great Ultimate (*taiji*) and the Square row the Celestial Element, one of four Elements (*siyuan*); Heaven, Earth, Human, and Substance.

Hence the first operation is stated saying "to place one rod in the celestial element" or "to place the celestial element unit".

The second step is to apply several operations to this configuration: The configuration $\begin{bmatrix} 27 \\ -1 \end{bmatrix}$ represents the virtual long side and the configuration $\begin{bmatrix} 0 \\ 27 \\ -1 \end{bmatrix}$ represents the virtual area. (In modern terminology, $27 - x$ represents the long side and $x(27 - x) = 27x - x^2$ represents the area.)

The third step is called cancellation: The virtual area is canceled by the given value which is placed in the Reality row $\begin{bmatrix} 180 \end{bmatrix}$ to form the equation $\begin{array}{|r|} \hline \\ \hline -180 \\ \hline 27 \\ \hline -1 \\ \hline \end{array}$. (In modern terminology, we form the equation by setting $(27x - x^2) - 180 = 0$; that is, $-180 + 27x - x^2 = 0$.)

The fourth step is to find a solution of this quadratic equation by the procedure of extraction, which was explained above.

VIII.5 Evaluation of a polynomial

Today when we want to calculate the value of a polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

at $x = q$, we substitute x with q to obtain $f(q)$. But in traditional Japanese mathematics, the equation was represented on the counting board:

	Quotient
a_0	Reality
a_1	Square
a_2	Side
a_3	Corner

(34)

Putting q in the Quotient row they applied the procedure of extraction with q and obtained the value $f(q)$ in the Reality row. For example, if they wanted to calculate the values of $f(x)$ at $x = 0.1, 0.2, \cdots, 0.9, 1$, first placing 0.1 in the Quotient row they applied the procedure of extraction with 0.1 to the board (34) and obtained

$f(0.1)$ in the Reality row. The counting board becomes

q	Quotient
a'_0	Reality
a'_1	Square
a'_2	Side
a'_3	Corner

(35)

Using this board (35) they added $q' = 0.1$ to the Quotient row and applied the procedure of extraction with $q' = 0.1$. Then they obtained $f(0.2)$ in the Reality row on the counting board

$q+q'$	Quotient
a''_0	Reality
a''_1	Square
a''_2	Side
a''_3	Corner

(36)

Repeating this calculation, they found the values $f(0.1)$, $f(0.2)$, $f(0.3)$, $\cdots$ successively. Our guess is that Takebe Katahiro found the Square row becomes empty when the Reality row becomes the smallest (or the largest). Then he formulated the calculation of the Square row in the procedure of extraction and found the equation $V'(y) = 0$ without knowing differentiation.

Chapter 6 can be considered the first instance in Japanese mathematics which essentially utilized the fact that the derivative $a_0 + a_1x + a_2x^2 + a_3x^3$ is equal to

$$a_1 + 2a_2x + 3a_3x^2. \tag{37}$$

But in the world of Japanese mathematics there was no Cartesian plane, consequently no notion of the graph of a function. They could not visualize the analytic expression of a gradient. This means, the mathematicians of the *Edo* period could not understand (37) as the derivative.

Takebe Katahiro claims in this chapter that he found (37) by numerical experiment. In all likelihood he calculated the values of $f(x)$ at various points by the above mentioned method and found that the Square row vanished when the Reality row attained the maximum (or the minimum).

He already developed the "bōshohō" (method of side-writing) to describe the procedure to calculate the Square row once given a new increment in the Quotient row.

With present day knowledge, it is trivial that the Side row vanishes when the Reality row become extreme. In fact, (26) is nothing but the Taylor expansion

$$f(x) = f(q) + f'(q)(x-q) + \frac{f''(q)}{2!}(x-q)^2 + \frac{f'''(q)}{3!}(x-q)^3. \tag{38}$$

References

[ChmlaEa2004] Chemla, Karine and Guo Shuchun: *Les Neuf Chapitres, Le Classique mathématique de la Chine ancienne et se commentaires*, Dunod, Paris, 2004.

[Endo1896] Endō Toshisada (遠藤利貞):『増修日本数学史』(*History of Japanese Mathematics, Augmented Edition*)、恒星社厚生閣 (Kōseisha Kōseikaku)、2003. The original edition was in 1896.

[Fujii2002] Fujii Yasuo (藤井康生): 「『研幾算法』術文の注」(Commentary on Procedures in the "Kenki Sanpō"), 数学史研究 (Journal of History of Mathematics, Japan), 175, 176, 2002-2003, pp. 19-72.

[Fujiwara1941] Fujiwara Matsusaburō (藤原 松三郎): 「建部賢弘の弧率と我国最初の三角函数表」("Takebe Katahiro's *Koritsu* and the table of trigonometric functions, first in Japan", in Japanese), 『東京物理学校雑誌』(Tōkyō Butsuri Gakkō Zasshi), 600(1941), pp.402–408.

[Fujiwara1945] Fujiwara Matsusaburō (藤原 松三郎): 「建部賢弘の著と考へられる算暦雑考 (和算史の研究其九)」(On the *Sanreki Zakkō* presumably written by Takebe Katahiro, Studies of history of Japanese mathematics, 9", in Japanese), 『科学史研究』 (Kagakushi Kenkū), 1 (1945), pp.84–92.

[Fujiwara1954] Nihon Gakushiin, Nihon Kagakushi Kankōkai (ed.) (日本学士院日本科学史刊行会編): 『明治前日本数学史』新訂版 (*History of Japanese Mathematics before the Meiji Restoration*, New Edition), in 5 volumes, 野間科学医学研究資料館 (Noma Kagaku Igaku Kenkyū Shiryō kan), 1979. (First edition, 岩波書店 (Iwanani Shoten), 1954 – 1960.)

[Fujiwara2007] Fujiwara Matsusaburō (藤原松三郎): 『東洋数学史への招待—藤原松三郎数学史論文集』(*Invitation to the History of Mathematics in East Asia — Collected Papers on Fujiwara Matsusaburō*)、東北大学出版会 (Tohoku University Press)、2007.

[Hayashi1911] Hayashi Tsuruichi (林 鶴一): 「関孝和ノ綴術及ビ其ノ建部賢弘ノ綴術トノ関係」("On the technique of linkage of Seki Takakazu and its relation with the technique of linkage of Takebe Katahiro"), 『東京数物記事』(Tōkyō Sūbutsu Kiji), II 6(1911).

[Hayashi1915] Hayashi Tsuruichi (林 鶴一): 「零約術ト我国ニ於ケル連分数論ノ発達」("Method of residual division and development of theory of continued fraction in Japan"), 『東北数学雑誌』(Tōhoku Sūgaku Zasshi), 6, 7(1915).

[Hayashi1937] Hayashi Hakase Icho Kankōkai, (ed.) (林博士遺著刊行会): 『林鶴一博士和算研究集録』上下 (*Collection of Research on Japanese Mathematics by Dr. Hayashi Tsuruichi* in 2 volumes), 東京開成館 (Tōkyō Kaiseikan), 1937.

[HirayamaEa1974] Hirayama Akira (平山 諦), Shimodaira Kazuo (下平 和夫), and Hirose Hideo (広瀬 秀雄) (ed.): 『関孝和全集』(*The Collected Works of Seki Takakazu*, in Japanese with English summary), 大阪教育図書 (Ōsaka Kōiku

Tosho), 1974.

[Horiuchi1994] Horiuchi, A.: *Les Mathématiques Japonaises à l'Époque d'Edo*, Vrin, 1994.

[Horiuchi1994b] Horiuchi, A.: "The *Tetsujutsu Sankei* (1722), an 18th century treatise on the methods of investigation in mathematics", in *The Intersection of History and Mathematics*, Science Networks-Historical Studies 15, edited by C. Sasaki et al., Birkhäuser, Basel, 1994, 149–163.

[Horiuchi2010] Horiuchi, Annick: *Japanese Mathematics in the Edo Period (1600-1868)*, Birkhäuser, 2010. [the English translation of [Horiuchi1994]]

[Kojima1963] Kojima, Takashi: *Advanced Abacus*, Tokyo: Charles E. Tuttle Co., Inc, 1963.

[Komatsu2000] Komatsu Hikosaburo (小松 彦三郎): 「綴術算経の異本と成立の順序」("Various editions of the *Tetsujutsu Sankei* and the order of their appearance"), 数理解析研究所講究録 (RIMS Kôkyûroku) 1130 (2000), pp. 229–244.

[Komatsu2004] Komatsu Hikosaburo (小松 彦三郎): 「綴術算経の異本と成立の順序補遺」("Addendum : Various editions of the *Tetsujutsu Sankei* and the order of their appearance"), 数理解析研究所講究録 (RIMS Kôkyûroku) 1392 (2004), pp. 69–70.

[Komatsu2007] Komatsu Hikosaburo (小松 彦三郎): 「大成算経校訂本作成の現状報告」("Report on the Present State of Editing the Restoration of the *Taisei Sankei*"), 数理解析研究所講究録 (RIMS Kôkyûroku) 1546 (2007), pp. 140–156.

[Li1984] Li Di (李 迪) (ed.) : 『中国の数学通史』(*Concise History of Mathematics in China*, translated into Japanese from Chinese by Ōtake, Shigeo (大竹 茂雄) and Lu Renrui (陸 人瑞)), 森北出版 (Morikita Shuppan) 2002. (Original Chinese edition, 『中国数学史簡編』, Shenyang (瀋陽), 遼寧人民出版 (Lianning Renmin Chuban), 1984)

[Martzloff1987] Martzloff, H.-C.: *A History of Chinese Mathematics*, Springer 1997. [the French original edition: *Histoire des mathématiques chinoises*, Masson, Paris 1987.]

[Mikami1913] Mikami Y.: *The Development of Mathematics in China and Japan*, Leipnig: Teubner, 1913; New York: Chelsea, 1974.

[Mikami1921] Mikami Yoshio (三上 義夫): 『文化史上より見たる日本の数学』(*Japanese Mathematics Considered from the Viewpoint of the History of Culture*), edited by Hirayama Akira (平山 諦), 恒星社厚生閣 (Kōseisha Kōseikaku, 1984); edited by Sasaki Chikara (佐々木 力), 岩波文庫 (Iwanami Bunko), 1999. This monograph was first published as a journal article in 1921.

[Morimoto1990] Morimoto Mitsuo (森本 光生) and Saitō Michiyo (斎藤 美千代): 「正多角形の周の長さによる円周率の近似計算」("Approximate calculation of the circular coefficient by circumference length of regular polygons"), 記号数式処理と先端的科学技術計算予稿集原稿 (manuscript of the abstract for the research meeting "Kigō Sīshiki Shori to Sentanteki") (1990), pp.1–10.

[Morimoto1990b] Morimoto Mitsuo (森本 光生): 『UBASIC による解析入門』(*An Introduction to Mathematical Analysis by means of UBASIC*), 日本評論社 (Nihon Hyōronsha), 1992.

[Morimoto2003] Morimoto, M.: Japanese mathematics in the 18th century, in *Proceedings of International Conference in Mathematics in honor of Father Bienvenido F. Nebres, S.J.*, 2003, see ICU web page, `http://science.icu.ac.jp/srr/`.

[Morimoto2004] 「『算学啓蒙諺解大成』について」、(“On the Great Colloquial Commentary of the *Suanxue Qimeng*”)、数理解析研究所講究録 (RIMS Kôkyûroku) 1392 (2004), pp.27–45.

[Morimoto2006] Morimoto Mitsuo (森本 光生): “Differentiation and Integration in Takebe Katahiro’s Mathematics”, 数理解析研究所講究録 (RIMS Kôkyûroku) 1513 (2006), pp.131–143.

[Morimoto2007] Morimoto Mitsuo (森本 光生): 「古法、四乗求背の術、六乗求背の元術について」(“On the old method, the 4-multiplication procedure for finding the back arc length, and the 6-multiplication original procedure for finding the back arc length”), 数理解析研究所講究録 (RIMS Kôkyûroku) 1546 (2007), pp.175 – 180.

[Morimoto2011] Morimoto Mitsuo (森本 光生): 「建部賢弘の数学哲学」(The Mathematical philosophy of Takebe Katahiro) 数理解析研究所講究録 (RIMS Kôkyûroku) 1739 (2011), pp.65 – 76.

[MorimotoEa2004] Morimoto Mitsuo (森本光生) and Ogawa Tsukane (小川束): “The mathematics of Takebe Katahiro: his three formulas of an inverse trigonometric function”, Sugaku Exposition, 20(2), 2007, pp.237-252. This article originally appeared in Japanese in Sūgaku 56(3), 2004, pp.308-319.

[Murata1981] Murata Tamotsu (村田 全): 『日本の数学、西洋の数学』(*Mathematics of Japan viz. Mathematics of the West*), 中央公論社 (Chūō Kōronsha)、1981.

[Murata1982] Murata Tamotsu (村田 全): 「建部賢弘の数学とその思想」(“Mathematics of Takebe Katahiro and his thought”), 『数学セミナー』 (Sūgaku Seminā), August 1982, pp.70–75, September, pp.69–75, October, pp. 62–67, November, pp.63–69, December, pp.60–64, January 1983, pp.76–81.

[Nonaka2010] 「建部賢弘『綴術算経』における数学思想」(Takebe Katahiro’s mathematical thought in the *Tetsujutsu Sankei*”) 数理解析研究所講究録 (RIMS Kôkyûroku) 1677 (2010), pp.83 – 92.

[Ogawa1994] Ogawa Tsukane (小川 束): 『関孝和『発微算法』現代語訳と解説』(*Seki Takakazu’s Hatsubi Sanpō, Its translation to modern Japanese with annotation*), 大空社 (Ōsorasha), 1994.

[Ogawa1996] Ogawa, T.: “A Process of Establishment of Pre-Modern Japanese Mathematics”, Historia Scientiarum, ser.2.5(1996)：255-262.

[Ogawa1997] Ogawa Tsukane (小川 束): 「円理の萌芽—建部賢弘の円周率計算—」(“The beginning of the studies on the circle, — calculation of the circular coefficient by Takebe Katahiro —”), 数理解析研究所講究録 (RIMS Kôkyûroku) 1019 (1997), pp.77 – 97.

[Ogawa1998] Ogawa Tsukane (小川 束): 「建部賢弘の『綴術算経』立元第二について」(“On Chapter 2 of Takebe Katahiro’s *Tetsujutsu Sankei*, Element Placement”), 四日市大学環境情報論集 (Yokkaichi Daigaku Kankyō Jōhō Ronshū), 2.1 (1998), pp.59–79.

[Ogawa1998b] Ogawa Tsukane (小川 束): 「建部賢弘の極値計算について」(“On the calculation of extreme value by Takebe Katahiro”), 数理解析研究所講究録 (RIMS Kôkyûroku) 1064 (1998), pp.129–147.

[Ogawa2000] Ogawa Tsukane(小川 束): 「近世日本数学史現われた無限級数の特質について」(“On the characteristic of the infinite series in the history of mathematics in modern Japan”), 数理解析研究所講究録 (RIMS Kôkyûroku) 1130 (2000), pp.212–219.

[Ogawa2001] Ogawa, T.: “A review of the history of Japanese mathematics”, *Revue d’histoire des mathématiques*, **7**(2001), 101–119.

[Ogawa2002] Ogawa Tsukane (小川 束): 「『綴術算経』の「探算脱術第七」について」(“On procedure of arithmetic removal in Chapter 7 the *Tetsujutsu Sankei*), 数理解析研究所講究録 (RIMS Kôkyûroku) 1257 (2002), pp.205–209.

[Ogawa2004] Ogawa Tsukane (小川 束): 「狩野本『綴術算経』について」(“On the *Tetsujutsu Sankei* in the Kanō Collection), 数理解析研究所講究録 (RIMS Kôkyûroku) 1392 (2004), pp.60–68.

[Ogawa2005] Ogawa Tsukane (小川 束): 「建部賢弘の『算学啓蒙諺解大成』における「立元の法」に関する註解について」(“On Takebe Katahiro’s comments on the rule of element placement in the *Great colloquial Commentary on the Suanxue Qimeng*”), 数理解析研究所講究録 (RIMS Kôkyûroku) 1444 (2005), pp.73–72.

[Ogawa2006] Ogawa Tsukane (小川 束): 「近世日本数学における表現形式 — 『大成算経』の隠題をめぐって」(“On the expression form of Mathematics in Modern Japan — About the concealed problem in the *Taisei Sankei*”), 数理解析研究所講究録 (RIMS Kôkyûroku) 1513 (2006), pp.112–120.

[Ogawa2007] Ogawa Tsukane (小川 束): 「『綴術算経』の「自質説」について — 現代語の試み」(“On the “Theory of Proper Character” in the *Tetsujutsu Sankei* — Trial Translation into Modern Japanese”), 数理解析研究所講究録 (RIMS Kôkyûroku) 1536 (2007), pp.163–174.

[OgawaEa2003] Ogawa Tsukane (小川 束) and Hirano Yōichi (平野 葉一): 『数学の歴史』(*History of Mathematics*), 朝倉書店 (Asakura Shoten), 2003.

[OgawaEa2008] Ogawa Tsukane (小川 束) , Sato Ken’ichi, Sr. (佐藤 健一), Takenouchi Osamu (竹之内 脩), and Morimoto Mitsuo (森本光生): 『建部賢弘の数学』(*The Mathematics of Takebe Katahiro*), 共立出版 (Kyōritsu Shuppan), 2008.

[Ozaki2004] 「『大成算経』巻之四三要（象形、満干、数）の謎」(On Volume 4 of the *Taisei Sankei*, Three Essentials (Symbols and Figures, Flow and Ebb, and

Numbers)）数理解析研究所講究録 (RIMS Kôkyûroku) 1392(2004), 186–196.

[Qian1990] Qian Baozong (銭 宝琮):『中国数学史』(*History of Chinese Mathematics*, (translation into Japanese from Chinese by Kawahara, Hideki (川原 秀城)), みすず書房 (Misuzu Shōbō), 1990.

[Sato1996] Satō Ken'ichi (佐藤 賢一): 「関孝和著『発微算法』の研究 異版の存在について」("Studies on Seki Takakazu's *Hatsubi Sanpō* — Existence of different versions—"), 科学史研究 (Kagakushi Kenkyū), II 35(1996), pp.179–187.

[Sato1996b] Satō Ken'ichi (佐藤 賢一): 「建部賢弘著『研幾算法』の研究」("Studies on Takebe Katahiro's *Kenki Sanpō*"), 科学史・科学哲学 (Kagakushi Kagakutetsugaku), 13(1996), pp.26–40.

[Sato2005] Satō Ken'ichi (佐藤 賢一): 『近世日本数学史 関孝和の実像を求めて』(*History of Mathematics in Modern Japan, Seeking a Real Image of Seki Takakazu*), 東京大学出版会 (University of Tokyo Press), 2005.

[SatoS1995] Satō, Ken'ichi, Sr. (佐藤 健一): 『建部賢弘の『算暦雑考』—— 日本初の三角関数表——』(*Takebe Katahiro's Sanreki Zakkō — Table of trigonometric functions (first in Japan) —*), 研成社 (Kenseisha), 1995.

[ShenEa1999] Shen Kangshen, John N. Crossley and Anthony W.-C. Lun: *The Nine Chapters on the Mathematical Art*, Oxford University Press, Science Press, Beijing, 1999.

[Shibata1935] Shibata Hiroshi (柴田 寛): 「零約術に就て」("On the method of residual division"), 東京物理学校雑誌 (Tōkyō Butsuri Gakkō Zasshi), 519(1935) pp.47–55.

[Shibata1935b] Shibata Hiroshi (柴田 寛): 「関の近似法に就て」("On Seki's method of approximation"), 東京物理学校雑誌 (Tōkyō Butsuri Gakkō Zasshi), 524(1935) pp.281–287.

[Shimodaira2006] Shimodaira Kazuo (下平 和夫): 『関孝和』(*Seki Takakazu*), 研成社 (Kenseisha), 2006.

[SmithEa1914] Smith, D. E. and Mikami, Y.: *A History of Japanese Mathematics*, Chicago: Open Court, 1914; Mineola, Chelsea, 1974.

[Sugiura1982] Sugiura Mitsuo (杉浦 光夫): 「円理——和算の解析学について——」("Studies on the circle — analysis in Japanese mathematics —"), 比較文化研究 (Hikaku Bunka Kenkyū), (1982), pp.1–20.

[Suzuki2005] Suzuki Takeo 鈴木武雄: 「建部賢弘著『綴術算経』と『不休綴術』の成立」(The origin of the *Tetsujutsu Sankei* and the *Fulyū Tetsujutsu* of Takebe Katahiro), 数学教育研究 (Sūgaku Kyōiku Kenkyū), 35(2005), pp. 237 - 255.

[Takenouchi2004] Takenouchi Osamu (竹之内 脩): 『研幾算法と研幾算法演段諺解』("The *Kenki Sanpō* and the *Colloqial Commentary on Series of Operations in the Kenki Sanpō*"), 近畿和算ゼミナール報告集 (Kinki Wasan Zemināru Hōkokushū) 9(2004).

[Takenouchi2006] Takenouchi Osamu (竹之内 脩): 「研幾算法第一問」("Problem 1 of the *Kenki Sanpō*")、数理解析研究所講究録 (RIMS Kôkyûroku) 1513(2006), pp.121–204.

[TakenouchiEa2004] Takenouchi Osamu (竹之内 脩) and Morimoto Mitsuo (森本 光生): "Selected Mathematical Works of Takebe Katahiro (1664 – 1739)", preliminary edition, Wasan Institute, July 2004

[UenoEa2008] Ueno Kenji (上野健爾), Ogawa Tsukane (小川束), Kobayashi Tatsuhiko(小林龍彦), and Sato Ken'ichi (佐藤賢一): 『関孝和論序説』(*Introduction to the Study on Seki Takakazu*), 岩波書店 (Iwanami Shoten), 2008.

[Wada1983] Wada Hideo (和田 秀男): 『高速乗算法と素数判定法 (マイコンによる円周率の計算)』(*Method of Rapid Multiplication and Primality Test (Calculation of the Circular Constant by Personal Computer)*), 『上智大学数学講究録』(Sophia Kōkyūroku in Mathematics) 15 (1983).

[WasanInst2000] Wasan Institute: *Jinkōki*, Tokyo Shoseki, 2000. (An English translation of the *Jinkōki* translated by Takenouchi Osamu et al.)

[Xu2002] 「建部賢弘的数学認識論——論『大成算経』中的 "三要"」(Takebe Katahiro's epistemology of Mathematics) 21(2002), pp. 232-243.

[Yokotsuka2004] Yokotsuka Hiroyuki (横塚 啓之): 「建部賢弘の著と考えられる『弧背截約集』について 」("On the Kohai-setsuyaku-shuū regarded as the work of Katahiro Takebe"), 数学史研究 (Journal of History of Mathematics, Japan) 182(2004), pp.1-39.

[Yokotsuka2006] Yokotsuka Hiroyuki (横塚 啓之): 「建部賢弘の著と考えられる『弧背截約集』と『弧背率』·『弧背術』の関係— 建部賢弘の元禄時代と享保時代の円理の研究」("The *Kohai Setsuyaku Shū* considered as a work of Takebe Katahiro and its relation with the *Kohairitsu* and the *Kohaijutsu* — Takebe Katahiro's Research on the Circle Principle during the Periods Genroku and Kyōhō"), 数理解析研究所講究録 (RIMS Kôkyûroku) 1513 (2006), pp.144-151

Index of Names

Index of Books

Index of Subjects

58v

[7]先小斜一中斜二大斜三ヲ以テ基數トス[8]是中斜ト小斜ト共ニ大斜ノ溴ニ合者[9]中股空ナリ[10]第一トス[11]一ヲ以テ三斜
ニ遍ク加テ小斜二中斜三大斜四ヲ得[12]是ヲ以テ
中股ヲ求ルニ其數不整[13]亦一ヲ以テ累テ三斜ニ
遍ク加テ小斜三中斜四大斜五ヲ得ル[14]是即句股ノ數ナリ
[15]是ヲ以テ中股ヲ求ルニ數整フ[16]第二トス[17]亦一ヲ
以テ累テ三斜ニ遍加テ小斜四中斜五大斜六ヲ
得ル[18]是ヲ以テ中股ヲ求ルニ數不整[19]亦一ヲ以テ
累テ三斜ニ遍加テ小斜五中斜六大斜七ヲ得ル
[20]是ヲ以テ中股ヲ求ルニ又不整[21]亦一ヲ以テ累テ

59r

三斜ニ遍ク加ヘテ小斜六中斜七大斜八ヲ得ル
[22]是ヲ以テ中股ヲ求ルニ數亦不整[23]如此一ヲ以テ
逐遍累加シテ三斜ノ數ヲ得テ中股ヲ求探ルニ
小斜一十三中斜一十四大斜一十五ニ到テ中股ノ數整
[24]第三トス[25]次ニ小斜五十一中斜五十二大斜五十三ニテ
中股ノ數整フ[26]第四トス[27]第五以上ノ數畧之
[28]於此其件件中股ノ整フ者ヲ以テ三斜ノ數ヲ探
ルニ其件ノ中斜ヲ四因シテ前件ノ中斜ヲ減去
シテ餘ヲ次件ノ中斜トス[29]即一算ヲ以テ中斜ヲ
減シテ小斜ヲ得又一算ヲ中斜ニ加ヘテ大斜ヲ

59v

得ル[30]如直ニ小斜ヲ求ルハ其件ノ小斜ヲ四因シテ二算ヲ加ヘ前件ノ小斜ヲ減去シテ次件ノ小斜トス[31]亦直ニ大斜ヲ求ハ其件ノ大斜ヲ四因シテ二算ト及前件ノ大斜トヲ減去シテ次件ノ大斜トス
○[32]又直ニ中股ヲ求ル者ハ其所求ノ前件ノ
大斜ニ一算ヲ加ヘ折半シテ得數ヲ其件ノ中股
不成ノ分子トス[33]其件ノ大斜ノ如ク而一ニシテ
中股不盡ノ數ヲ得ル[34]又不成ノ分子ヲ以テ其件
ノ中斜ヲ減シテ餘ヲ中股ノ成數トス[35]其不盡ヲ
帶テ即中股ノ全數也
[36]右ノ術ハ中根上右衛門カ會スル所ニシテ數
ヲ探リテ數ノ據ヲ得ル者也[37]元來理ヲ備ルカ

60r

ユヘニ所求ヲ得ルト雖其理ヲ察スル。トキ
潜伏シテ探リ得難シ[38]如此ハ其理ヲ索ルヿヲ
不為唯數ノ成儘ニ用ルヲ算ノ道ニ循フトス
[39]然ヲ人或叵測シテ理ノ外ナリト意ハ是瞽也
[40]或疑惑シテ強テ其理ヲ察セント欲ハ是癡也
[41]附録畢
[42]乙巳夏至十三日

56v

ヿ無ク咸算ノ中エ入トキハ自心ト道ト混一ニ
シテ可議ハ可議シテ我ニ従ヒ不可議ハ不可議
シテ又我ニ従フ是[27]乃道ニ體スルノ一端也矣夫[28]
算ノ道ヲ心ニ知テ言ニ説者ハ不實ナリ[29]道ニ體
シテ事ニ行フ者ハ真實也[30]此道ニ體スル真實ハ
敢テ不可思議者也[31]而ルニ其思議スヘカラサル
真實ニ於テ自是ヲ修スルニ吾生得ノ質ニ随フ
一个ノ則有ヿヲ肯得タリ[32]然レトモ吾道猶未熟
[33]故ニ不説之也[34]其可言ヲ肯シテ後ニ言ヿ有ン歟
[35]是卽吾偏質也[36]蓋純粋ノ質ニシテハ纔テ一字ト

57r

シテ可説ヿ無シ何[37]ヲカ説ヿ有ン其[38]説ヿ有ルハ
卽是生得ノ偏質ヲ説者也[39]凡生得粋偏厚薄ノ質
人人齊者有ヿ無シ以[40]是吾算ノ質ニ従フ所以ヲ
説ヿ正ニ如此ト雖人モ亦質ニ従フ所以ハ是ノ
如シト云ニ非ス故[41]ニ如算ヲ學フ者此説ヲ聴テ
徒ニ是トスルヿ無レ[42]又空シク非トスルヿ無レ
[43]唯人人自己ノ生得ル質ヲ實ニ識得テ質ニ従テ
算ノ數ノ真實質ニ従フ所以ヲ説ヘキ也

57v

綴[44]術算經終

58r

附録

三[1]斜差各一整中股數

[2]假如有三斜大斜[3]中斜差與中斜小斜差各一欲[4]使
中股數整問[5]件件三斜及中股各幾何

[6]

小斜	中斜	大斜	中股
一	二	三	空
三	四	五	二二五之二
一十三	一十四	一十五	一十一五之一
五十一	五十二	五十三	四十四五十三之八
一百九十三	一百九十四	一百九十五	一百六十七六十五之九

54v

其三差矣術ヲ以テ半圓ニ合スルトキハ矢ノ
多キ者ニ於テ十許位ヲ盡スヘキ丁ヲ案シテ
即六件ノ限ヲ立テ率數ヲ求テ總術トス委ク[71]
弧率ニ載之
[72]右弧數逐差求除ノ段數ハ數ニ據テ數ヲ探ル者
也[73]背ヲ求ル術ハ數ニ據テ法ヲ探ル者也[74]蓋圓周
弧背等ハ數ニ於ル術ニ於ル皆理ニ據テ探時ハ
必得ヘカラス[75]純數ニ據テ探テ即可得[76]是弧圓ノ
質ナリ
[77]術例尾

55r

自質説　一條
[1]筭ノ數ノ心ニ從フトキハ泰シ[2]不從トキハ苦ム
[3]所謂心ニ從フハ即質ニ從フナリ[4]其從フ所以ハ
其事未會以前ニ必可得ヲ肯スル心有エヘ疑フ
丁無シテ泰ニ居ル[5]泰ニ居ルエヘ常ニ爲テ不止
[6]常ニ爲テ不止エヘ不成得ト云丁ナシ[7]不從者ハ
其事未會以前ニ可得ヲモ不可得ヲモ料ル丁無
シテ疑フ[8]疑フエヘニ苦ミ屈ス[9]苦ミ屈スルエヘ
成得丁難シ[10]吾筭ヲ學テヨリ常ニ安行ナランヿ
ヲ意テ筭法ニ苦ムヿ久シ[11]蓋是未自己ノ質分ヲ

55v

不盡エヘセリ徐六旬ニ及ハントスル比[12]自生レ
得ル本質ノ偏駁ナル丁ヲ實ニ識得テ筭ノ數ノ
質ニ從フ丁ヲ肯セリ[13]嗚呼自己粹偏ノ本質ハ人々
生レ得ル儘ニシテ學ヒ盡スト雖更ニ増長スル
丁無ク又廢忘スト雖些モ損消スル丁靡シ[14]乃其
偏質ヲハ思議スヘシ[15]粹質ヲハ思議スヘカラス
[16]人々自此質分ヲ不盡ハ敢テ筭ノ質ニ從フ眞實
ヲ會スヘカラス[17]然ルニ[18]人皆質分ノ粹偏生得ノ
自然タル丁ヲ不曉學盡シテ後ハ咸通明ニシテ
カヲ用丁無ト爲リ[19]惑ヘル哉[20]如此ハ純粹ノ質ハ

56r

學テ得ル者ト思ヘル也[21]如何ソ學テ純粹ノ質ニ
變成スル丁有ンヤ[22]蓋其質分ヲ盡シ道ニ體スル
トモ生得ノ質ハ便生得ノ質ニシテ動ク丁無ク
變スル丁無ク亦可惑丁モ無ク還可明丁モ無ク
而モ毎ニ事ニ臨テハ難易ニ從テカヲ不用ト云
丁無耳[23]亦嘗聞或某藝ヲ否ト[24]是ハ此本質ノ純粹
ナル者ヲ謂フ歟[25]熟思フニ藝ヲ以テ己ニ從ヘテ
自心ノ中ニ容ルトキハ可議ト不可議トノ分有
エヘ其可議限ハ我ニ從フト雖不可議ニ到テハ
我ニ不從ヿ有リ[26]吾ハ謂フ自己ヲ以テ些モ忤フ

52v

術ヲ以テ六百七十四萬三千〇〇八乘二千六
百一十七萬六千二百九十三除ノ極限ヲ探リ
索ム[48]是矢自乘ノ段數ヲ探リ索ルニ矢猶多クシテ眞數伏テ得難シ故[49]ニ矢一塵ノ半背
冪九十餘位ヲ求テ悉ク探リ得タリ[50]其數繁多ナルユヘ畧之
二[51]汎差ヲ置テ矢冪ヲ乘シテ實トス[52]矢自乘六百
七十四萬三千〇〇八乘二千六百一十七萬六千
二百九十三除シテ徑自乘ノ數ニ加ヘ得ル內矢徑
相乘一千六百九十六乘一千四百一十九除スル
數ヲ減シテ餘ヲ法トシテ實ヲ除シ亦四十三乘
九百八十除シテ四乘ノ三汎差トス

53r

如[53]四差ヲ求ムル者ハ三汎差ヲ以テ三定差ヲ
却減シテ餘負ヲ四定差トス[54]卽其首位ヲ視テ
三汎差ノ首位ヨリ下レル位數ヲ料テ三汎差
ニ矢再乘冪ヲ乘シテ得ル數ヲ豫徑再乘冪ニ
可除ヿヲ探リ先乘除ノ段數ヲ索メ術ニ依テ
矢徑冪相乘次ニ矢冪徑相乘次ニ矢再自乘等
ノ各段數ノ極限ヲ探リ索テ七乘ノ減四差ヲ
得ル[55]七乘ノ差ヲ求テハ次ニ十一乘ノ差次ニ
十六乘ノ差次ニ二十二乘以上ノ差ヲ求ル也
[56]最繁多ナルユヘ畧之

53v

[57]

定半背冪	汎半背冪	一定差	一汎差	二定差	二汎差	三定差	三汎差
九乙	乙	九〇	參〇	五〇	五〇	〇〇	〇〇
〇絲	絲	〇絲	參絲	七絲	六絲	〇絲	〇絲
六〇		六〇	參〇	參〇	五〇	七〇	七〇
六〇		六〇	參〇	參〇	五〇	八〇	八〇
六〇		六〇	參〇	參〇	參〇	〇〇	〇〇
六〇		六〇	參〇	參〇	二〇	〇〇	〇〇
七〇		七〇	參〇	參〇	九〇	四〇	四〇
二〇		二〇	參〇	九〇	二〇	六〇	六〇
八參		八參	參參	四〇	七〇	七〇	七〇
二參		二參	參參	九〇	〇〇	八〇	八〇
參參		參參	參參	〇●	四●	五●	五●
四參		四參	參參	乙〇	八〇	二〇	二〇
七參		七參	參參	四〇	九〇	四〇	四〇
七參		七參	參參	四〇	四〇	九〇	九〇
六五		六五	參參	參乙	五乙	八〇	八〇
九乙		九乙	參參	六七	乙七	四〇	四〇
四乙		四乙	參參	乙七	六七	四〇	四〇
七乙		七乙	參參	四七	六七	八〇	八〇
九乙		九乙	參參	六七	參七	二〇	二〇
五乙		五乙	參參	二七	八七	四〇	四〇
九二		九二	參參	六八	五八	〇●	〇●
五二		五二	參參	二九	參九	八〇	九〇
八五		八五	參參	五二	八二	六〇	乙〇
七參		七參	參參	四〇	四〇	九〇	七〇
五九		五九	參參	二六	七六	五〇	七〇
强六		强六	强參	强參	强參	弱〇	弱〇

[58]此元術矢徑相乘爲汎半背冪[59]置矢自乘三除之爲
一差[60]置一差矢乘爲實[61]置矢九乘一十四除用減徑

54r

餘爲法[62]實如法而一亦八乘一十五除之爲二差[63]置
二差矢冪乘爲實[64]置矢自乘六百七十四萬三千
〇八乘二千六百一十七萬六千二百九十三除加
徑自乘之數得內減矢徑相乘一千六百九十六乘
一千四百一十九除數餘爲法[65]實如法而一亦四十
三乘九百八十除之爲三差[66]置汎半背冪以一二三
差各加之爲定半背冪[67]開平方除之得半背也[68]七乘差以上本術畧之
[69]此術二差ヲ用レハ盡ルヿ原數ノ五差ニ及ヒ
三差ヲ用レハ盡ルヿ原數ノ八差ニ及フ[70]故ニ

50v

ニ矢ヲ乘ズル者徑ニ除テ二沉差ヲ求テ不精
[26]又矢徑ノ差ニ除シ求メテモ猶未精[27]故ニ矢ニ
段數ヲ乘シ徑ヲ減スル餘ヲ以テ除スル時ハ
精キヲ得ヘキヿヲ探テ卽一沉差ニ矢ヲ乘シ
亦八乘一十五除シテ得ル數ヲ又二定差ヲ以テ
除之徑ヲ却減シテ餘負六微四二八五七一八
六七三強ヲ矢ヲ以テ約テ六分四二八五七一
八六七三強ヲ得ル[28]矢應減沉段數トス[29]零約ノ術
ヲ以テ九乘一十四除ノ極限ヲ探索ム
[30]一沉差ヲ置矢ヲ乘シテ實トス[31]矢ヲ置九乘一十

51r

四除シテ以テ徑ヲ減シ餘ヲ法トシテ實ヲ除シ
亦八乘一十五除シテ得ル數ヲ再乘二沉差トス
[32]用テ二定差ヲ減シテ餘正ヲ三定差トス
[33]三定差ヲ視ルニ二沉差ノ首位ヨリ十四位下
レルニ依テ二沉差ニ矢冪ヲ乘シ得ル數ヲ豫
徑冪ニ除テ三差ヲ求ヘキヿヲ探テ卽二沉差
ニ矢冪ヲ乘シ徑冪ニ除シ是ヲ以テ三定差ヲ
約テ四釐三八七七六〇三四六三二強ヲ得ル
[34]零約ノ術ヲ以テ四十三乘九百八十除ノ極限
ヲ探リ索ム[35]於此其二沉差ニ矢冪ヲ乘スル者

51v

徑冪ニ除テ三差ヲ求テ不精[36]亦矢徑ノ差冪ニ
除シテ求テモ猶未精[37]故ニ矢徑相乘ト矢自乘ト
各段數ヲ乘シ正負ニ依テ徑冪ニ加減スル數
ヲ以テ除スルトキハ精キヲ得ヘキヿヲ探ル
[38]卽二沉差ニ矢冪ヲ乘シ亦四十三乘九百八十
除シテ得ル數ヲ又三定差ヲ以テ除之徑冪ヲ
却減シテ餘負一絲一九五二〇七六三五二八
二四九九二四九六強ヲ段積實トス[39]是徑冪ノ
首位ヨリ六位下レルニ依テ矢徑相乘ノ數ヲ
用ヘキヿヲ會ス[40]卽矢徑相乘ノ數ヲ以テ約テ

52r

一段一九五二〇七六三五二八二四九九二四
九六強ヲ得ル[41]矢徑相乘應減沉段數トス[42]零約ノ
術ヲ以テ一千六百九十六乘一千四百一十九
除ノ極限ヲ探索ム[43]亦矢徑相乘シ一千六百九
十六乘一千四百一十九除シテ得ル數段積實ヲ
却減シテ餘正二渺五七五九九八一二三七
三強ヲ得ル[44]是徑冪ノ首位ヨリ十三位下ルニ
依テ矢自乘ノ數ヲ用ヘキヿヲ會ス[45]卽矢自乘
スル數ヲ以テ約テ二分五七五九九八一二二
三七三強ヲ得ル[46]矢自乘應加沉段數トス[47]零約ノ

48v

又隨テ多キヲ得テ其術容易ナラス[94]定率ト爲
難シ[95]故ニ法ヲ蹙約テ捷徑ノ術ヲ探ニ其矢ヲ
以テ一差ニ累乘スル者矢徑ノ差ヲ以テ累除
シテ逐差ヲ求ル如ハ其損消スルヿ稍多シト
雖猶未疾[96]更ニ亦玄ク探リテ矢ニ段數ヲ乘シ
徑ヲ減スル餘ヲ以テ累除シテ逐差ヲ求ルニ
到テ其損消スルノ數急ニ疾ヿヲ得タリ[97]故ニ
以テ定率ノ元術トス[98]其矢徑ノ差ヲ以テ累除
スル者ハ曽用ルヿ無ト雖其術ヲ列テ探索ノ
階梯トスル也

49r

[99]矢徑差除求差者
[00]起術矢徑相乘爲汎半背冪[01]矢自乘三除之爲加一
差[02]置一差矢乘矢徑差除亦八乘一十五除之爲加
二差[03]置二差矢乘矢徑差除亦五乘一十四除之爲
減三差[04]置三差矢乘矢徑差除亦一十二乘二十五
除爲加四差[05]置四差矢乘矢徑差除亦二百二十三
乘三百九十八除之爲減五差[06]六差以上微之[07]置汎半背冪
以逐差各加減之爲定半背冪也[08]探數累之
[09]此術半圓ニ合スル時ハ矢ノ多キ者ニ於テ其
二差ヲ用レハ三位ヲ盡シ三差ヲ用レハ四位

49v

ヲ盡シ四差ヲ用レハ五位ヲ盡ス[10]一差ヲ增シ
用ル毎ニ一位ヲ盡者ナリ[11]是吾徃歳所立六乘
求背ノ元術ニ符合ス[12]元來六乘ニシテ七位ヲ
盡スヘキヿヲ意テ其法ヲ立ト雖多乘ニシテ
且精密ナラス[13]故ニ亦不用シテ廢ス
[14]古法ニ矢自乘シ矢冪ノ法ヲ乘シ弦冪ニ加テ
汎背冪トス[15]倍矢ヲ以テ徑ヲ減スル餘ニ矢冪
ヲ乘シ徑矢ノ差ニ除シ又折半シテ得ル所ヲ
以テ汎背冪ヲ減シ餘ヲ定背冪トスル者右ノ
二差ヲ用ル術ニ自符合ス

50r

[16]探除法用據矢段數
[17]矢徑相乘汎半背冪トス[18]用テ定半背冪ヲ減シテ
餘ヲ一定差トス○[19]矢自乘三除シテ一汎差トス
[20]用テ一定差ヲ減シ餘正ヲ二定差トス[21]以上探術同前
[22]二定差ヲ視ニ一汎差ノ首位ヨリ六位下レル
ニ依テ一汎差ニ矢ヲ乘シ得數ヲ豫徑ニ除テ
二差ヲ求ヘキヿヲ探ル[23]即一汎差ニ矢ヲ乘シ
徑ニ除シ是ヲ以テ二定差ヲ約テ五分三三三
三三六七六一九一弱ヲ得ル[24]零約ノ術ヲ以テ
八乘一十五除ノ極限ヲ探索ム[25]於此其一汎差

46v

其[65]逐差ヲ求ル乘除之數ヲ視ルニ乘法ハ一差一
ヨリ起テ逐段一筭ヲ増ノ元數[66]（二差ハ二三差ハ三四差ハ四以上也）ヲ
以テ各自乘シテ奇段ハ[67]（一差三差五差以上ノ者ナリ）直偶段ハ
[68]（二差四差六差以上ノ者ナリ）倍スル數ナルヿヲ探會ス◯[69]除法
ハ左數ハ一差三ヨリ起テ逐段二筭ヲ増ノ元數
[70]（二差ハ五三差ハ七四差ハ九以上也）右數ハ奇段ハ一差一ヨリ起テ一
筭ヲ逐増スルノ元數[71]（三差ハ二五差ハ三七差ハ四以上也）偶段ハ二差
三ヨリ起テ二筭ヲ逐増スルノ元數[72]（四差ハ五六差ハ七八差ハ九以
上也）各左右ノ元數ヲ以テ相乘スルノ數ナルヿヲ
探會ス

47r

[73]

一差	二差	三差	四差	五差	六差	七差	八差	九差	十差
奇段	偶段	奇段	偶段	奇段	偶段	奇段	偶段	奇段	偶段
乘法一	乘法八	乘法九	乘法三十二	乘法二十五	乘法七十二	乘法四十九	乘法一百二十八	乘法八十一	乘法二百
元數一自乘	元數二自乘倍	元數三自乘	元數四自乘倍	元數五自乘	元數六自乘倍	元數七自乘	元數八自乘倍	元數九自乘	元數十自乘倍
除法三	除法一十五	除法一十四	除法四十五	除法三十三	除法九十一	除法六十	除法一百五十三	除法九十五	除法二百三十一
左元數三右元數一相乘	左元數五右元數三相乘	左元數七右元數二相乘	左元數九右元數五相乘	左元數十一右元數三相乘	左元數十三右元數七相乘	左元數十五右元數四相乘	左元數十七右元數九相乘	左元數十九右元數五相乘	左元數廿一右元數十一相乘

47v

乘[74]除ノ段數ヲ以テ原術ノ如ク逐差ヲ累テ半背
冪ヲ求ル時ハ碎抹スルヿヲ不用シテ直ニ眞數
ヲ得ル是[75]乃弧背自然ノ質ヲ盡ス者也是[76]ニ依テ
會シ得ヘシ[77]弧圓ハ不盡ヲ以テ質トス故[78]ニ其術
モ亦不盡ヲ以テ求ムヘキヿヲ[79]盡數ニ盡ルアリ
不盡アリ術[80]ニ盡ルアリ不盡アリ質[81]ニ盡ルアリ
不盡アリ所[82]謂四之一五之一ノ如キハ數ノ盡也
三[83]之一七之一ノ如キハ數ノ不盡也加[84]減因乘ノ
如キハ術ノ盡也歸[85]除開方ノ如キハ術ノ不盡也
方[86]圓直積ノ如キハ質ノ盡也[87]圓周弧積ノ如キハ

48r

質ノ不盡也乃[88]弧圓ノ屬ハ質不盡シテ術モ不盡
術不盡シテ數モ復不盡者也而[89]ルヲ人皆其質ヲ
識ヿ無ク疑惑シテ譬ハ句股ノ弦ヲ求メ或錐壔
ノ積ヲ求ルカ如ク質ヲ盡シ術ヲ盡テ求ンヿヲ
意ヘリ豈[90]敢テ得ヿ有ンヤ
[91]原術ハ弧ノ質ニ順フ自然ノ法タリ其[92]矢ノ微
ナル半背冪ヲ求ル者ハ逐差ノ數損消スルヿ
最急ナルユヘ頻ニ眞數ヲ究得ルト雖如矢ノ
多キ者ニ於テハ逐差損消スル數緩遲ニシテ
差ヲ累ルヿ甚衆キニ到ル然[93]ルトキハ乘數モ

44v

三ヲ除シテ得ル數ヲ五泯差トス[44]用テ五定差ヲ減シ
テ餘ヲ六定差トス
[45]六定差ヲ視ルニ五泯差ノ首位ヨリ六位下レル
ニ依テ五泯差ニ矢ヲ乘シ徑ニ除シテ六差ヲ
可求ヿヲ採ル[46]卽チ五泯差ニ矢ヲ乘シ徑ニ除シ
得ル所ヲ以テ六定差ヲ約メテ七分九一二〇九
四三七三六三强ヲ得ル[47]零約ノ術ヲ以テ七十
二乘九十一除ノ極限ナルヿヲ採リ索ム
[48]五泯差ヲ置矢ヲ乘シ徑ニ除シ亦七十二乘九十
一除シテ得ル數ヲ六泯差トス[49] 求ル七差以上者畧之

45r

[50]

定背羃	泯背羃	一定差	一泯差	二定差	二泯差	三定差	三泯差	四定差	四泯差
九乙	乙	九○	参○	五○	七○	七○	七○	○○	○○
○絲	絲	○絲	参絲	七絲	七絲	九絲	乙絲	八絲	八絲
六○		六○	参○	参○	七○	五○	四○	乙○	乙○
六○		六○	参○	参○	七○	五○	二○	二○	二○
六○		六○	参○	参○	七○	五○	八○	六○	六○
六○		六○	参○	参○	七○	五○	五○	九○	九○
七○		七○	参○	参○	七○	六○	七○	九○	八○
二○		二○	参○	九○	七○	乙○	乙○	○○	四○
八参		八参	参参	四○	七○	七○	四○	二○	乙○
二参		二参	参参	九○	七○	乙○	二○	八○	二○
参参		参参	参参	○●	七●	二●	八●	参●	六●
四参		四参	参参	乙○	七○	参○	五○	七○	九○
七参		七参	参参	四○	七○	六○	七○	九○	八○
七参		七参	参参	四○	七○	六○	乙○	五○	四○
六五		六五	参参	参乙	七乙	五○	四○	乙○	乙○
九乙		九乙	参参	六七	七七	八○	二○	五○	二○
四乙		四乙	参参	乙七	七七	参○	八○	五○	六○
七乙		七乙	参参	四七	七七	六○	五○	乙○	九○
九乙		九乙	参参	六七	七七	八○	七○	乙○	八○
五乙		五乙	参参	二七	七七	四○	乙○	参○	四○
九二		九二	参参	六八	七七	八乙	四乙	四●	乙●
五二		五二	参参	二九	七七	四乙	二乙	乙○	二○
八五		八五	参参	五二	七七	七四	八四	九○	六○
七参		七参	参参	四○	七七	六二	五二	○○	九○
五九		五九	参参	二六	八七	四八	七八	七○	八○
强六		强六	强参	强参	弱七	强五	强五	强○	强○

45v

五定差	五泯差	六定差	六泯差
○○	○○	○○	○○
○絲	○絲	○絲	○絲
○○	○○	○○	○○
○○	○○	○○	○○
●○	●○	●○	●○
○○	○○	○○	○○
○○	○○	○○	○○
六○	六○	○○	○○
乙○	乙○	○○	○○
五○	五○	○○	○○
六●	六●	○●	○●
八○	八○	○○	○○
乙○	○○	○○	○○
乙○	六○	四○	四○
○○	乙○	八○	八○
二○	五○	七○	七○
八○	六○	乙○	乙○
乙○	八○	参○	参○
二○	○○	二○	乙○
九○	六○	参○	九○
二●	乙●	乙●	乙●
九○	五○	参○	五○
二○	六○	五○	七○
○○	八○	二○	○○
九○	乙○	八○	参○
弱○	弱○	强○	强○

[51]此原術矢徑相乘爲泯半背羃[52]矢自乘三除之爲一
差[53]置一差矢乘徑除亦八乘一十五除之爲二差[54]置
二差矢乘徑除亦九乘一十四除之爲三差[55]置三差
矢乘徑除亦三十二乘四十五除之爲四差[56]置四差
矢乘徑除亦二十五乘三十三除之爲五差[57]置五差
矢乘徑除亦七十二乘九十一除之爲六差[58] 七差以上準之

46r

[59]以逐差各加泯半背羃爲定半背羃
[60]此術半圓ニ合スルトキハ矢ノ多キ者ニ於テ
二差ヲ用ル者ハ二位ヲ盡シ三差ヲ用ル者ハ
三位ヲ盡シ四差ヲ用ル者ハ四位ヲ盡ス[61]其一
差ヲ増用ル毎ニ一位ヲ盡セリ[62]卽是關氏所立
四乘求背ノ術ニ自符合ス[63]然レトモ邊ニ近キ
背ヲ以テ自然ノ數ヲ採リ索ルヿヲ不會[64]偏ニ
半圓ニ合スルヿヲ要シテ率數ヲ造テ其四乘
ニシテ四位ヲ盡スヿヲ不意エヘ不啻ナリト
シテ其術廢シヌ

42v

[20]古法ニ矢自乘シ乘法五段八六九六强ヲ乘シ
弦羃ニ加テ背羃トスル者自カラ是ニ符合ス
[21]古ヨリ三除ノ數ヲ不會偏[22]ニ半圓ニ合スルヲ
求ルユヘ矢羃ノ乘法ヲ用ユ
[23]矢ヲ自乘シ三除シテ得ル數ヲ一汎差トス[24]用テ
一定差ヲ減シテ餘ヲ二定差トス
[25]二定差ヲ視ニ一汎差ノ首位ヨリ六位下レル
ニ依テ一汎差ニ矢ヲ乘シ徑ニ除シテ二差ヲ
求ヘキヿヲ探ル卽[26]一汎差ニ矢ヲ乘シ徑ニ除
得ル所ヲ以テ二定差ヲ約テ五分三三三三三

43r

六七六一九一弱ヲ得ル[27]零約ノ術ヲ以テ八乘
一十五除ノ極限ナルヿヲ探索ム
[28]一汎差ヲ置矢ヲ乘シ徑ニ除シ亦八乘一十五除
シテ得ル數ヲ二汎差トス[29]用テ二定差ヲ減シテ
餘ヲ三定差トス
[30]三定差ヲ視ニ二汎差ノ首位ヨリ六位下レル
ニ依テ二汎差ニ矢ヲ乘シ徑ニ除シテ三差ヲ
可求ヿヲ探ル[31]卽二汎差ニ矢ヲ乘シ徑ニ除得
所ヲ以テ三定差ヲ約メテ六分四二八五七六
微强ヲ得ル[32]零約ノ術ヲ以テ九乘一十四除ノ

43v

極限ナルヿヲ探索ム
[33]二汎差ヲ置矢ヲ乘シ徑ニ除シ亦九乘一十四除
シテ得ル數ヲ三汎差トス[34]用テ三定差ヲ減シテ
餘ヲ四定差トス
[35]四定差ヲ視ニ三汎差ノ首位ヨリ七位下レル
ニ依テ三汎差ニ矢ヲ乘シ徑ニ除シテ四差ヲ
求ヘキヿヲ探ル卽[36]三汎差ニ矢ヲ乘シ徑ニ除
得ル所ヲ以テ四定差ヲ約テ七分一一一一一
六四九八三二强ヲ得ル[37]零約ノ術ヲ以テ三十
二乘四十五除ノ極限ナルヿヲ探索ム

44r

[38]三汎差ヲ置矢ヲ乘シ徑ニ除シ亦三十二乘四十
五除シテ得ル數ヲ四汎差トス[39]用テ四定差ヲ減
シテ餘ヲ五定差トス
[40]五定差ヲ視ニ四汎差ノ首位ヨリ六位下レル
ニ依テ四汎差ニ矢ヲ乘シ徑ニ除シテ五差ヲ
可求ヿヲ探ル[41]卽四汎差ニ矢ヲ乘シ徑ニ除シ
得ル所ヲ以テ五定差ヲ約テ七分五七五七六
三五六九七七弱ヲ得ル[42]零約ノ術ヲ以テ二十
五乘三十三除ノ極限ナルヿヲ探索ム
[43]四汎差ヲ置矢ヲ乘シ徑ニ除シ亦二十五乘三十

40v

據テ數ヲ探ル者也[75]增約ノ術ヲ累遍シテ極數ヲ
求ルハ數ニ據テ數ヲ探ル者也[76]零約ノ法ニ依テ
率數ヲ求ルモ又數ニ據テ數ヲ探者也[77]其增約及
零約ハ各法術ニ依ト雖本是數ニ據テ探會シテ
其法ヲ立ユヘ皆數ニ據テ數ヲ探者トス

探弧數　第十二

[1]弧背ノ形質ヲ探ルニ半圓ニ近キ者ハ眞數伏レ
邊ニ近キ者ハ眞數顯ル[2]是半圓ニ近ハ緯ニ屬テ
其規急ナリ[3]邊ニ近キハ經ニ屬テ其規舒ナルニ
依レリ[4]故ニ矢ノ極テ微ナルヲ以テ其數ヲ探テ

41r

術ヲ索ルナリ
[5]始徑一尺ニシテ矢一寸矢二寸矢三寸矢四寸
等ノ定背ヲ碎約シ求メ續テ又矢四寸五分矢
四寸九分等ノ定背ヲ求テ其數ヲ探ニ半圓ニ
近者ニ於テ敢テ據ト爲ルヿヲ不會[6]故ニ往歲
關氏弧率ヲ造改ヿ再次吾亦重テ造改ヿ一次
共皆不精シテ其術廢シヌ[7]其矢一寸ノ半背冪
一十寸○三強ト矢一分ノ半背冪一寸○○三
三強ノ數ニ依テ豫矢ノ極微ナル者必眞數ノ
顯ヘキヿヲ察シテ矢一忽ノ半背冪ノ定數ヲ

41v

求得テ其質ヲ探會セリ
[8]矢一忽ノ弧ヲ截テ二斜ト造シ次ニ截テ四斜ト
造シ次ニ截テ八斜ト造シ次ニ截テ一十六斜ト
造逐[9]如此截斜ノ數ヲ倍シテ各截半背冪ヲ求メ
累遍增約ノ術ニ依テ定半背冪一絲○○○○
○三三三三三五一一一一二二五三九六九
○六六六六七二八二三四七七六九四七九五九
五八七五強ヲ得ル碎[10]約ノ法圓周冪ヲ求ニ同シ但[11]背ノ截數六十四斜ニ到テ截半背冪ヲ求メ增約ノ術ヲ以テ眞數ヲ究ルヿ五十餘位ヲ得也[12]其截數畧之
[13]矢一寸ナル者ハ半背冪一十ノ數矢一分ナル

42r

者ハ半背冪一寸ノ數矢一忽ハ半背冪一絲ノ
數ニ依テ矢徑相乘ノ元數ナルヿヲ探會ス[14]是
卽二斜ノ截背冪ニ符合ス
[15]矢徑相乘得ル數ヲ泥半背冪トス[16]用テ定半背冪
ヲ減シテ餘ヲ一定差トス
[17]一定差ヲ視ニ半背冪ノ首位ヨリ七位下ルニ
依テ矢自乘ノ數ヲ以テ可求ヿヲ探ル[18]卽矢ヲ
自乘シテ得ル數ヲ以テ一定差ヲ約メ三分三三
三三三五一一一一一強ヲ得[19]零約ノ術ヲ以テ
三之一ノ極限ナルヿヲ探索ム

38v

強率トス[45]如此逐テ次商ヲ以テ其等ノ徑周率ニ
来シ前等ノ徑周率ヲ加テ次等ノ徑周率トシテ
強弱漸親ノ率ヲ求ム[46]其零約ノ[47]諸率ノ數載ル于圓率ニ故今畧此
[48]始關氏零約術ヲ用ルニ徑一周三ヲ累加シテ
各徑周ノ率トシ毎ニ徑率ヲ以テ周率ヲ除シ
得ル所ノ數定周ヨリ少キニ到ルトキハ徑一
周四ヲ加逐一ニ是ヲ求ム[49]賢明其術ノ煩キヲ
厭テ本術ヲ探リ設タリ[50]是示首ヨリ本術ヲ案
スルニ非ス[51]先逐一ニ求ル術ヲ用テ後玄探テ
眞法ヲ會セリ

39r

[52]其零約ノ眞術如此トイヘトモ更ニ又暦筭ニ
朔餘分ヲ以テ日法ヲ定ルカ如キハ逐テ精密
ノ數ヲ究盡ヿヲ不求唯[53]秒數ノ尾位ヲ調ヿヲ
要ス[54]故ニ或一等二等ノ強弱ノ二率ヲ以テシ
或ハ二等三等或三等四等ノ強弱ノ二率ヲ用テ
逐累加シテ位數不繁ノ間率數件ヲ求メ宜ヲ
料テ擇取テ用ルナリ
[55]凡暦筭ニ於テハ別ニ一科ノ法則有リ[56]不知ハ
有ヘカラス[57]假ヘハ立術ノ如キハ眞理ノ儘ニ
術ヲ設ルトキハ布筭甚難クシテ用ルニ不堪

39v

[58]故ニ眞數ヲ求ムヘキ程限ヲ料テ別ニ簡易ノ
假術ヲ探設テ用之又[59]求數ノ如ハ眞術ニ依テ
微芒ノ數ヲ極ルヿヲ不要[60]純尾位ヲ究ムヘキ
多寡ノ位數ヲ料テ別ニ假術ヲ設テ數ノ不繁
ヲ索テ用之[61]其零約ノ數ニ於テモ如此[62]眞率ヲ
不取シテ間率ヲ用ルヿ有也
[63]隋書古之九數圓周率三徑率一[64]其術踈舛[65]自劉
歆張衡劉徽王審皮延宗之徒各設新率[66]未臻折
衷[67]宋南徐州從事史祖冲之更開密法[68]以圓徑一
億爲一丈圓周盈數三丈一尺四寸一分五釐九

40r

毫二秒七忽朒數三丈一尺四寸一分五釐九毫
二秒六忽[69]正數在盈朒二限之間[70]密率圓徑一百
一十三圓周三百五十五約率圓徑七圓周二十
二ト[71]曩關氏圓ヲ砕抹シテ定周ヲ求メ零約ノ
術ヲ以テ徑周ノ率ヲ造レリ[72]爾シヨリ後二十
餘年ヲ歴テ隋志ヲ觀ルニ周數率數咸邂逅ニ
符合スル者有リ[73]咨祖子也關子也邦ヲ異ニシ
時ヲ殊ニスト雖眞理ニ會スルヿ相同シ可謂
妙ナリト
[74]右圓ノ數砕抹ノ術ヲ以テ截周冪ヲ求ルハ理ニ

36v

トス◯[16]一遍約周羃八角以上ヲ以テ逐テ前段ト相減シテ餘ヲ各二差トス[17]後差ヲ以テ前差ヲ除シ探ルニ逐差ノ數一十六分之一ノ極限ナルヿヲ會ス[18]卽増約ノ術ニ依テ約法ノ内一ヲ減シテ餘一十五ヲ以テ各二差ヲ約メ各其段ノ一遍約周羃ニ加テ二遍約周羃トス◯[19]二遍約周羃一十六角以上ヲ以テ逐テ前段ト相減シテ餘ヲ各三差トス[20]後差ヲ以テ前差ヲ除シ探ルニ逐差ノ數六十四分之一ノ極限ナルヿヲ會ス[21]卽増約ノ術ヲ以テ三遍約周羃ヲ求ム[22]其四遍約周羃ヲ求ル者

37r

増約ノ數二百五十六分之一五遍ハ一千〇二十四分之一[23]如此増約ノ法ハ逐四因ノ數ナルヿヲ探リ會シテ約周羃ノ遍ヲ累ル増約ノ術ヲ用テ定周羃ヲ求ル也[24]〔其増約諸數載于圓率[25]故今畧此〕

始[26]關氏増約ノ術ヲ以テ定周ヲ求ルヿヲ理會シテ一遍ニシテ止ム[27]故ニ十三萬千七十二角ニ到ル截周ヲ求テ十五六位ノ眞數ヲ究メ得タリ[28]今累遍増約ノ術ヲ用ルヿヲ探リ會シテ千二十四角ニ到ル截周羃ヲ求テ四十餘位ノ眞數ヲ究ム[29]是示首ヨリ増約累遍ヲ用ルヿヲ

37v

察スヘカラス[30]一遍ノ増約ヲ用テ後玄ク探テ累遍スルヿヲ會セリ

[31]碎約ノ術ヲ用テ徑一尺ノ定周三尺一寸四一五九二六五三五八九七九三二三八四六二六四三三八三二七九五〇二八八四一九七一二强ヲ求得テ零約ノ術ヲ以テ徑周ノ率ヲ造ル

[32]元數一ヲ置卽尺ノ位ト定ム[33]以テ定周ヲ除テ[34]毎ニ〔少ヲ以テ多ヲ除也〕得商ト不盡ヲ第一トス[35]第一ノ不盡ヲ以テ元數一ヲ除テ得商ト不盡ヲ第二トス[36]第二ノ不盡ヲ以テ第一ノ不盡ヲ除テ得商ト不盡ヲ

38r

第三トス[37]第二ノ不盡ヲ以テ第三ノ不盡ヲ除テ得商ト不盡ヲ第四トス[38]第四ノ不盡ヲ以テ第三ノ不盡ヲ除テ得商ト不盡ヲ第五トス[39]如此其段ノ不盡ヲ以テ前段ノ不盡ヲ除テ逐商ヲ求ム◯

[40]元數ノ一ヲ徑率トシ第一ノ商ヲ周率トス[41]是ヲ一等ノ弱率トス[42]第二ノ商ヲ以テ一等ノ徑周率ニ乘シ周率ニ元數ノ一ヲ加ヘテ二等ノ强率トス

[43]第三ノ商ヲ以テ二等ノ徑周率ニ乘シ一等ノ徑周率ヲ加テ三等ノ弱率トス[44]第四ノ商ヲ以テ三等ノ徑周率ニ乘シ二等ノ徑周率ヲ加テ四等ノ

34v

四歩ヲ置以テ廉法ニ乘シ方法ニ加次商ヲ以テ
方法ニ乘シ實二百五十六歩ヲ減シテ一十歩ヲ
餘ス[23]逐テ如此探テ三商四商以上ヲ求ル也
[24]初商一十ヨリ逐増シ探テ過不及ノ間ニ於テ
三十ノ定商ヲ知ルト雖技熟スルトキハ數件
ヲ探ルヿヲ不為又法術ニ依ヿ無ク直ニ三十
ナルヿヲ察ス[25]亦次商ノ如ハ一歩ヨリ逐増シ
探リテ過不及ノ間ニ於テ四歩ノ定商ヲ知ト
雖方級ノ數ヲ以テ實級ノ餘數ヲ除シ探リテ
次商トスルノ法術ヲ立テ求ルユヘ直ニ四歩

35r

ナルヿヲ察ス[26]是不探シテ直ニ知ニ似タリト
雖實ハ直ニ知ニ非ス[27]探ルヿヲ一旦ニスル也
[28]初入ノ者探ルヿヲ不累シテ首ヨリ定商ヲ得
ヘカラス[29]純探リ探テ得ヿヲ知テ後技熟シテ
一旦ニ定商ヲ知ヿヲ會スヘシ
[30]右開平方ハ理ニ據テ法術ヲ立法術ニ依テ數ヲ
探ル者ナリ[31]然ルニ開平方ノ法術ノ儘ニ其理ヲ
索ルトキハ直ニ顯シ難シトイヘトモ本是理ヲ
察シテ立ル所ノ法術ナルユヘ理ニ據テ數ヲ探ル
者ト為リ

35v

探圓數　第十一

[1]徑一尺ノ圓ヲ截テ四角ト造テ截周羃ヲ求ム亦[2]
截テ八角ト造テ截周羃ヲ求ム亦[3]截テ一十六角
ト造テ截周羃ヲ求ム[4]亦截テ三十二角ト造シ亦
六十四角ト造シ亦一百二十八角ト造[5]以上逐テ
角數ヲ倍シテ各截周羃ヲ求メテ其數ヲ視ルニ
角數倍スルニ隨テ徐真數ニ近キヲ得ルト雖敢
真數ヲ究ルヿ無[6]故ニ逐角ノ截周羃ヲ以テ遞ニ
相減シテ其差漸損スル數ヲ探リテ增約ノ術ヲ
以テ真數ヲ究得也[7]
其求截周羃術及所求之數截干圓率[8]故今畧之

36r

[9]始關氏角面羃ヲ開平方ニシテ各角面ヲ求テ
截周ヲ用ユ[10]今角面羃ヲ以テ截周羃ヲ求ル者
開平方ノ功ヲ省也[11]是首ヨリ羃數ヲ用ルヿヲ
察スルニ非ス[12]先截周ヲ用テ後玄探テ羃數ヲ
用ルヿヲ會ス
[13]其截周羃四角以上ヲ以テ逐テ前段ト相減シテ
餘ヲ各一差トス[14]後差ヲ以テ前差ヲ除シ探ルニ
逐差ノ數四分之一ノ極限ナルヿヲ會ス[15]即增約
ノ術ニ依テ約法ノ内一ヲ減シテ餘三ヲ以テ各
一差ヲ約メ各其段ノ截周羃ニ加テ一遍約周羃

32v

以テ求ムト雖徑ニ眞積ノ極數ヲ得ルナリ蓋[14]其
片積ヲ求ルヿ臺積ヲ求ムル術ニ非ス是[15]球積ヲ
碎抹スルニ於テ一奇術ニシテ即球積碎抹ノ質
ニ順フ者ナリ
[16]圓ノ屬ヲ碎抹スルヿ純形質ニ順フヿヲ索ム
ヘシ[17]忤フヿヲ為ヘカラス所[18]謂削片スヘキヲ
細截スルトキハ忤フ[19]周圍ニ依テ截碎ヘキヲ
直徑ニ依テ截碎時ハ忤フ[20]縦截スヘキヲ横截
スルトキハ忤フ是[21]等ノ如キ形質ニ不順者ハ
眞數ヲ求メ得ルト雖或極數ヲ探ルニ停滞シ

33r

或術理ヲ會スルニ否塞ス[22]其形質ニ順フヿヲ
求ルハ先理ヲ察シテ數ヲ索メ數ニ依テ玄ク
探テ是ヲ會シ得ル者ナリ[23]故ニ碎抹ヲ用ント
欲ハ眞數ヲ得ルノ一偏ニ陥テ其順ト不順ノ
理ヲ失スルヿナカレ
[24]右碎抹ハ皆理ニ據テ數ヲ探ル者也[25]但其形質ニ
順フヿヲ索ルニ到テハ必數ニ據テ數ヲ探リテ
識ヘキナリ
探開平方數　第十
[1]假如有積一千一百六十六歩[2]問開平方幾何[3]荅

33v

曰方三十四歩 餘積一十歩
[4]解題本術[5]置積為實以[6]一為廉法開[7]平方除之得方
面也
[8]積ヲ實ニ置廉法ニ一ヲ置テ位ヲ超テ[9]諸級之進退挍畧之
初商十ノ位ナルヿヲ知ル[10]其初商一十ニシテハ
一一如一百ナルユヘ實ヨリモ少シ[11]二十ニシテ
ハ二二如四百ナルユヘ亦實ヨリモ少シ[12]三十ニ
シテハ三三如九百ナルユヘ猶實ヨリモ少シ[13]又
四十ニシテハ四四一千六百ナリ[14]却テ實ヨリモ
多[15]故ニ三十餘ナルヿヲ知テ初商三十ヲ置是ヲ

34r

廉法ニ乘シテ方法ニ置初商ヲ以テ方法ニ乘シ
實九百ヲ減シテ餘二百六十六歩[16]亦初商ヲ以テ
廉法ニ乘シ方法ニ加方法ニ六十歩ヲ得ル[17]於此
次商ヲ探ニ一歩ニシテハ實ヲ減スルヿ六十一
歩ナルユヘ少シ[18]二歩ニシテハ實ヲ減スルヿ一
百二十四歩ナルユヘ亦少シ[19]三歩ニシテハ實ヲ
減スルヿ一百八十九歩ナルユヘ亦少シ[20]四歩ニ
シテハ實ヲ減スルヿ二百五十六歩ナルユヘ猶
少シ[21]五歩ニシテハ實ヲ減スルヿ三百二十五歩
ナリ却テ多シ[22]故ニ四歩有餘ナルヿヲ知テ次商

30v

術ヲ得ル者也蓋[51]此術數ニ據ル者ハ事ヲ用ルヿ
最繁多ナリト雖徑ニシテ入易シ理[52]ニ據ル者ハ
理ヲ取ルヿ緐テ容易ナリト雖幽ニシテ入難シ
[53]今其兩ノ術ヲ舉テ其義ヲ論シテ各會スルヿノ
同一ナルヿヲ證ス

員數　四條

探碎抹數　第九

[1]理ニ據テ索ル如キハ立元ノ法則有テ萬術ヲ
貫ケリ數[2]ニ據テ探ル者ハ碎抹ノ術ヲ出ルヿ
無ク而モ探ルニ定則無シテ萬法各其徑ヲ異

31r

ニス乃[3]碎抹ハ求數察理ノ原探索ノ事ニシテ
法術ヲ探ルノ法則也故[4]ニ形質ニ順テ碎抹シ
數ヲ索ルニ尋テ玄ク探ル時ハ法術ヲ不會ト
云ヿナシ以[5]是其趣意ヲ演テ肝要タルノ義ヲ
星ス也
假[6]令圓ノ周ヲ碎抹スル者若直徑ヲ等横ニ細截
シテ毎截上下ノ弧弦ノ盡ル處ノ左右ノ斜弦ヲ
求メ截數ノ如ク其斜弦ヲ累併セテ截周ヲ求ル
時ハ其所截ノ徑ノ分等シテ周ノ分不等然[7]レハ
截數ヲ逐倍シテ件件ノ截周ヲ求探トモ其質ニ

31v

恊エヘ極數ヲ求ルニ滯リ且圓ノ質ヲ識據ヲ得
ヿナシ故[8]ニ四角以上逐倍ノ角形ニ截テ截周ヲ
求ルトキハ卽周ヲ等ク細截スル者ニシテ其數
求周ノ質ニ順エヘ角數逐倍スル件件ノ截周ヲ
求增約累遍ノ法術ヲ探會シ速ニ極數ヲ求得テ
圓ノ質ヲ識ノ據ヲ得ナリ
[9]球積ヲ碎抹スル者ハ球徑ヲ等ク細片シテ毎片
圓臺ノ形ト造シ片厚ヲ累テ弧矢トシテ毎片ノ
弧弦ヲ求テ便毎片上下ノ臺徑ニ取用シ片厚ヲ
便臺高トシ圓臺ノ積ヲ求ル術ニ依テ片片ノ積

32r

ヲ求メ片數ノ如ク臺積ヲ累併テ截積トス但[10]臺積ヲ
求ル者圓率ヲ用ルヿヲ截テ方積ニ通用ス又[11]其片數ヲ逐倍シテ件件
ノ截積ヲ求メ其數ヲ探テ增約ノ數ヲ索メ術ノ
如クシテ眞積ノ極數ヲ得ナリ是[12]卽求積ノ理ニ
恊ヿ無エヘ極數ヲ得ルニ滯ルヿ無ト雖更ニ又
玄ク探ルニ其臺積ヲ求ルノ術理ニ中ルニ似テ
數猶不中也[13]故ニ毎片ノ上徑冪ト下徑冪ト相併セ
高ヲ乘シ折半シテ通片積トシ片數ノ如ク累子
併テ通截積ヲ得又片數ヲ逐倍スル件件ノ截積
ヲ求テ增約ノ術ニ依テ求ルトキハ片數最少ヲ

28v

省者ニシテ本術ハ徑自乘圓周ノ法ヲ乘シテ
直ニ球面積ヲ得ルナリ
[19]解題本術[20]置球徑自乘以[21]圓周率乘之如徑率而一
得面積也
[22]關氏曰萬法ヲ理會スルハ形ヲ見道條ヲ立ヲ
以テ原要トス[23]是ハ此探ヿヲ不爲シテ首ヨリ
眞術ヲ會スルノ奥旨也ト[24]乃後ノ術球ノ形ヲ
察シテ中心ヲ極トシ錐形ニ見造ハ卽形ヲ見
道條ヲ立ルニシテ探ルヿ無ク直ニ眞術ヲ理
會スル也[25]故ニ始ノ術ヲ以テ下等ナリト爲リ

29r

[26]吾元來質ノ魯ナルヨリ觀ルニ總テ理ニ據テ
直ニ眞法ヲ會セントスレハ此術ノ如キ速ニ
理ノ察シ易キニ逢トキハ最容易ナリト雖或
理ノ據無ニ逢時ハ必得ヘカラス[27]是ノ如キハ
純數ヲ以テ探リ探テ據有ルヿヲ會シ其據ニ
就テ眞法ヲ成得ル者也[28]以此強ニ下等ナリト
不爲[29]蓋探ラサレハ會シ難キヿ有ルハ是質ノ
偏駁ナルニ依ユヘ乎[30]如純粋ナラハ數ト理ノ
據ヲ別ツヿ無ク事事不探シテ直ニ會セン耳
[31]然レトモ是ハ吾質偏駁ナルユヘ脩シ盡テモ

29v

更ニ至ルヿ無處也[32]凡員數ニ於ル術理ニ於ル
法則ニ於ル總テ咸元來自然ニ具レル者ナリ
[33]是ヲ會セルハ敢テ新ニ其道ヲ蹊タルニ非ス
[34]自然ノ道ニ合會スルナリ[35]然ルトキハ其探テ
會スルモ亦可ナラン歟[36]熟意フニ關氏カ生知
ナルヿ世ニ冠タリ[37]然モ常ニ謂ラク圓積ノ類
甚難シ不可得者ト[38]嗚呼是安行ニ住セル故乎
[39]吾ハ言フ圓積ノ類ト雖力ヲ用テ必得ル者ト
[40]卽是苦行ニ止ルユヘナリ[41]其關氏カ不可得ト
謂ハ安行ニ住シテ安行ナルユヘ探ルヿ無シテ

30r

直ニ得ヲ意トスルニ依[42]必シモ不得ニハ非シ
[43]或事ヲ不盡處歟[44]吾生得ノ質魯ナルユヘ安行
ニ住シテ安行ヲ得ルノ地ニ至ルヿ無シ[45]常ニ
苦行ニ止テ而モ泰キニ居ル道ヲ肯スルヿ有
[46]故ニ探索テ必得ト爲リ[47]是ヲ以テ省意フニ吾
生得ノ本質孝和ニ比スレハ減ルヿ十ニシテ
一ナルヿヲ
[48]右球面ノ積ヲ求ル術始ニ設ル所ハ片探テ數ヲ
求メ數ニ據テ又術ヲ探ル者ナリ[49]後ニ設ル所ハ
數ヲ探リ術ヲ探ルヿヲ不爲[50]直ニ理ヲ察シ直ニ

26v

難シト不為ハ是算ノ實ニ我心ニ從ユヘナリ
[35]凡求數ニモアレ施術ニモアレ探法ニモアレ
總テ一些モ難シト意ヿ有ハ心ニ不從處有テ
真實ノ不至ニ依レリ[36]其心ニ從フト不從トノ
意ノ實ヲ識者ハ賢明乎[37]夫思慮ノ慧利ナルニ
依ルヿ無ク亦氣情ノ壯盛ナルヲ用ルヿ無ク
恭ニ居テ常ニ為テ不止者ハ即柔ヲ以テ剛堅
ヲ碎キ寡ヲ以テ衆多ヲ量ルノ力ナリ
[38]右算脫ハ類數ヲ設テ碎キ探リ數ニ據テ法術ヲ
會スル者ナリ[39]元來其理ヲ備ルヿ有ト雖敢理ヲ

27r

察シテ得ヘカラス[40]唯其數ノ成處ニシテ數ヨリ
自心ヲ導ヿヲ得テ是ヲ會スルナリ

探求球面積術　第八

[1]假如有球[2]徑一尺問[3]面積幾何[4]答曰面積三百一
十四寸一五九二六五三五九弱
[5]剔片ノ術ニ依テ[6]細抹スル者ハ質ニ不順ユヘ不用徑一尺〇〇一
釐ノ球積ヲ求テ徑一尺ノ球積ヲ脫スルトキハ
片實積一寸五七二三六七六四六七二強ヲ得[7]片厚五毫ヲ以テ除テ
片面積三百一十四寸四三五二九三四四強ヲ得ル[8]亦徑一尺〇〇
〇〇一絲ノ球積ヲ求テ徑一尺ノ球積ヲ脫スル

27v

時ハ片實積一毫五七〇八一二〇三四八一強ヲ得ル[9]片厚五忽ヲ以テ
除テ片面積三百一十四寸一六二四〇六九六二強ヲ得ル[10]又徑一尺
〇〇〇〇〇〇一微ノ球積ヲ求テ徑一尺ノ球積
ヲ脫スル時ハ片實積一絲五七〇七九六四八三八七強ヲ得[11]片厚
五微ヲ以テ除テ片面積三百一十四寸一五九二九六七七五弱ヲ得ル
[12]是片厚ノ最微ナルニ從テ真數徐顯ル者也
[13]三件ノ片面積ヲ視テ損約ノ術ニ依テ球面ノ
真積三百一十四寸一五九二六五三五九弱ヲ
求得テ是ヲ探ニ積數ニ圓周ノ法ノ數ヲ顯ス
[14]故ニ必圓周ノ法ヲ乘シ求ヘキヿヲ會シテ即

28r

圓周ノ法ヲ以テ積ヲ約ルトキハ一百ノ數ニ
整フヿヲ得ル[15]是即徑自乘ノ數ナルヿヲ探リ
會シテ本術ヲ成者ナリ
[16]亦球心ヲ錐ノ尖ト見球半徑ヲ錐ノ高ト見球積
ヲ錐ノ積ト見テ積ニ錐法三ヲ乘シ錐高ヲ以テ
除テ錐面ノ積ヲ得ルヲ便球面ノ積トス
[17]球徑再自乘シテ圓周ノ法ヲ乘シ六ヲ而一ニ
シテ球積ヲ得ル[18]是ニ錐法三ヲ乘シ球半徑ニ
除テ球面ノ積ヲ得ルユヘ此術ヲ化スル時ハ
始ニ其球積ヲ求ル術ノ徑ヲ省キ又六而一ヲ

24v

此眞僊極積ヲ問ニ到テ其同題タルヿヲ不意[48]卽立元ノ法ニ依テ其理ヲ察シテ忽ニ其術ヲ探リ得タリ是[49]理ニ據ルト數ニ據ルト其時ニ臨ミ題ニ應シテ意ニ肯スルヲ以テ用ル者也[50]以是當知數ニ據テ探ル者ト理ニ據テ探者ト各其事ヲ異ニスト雖會シ得ルヿハ本是同一ナルヿヲ

探算脱術 第七

黒[1]白ノ棋子三十[2](一半ハ白子 一半ハ黒子)ヲ交備テ順算シテ毎ニ十ノ數ニ當ヲ脱去ル既[3]ニ黒子十四ヲ脱テ

25r

一黒子ヲ止ム於[4]此其所止一黒子ヨリ逆算シテ[5](順算モ同ト雖 姑ク古傳ニ從フ)十ノ數ニ當ル脱去時ハ却テ白子咸脱盡シテ黒子一ヲ止ム是[6]ヲ古ヨリ繼子立ト號ス今[7]其術ヲ探會シテ別隊ヲ求ム

[8]棋子ヲ以テ並布テ[9](皆黒子一他ハ 白子ヲ用)其整不整ヲ驗ニ二脱ハ白子ヲ止ルヿ一、三、七、十五、三十一等ニ整フ三[10]脱ハ白子ヲ止ルヿ三、五、八、三十等ニ整フ[11]四脱ハ白子ヲ止ルヿ一、四、八、十一、十五等ニ整フ五[12]脱ハ白子ヲ止ルヿ二、五、十一、十四、三十六等ニ整フ[13]六脱ハ一、二、七、十、十三等ノ數ニ整

25v

[14]其驗ヿヲ往返シ探ルニ必不整ノ數有リ[15]可整數有リ[16]其可整數ニ於テ亦整不整有ルヿヲ得ル卽[17]是據ト成テ本術ヲ會セリ

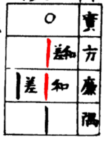
[18]求限本術 置[19]一[20](子擬 黒)於法實[21]ハ空[22]法ニ一實ニ脱數各累加之實滿法除去之實盡者法減一[23](子原擬 者黒也)餘爲正限數也

[24]算脱ノ術ハ兄賢明カ探會スル所ナリ賢[25]明カ生知孝和ニ亞リ其[26]稟受ノ氣情最怯弱ニシテ常ニ病日多カリシ曾[27]五斜ノ括術ヲ爲ント欲シテ甚繁雜セリ[28]タトヒ萬位ニ及フトモ曰ニ

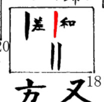
26r

百位ヲ造ハ徐百日ニシテ畢テントト言テ果テ月餘ニシテ悉成シ得タリ賢明[29]沒シテ後吾彼成得タルヲ意テ始テ實ニ肯スルヿヲ得タリ[30]旬日ナラスシテ黄赤道立成ノ元數ヲ求得テ中根上右衛門ニ授ク時[31]ニ五十七歳ナリキ亦[32]吾少カリシ時所問有テ宣明暦天正氣朔轉交四件ノ分數ヲ以テ積年ヲ求ル段數ヲ爲畢テ以爲多位ニシテ最難爲者ト[33]若今既ニ齡傾キ情氣徐一半ヲ損スルニ逮テ却テ許多ノ數ヲ求メ力ヲ用ルヿ壯ナリシ時ニ倍セリ[34]而ルニ

22v

理ニ據テ探ル者
[8]立天元一為闊○|[9]加差為長  [10]亦以闊減和為
高  [11]長闊
高相乘為積 
[12]是ヲ以テ元式トシテ其術意ヲ探ニ若題中ニ
積數ヲ云トキハ積數ヲ以テ元式ト相消シテ
積數即實級ニ止レリ[13]其實級ノ極テ多キ者ハ
方級ヲ開盡スヲ以テ限トスルユヘ立ル所ノ
闊ヲ即商トシテ元式ヲ用テ開出商數ノ法ニ
依テ方級ノ極限ヲ求ルヲ以テ相消スルノ度

23r

ヲ得ルナリ
[14]以闊便為商○|[15]置元隅級ニ負乘商加ニ元廉級為開
廉級一  [16]又乘商為應開  [17]亦置元隅級乘
變數得負 方級一遍數負 商加開廉級一
變數為開廉  [18]又乘商為應開  [19]加應開方
級二變數負 方級二遍數負 級一遍數
為方級  [20]寄左[21]置元方級數正與寄左相消[22]開出商ノ
極限數負 法ハ實級ニ到ル迄皆同加異減ヲ用ユ[23]故ニ是相消スルモ亦同加異
減ヲ用テ相併之世 得度 

23v

[24]是題數ヲ以テ求ヘシト雖本術ヲ為ンヿシ欲
シテ題辭ノ號ヲ書シテ其畫式ヲ求ル也
[25]解題本術 [26]置和以[27]差乘之為實正[28]亦置和減去差餘
倍之為方正[29]以三為廉負[30]開平方除之得闊加差得[31]
長以[32]闊減和得高[33]長闊高各相乘得積也[34]所得闊尺下帶不盡
[35]故以原式實三因方依舊廉三約開平方除之得一十四尺[36]三約得四尺三分尺之二
[37]右直堡ノ術ハ理ニ據テ術ヲ探ル者是ノ如ク也
[38]本術ノ儘ニ其理ヲ索ムルトキハ伏レテ顯ル處
無ト雖本是立元ノ法ヲ以テ理ヲ察シテ立ル所
ナルユヘ理ニ據テ術ヲ探ル者トセリ[39]凡法術ハ

24r

數ニ據テ立ルノミニ非ス[40]理ニ據テ立ル者ト雖
法術ノ儘ニ其理ヲ索ルトキハ伏レテ顯レサル
ヿアリ[41]如此ハ強テ其理ヲ察スルヿシ不為[42]理ヲ
法術ニ委子テ唯其法術ノ儘ニ從ヒ用ルヲ以テ
數ノ道ニ循トス
[43]曩ニ或授時暦月離遲疾差ヲ求ルニ立平定ノ
三差ヲ用ル者其損益ノ極限ノ數ヲ求ル術ヲ
問ヿ有[44]吾理ヲ察スルヿヲ不為[45]即類數ヲ碎テ
乍ニ實一方二廉三ノ數ノ據ヲ探得テ其術ヲ
會セリ[46]其探數略之[47]爾シヨリ後又題辭ヲ變シ造テ

20v

スルユヘ壅塞シテ敢テ會スルヿヲ不得[86]又必
理ニ據テ探ルヘキヲ强テ數ニ據テ探ラント
スレハ盡スヿ不能シテ必凝滯ス[87]然レトモ或
數ヲ索ルヿヲ不爲トキハ質ニ順フ所以ヲ識
ヘカラス[88]是ノ如キハ空ク思ヲ覃シ心ヲ役ス
トモ竟ニ何ノ益カ有ン[89]故ニ今數ト理ト兩ノ
徑ノ據ヲ分チ數ヲ索ルヿハ根本タルノ義ヲ
釋シテ探索ノ徑ニ不爽ヿヲ訓ス
術理　四條
探織工重互換術　第五

21r

[1]假如有織工[2]三人二十一日織錦四端[3]今織工七人
織四十五日[4]問錦幾端○[5]荅曰織錦二十端
[6]初ニ云フ錦四端ヲ置テ織工三人ヲ以テ除スル
時ハ一人二十一日ニ織錦一端三三三三三强ヲ得ル[7]累テ
二十一日ヲ以テ除スル時ハ一人一日ニ織錦六毫
三四九二一弱ヲ得ル[8]故ニ後ニ云フ四十五日ヲ乘スル
トキハ一人四十五日ニ織錦二端八五七一四三弱ヲ得ル
[9]累テ七人ヲ乘シテ七人四十五日ニ織錦二十端
ヲ得ル也
[10]元術如此トイヘトモ除ヲ累テ用ルニ數不整

21v

有ル者ハ還源ヲ失ス[11]故ニ先乘後除ノ法式ヲ
立テ括術ヲ爲リ
[12]解題本術[13]置初云錦四端[14]以後云織工七人乘之亦以日
數四十五乘之爲實[15]置初云織工三人[16]以日數二十一乘之
爲法除[17]之得織錦數也
[18]是乘スヘキヲ累テ乘シテ實トス[19]除スヘキモ
又累テ乘シテ法トシテ一般ノ除ヲ用ルヿ首
ヨリ會シ難シ[20]先毎一ノ數ヲ得ルノ術理ヲ碎
探テ後乘除ヲ括シテ本術ヲ爲ナリ
[21]右織工重互換ノ術ハ理ニ據テ術ヲ碎探ル者ニ

22r

シテ卽術ニ依テ互換ノ法式ヲ立ル也[22]如法術ノ
儘ニ其理ヲ索ルトキハ直ニ顯レ難シト雖本是
探索ル理ニ據テ立ル所ナルユヘ理ニ據テ術ヲ
探ル者トス
探直堡求極積術　第六
[1]假如有直堡[2]長闊差七尺闊高和八尺[3]欲使積至多
[4]問長闊高及極積各幾何○[5]荅曰闊四尺三分尺之二○
長一十一尺三分尺之二○高三尺三分尺之一○積一百八
十一尺二十七分尺之一十三
[6]數ニ據テ探ルヿヲ不爲[7]立元ノ法ヲ以テ直ニ

18v

[61]平立以上ノ限差法ヲ求ムルヿ各段底面ノ差
齊者ハ探得難シ故ニ[62]各段底面ノ參差タルヲ
設テ是ヲ探會スル也[63]亦三差ノ數加減有者ハ
四角垜ノ數ニ依テ所効無シ故ニ[64]暦法躔離ノ
差度ノ如キ各限ノ元積過不及有ル者ヲ設テ
正負ノ數ヲ得ルニ依テ應加應減シ定ルナリ
[65]更ニ不繁説
[66]解題本術 [67]倍底面加三底面乘之加一亦底面乘之
六而一得積也
[68]積數ハ必底面再自乘ノ數ニ當ルヿヲ識ルニ

19r

據テ更ニ又底面ト底面自乘ト底面再自乘ト
三科ノ數ヲ聚テ積數ニ比課シ求ムヘキヿヲ
探テ方程ノ法則ヲ會ス[69]卽底面ト底面自乘ト
底面再自乘ト及積數ヲ用テ四列三等ノ行ヲ
設布テ疊約シテ各段數ヲ得テ求ルモ同意也
[70]亦關氏方垜ノ總術ヲ立爲セリ [71]其自乘垜ノ術四角垜ニ相合ス [72]其術皆畧之
[73]右招差ハ同類ヲ設テ碎テ數ヲ求メ數ニ據テ法
則ヲ會スル者也[74]凡數ニ據テ會スル法術ハ理ヲ
察シテ盡シ得ヘカラス[75]故ニ強テ其理ヲ索ルヿ
ヲ不爲[76]唯其法ノ儘ニ術ヲ成ヲ以テ卽數ノ道ニ

19v

循トス
[77]凡探索ノ法必理ニ據テ探ヘキ有リ必數ニ據
テ探ルヘキ有リ亦兩ナカラ該ル有リ[78]其理ニ
據テ探ル者ハ數ヲ求ムルヿヲ不爲ト雖心ヲ
用ルヿ直實ナルトキハ必是ヲ會ス[79]且立元ノ
法則ノ精通タル者有テ如彼ヲ用ルヿ有ルニ
到テハ許多ノ功ヲ省エヘ力ヲ用ルヿ鮮シテ
輙ク得ル者ナリ[80]數ニ據テ探ル者ハ強ニ理ヲ
察スルヿヲ不爲[81]純數ヲ索テ幽ク探ル時ハ必
是ヲ會スル者ナリ[82]其探ル法ハ大率滿極干盡

20r

ノ際ニ於テ或增長シ或損消シ截抹剖折シテ
類數ヲ求メ其消息ノ機ニ從テ或理ノ據ヲ探
得或數ノ據ヲ探リ得其據ニ就テ千變萬化ニ
シテ法術ヲ成ナリ[83]蓋人法則ニ靠テ術ヲ施ス
ヿヲ學得ヿ有トイヘトモ質ヲ盡シテ法則ヲ
會スルニ意ヲ寄ルヿ有者鮮シ[84]故ニ心ヲ用テ
理ヲ察ルヿハ爲易力ヲ竭シテ數ヲ索ルヿハ
爲難[85]然ルニ理ニ據テ探ル者ト數ニ據テ探ル者ト
徑ノ異有ヿヲ辨スルヿ無シテ其必數ニ據テ
探ルヘキニ於テモ強ニ理ヲ以テ極盡サント

16v

シテ不齊故ニ[32]其段ト後段ト底面ノ差ヲ用テ
約ヘキヿヲ探リ會ス
[33]逐段其段ノ底面ト後段ノ底面ト相減シテ平限
差法トス[34]各段一ヲ得ル[35]以テ各段ノ定積差ヲ約
テ各段ノ平積トス[36]一限一个二之一二限二个六之一三限
二个六之五四限三个二之一五限四个六之一六限四个六之五ヲ得
[37]其平積ヲ以テ逐段後段ト相減シテ平積差トス
[38]一限三之二二限三之二三限三之二四限三之二五限三之二
ヲ得ル
[39]其平積差ノ數必底面ヲ以テ約ヘキヿヲ案ト

17r

雖或直ニ底面ニ約メ或ハ前後ノ底面ノ差ニテ
約ル時ハ其數皆參差トシテ不齊故ニ[40]其段ト
一段ヲ隔テ後段ト底面差ヲ用テ約ヘキヿヲ
探リ會ス[41]即亦是ニ依テ三乘積差ヲ求ル如ハ
二段ヲ隔ル底面差ヲ以テ三乘ノ限差法トシ
四乘ノ積差ヲ求ル如ハ三段ヲ隔ル底面差ヲ
以テ四乘限差法トス[42]五乘差以上准之逐乘ノ
限差法ヲ求ルヿヲ會ス
[43]逐段其段ノ底面ト一段ヲ隔テ後段ノ底面ト相
減シテ各段ノ立限差法トス[44]各段ニヲ得ル[45]以テ

17v

各段ノ平積差ヲ約テ各段ノ立積トス[46]一限三之一
二限三之一三限三之一四限三之一五限三之一ヲ得テ各
段相齊シ[47]故ニ三之一ヲ以テ立差トス◯[48]各限ノ
底面ヲ自乘シテ立差ヲ乘シ各其段ノ第一ノ定
積ヲ減シテ各段ノ第二定積トス[49]一限三之二二限
一个六之一三限一个三之二四限二个六之一五限二个三之二六限
三个六之一七限三个三之二ヲ得ル[50]逐テ其段ト後段ト相
減シテ定積差トス[51]一限二之一二限二之一三限二之一
四限二之一五限二之一六限二之一ヲ得ル[52]各段平限差
法ヲ以テ約テ各段ノ平積トス[53]一限二之一二限二之

18r

一三限二之一四限二之一五限二之一六限二之一ヲ得テ
各段相齊シ[54]故ニ二之一ヲ以テ平差トス◯[55]各限
ノ底面ニ平差ヲ乘シ以テ各其段ノ第二定積ヲ
減シテ第三ノ定積トス[56]一限六之一二限六之一三限
六之一四限六之一五限六之一六限六之一七限六之一ヲ得
テ各段相齊シ[57]故ニ六之一ヲ以テ便定差トス[58]是
立積ヲ求ルニ到テ各段平均ヲ得ユヘ必一限二限
三限ノ三等ノ數ヲ用テ求ムヘシト雖姑七等ノ
數ヲ設テ其各段遍ク
平均ヲ得ルヿヲ證ス
[59]三差ノ數各同分母ヲ求テ立差二平差三定差一
ヲ得ル[60]同母ノ六ヲ以テ約法トス

14v

四十二[16]以子餘除去母餘母再餘二十一[17]以母再餘
去子餘子再餘二十一[18]於此母子餘數相齊[19]便以二
十一為約法[20]以約原分母子得約數也
[21]約分ハ總テ數ノ繁キヲ治ルニ法ル[22]其術分母
ト分子トヲ以テ互ニ相去ルハ等數ヲ一旦ニ
探索ル者ナリ[23]凡萬題萬術皆員數ヲ言フ際ハ
或約分ヿヲ離ルヿ無シ[24]乃其術ヲ擴充スルニ
到テハ題問ノ趣意ニ依テ諸約ノ法課分ノ屬
其等甚多ト雖皆約分ニ本ツキテ互去ノ術ヲ
不該ト云ヿナシ[25]是最輕淺ニ似タリト雖至テ

15r

深重也故[26]ニ其事ヲ擧テ其義ヲ釋ス
[27]右約分ノ法互去ノ術ハ最簡易ナルユヘ其理ヲ
察スルトキハ憑處有リト雖元來約分ハ問辭ノ
品彙ニ依ルヿ無ク一向數ニ從テ立ル法術ナル
ユヘ數ニ據テ法ヲ探ル者トス

探招差法　第四

[1]假如有四角尖[2]垜底面一十九[3]問積幾何[4]〇荅曰積
二千四百七十箇
[5]四角垜ノ底面一ナル者ハ積ヲ計ルニ卽一ナリ
[6]一限トス〇[7]次ニ底面二ナル者ハ積ヲ計ルニ五

15v

ヲ得ル[8](卽一四ノ數相併スル者)二[9]限トス〇[10]次ニ底面三ナル
者ハ積ヲ計ルニ一十四ヲ得ル[11](卽一四九ノ數相併スル者)三[12]
限トス〇[13]次ニ底面四ナル者ハ積ヲ計ルニ三十
ヲ得ル[14](卽一四九十六ノ數相併スル者也[15]五限以上倣之)四[16]限トス〇[17]次ニ
底面五ナル者ハ積ヲ計ルニ五十五ヲ得ル[18]五限
トス〇[19]次ニ底面六ナル者ハ積九十一ヲ得ル[20]六限
トス〇[21]次ニ底面七ナル者ハ積一百四十ヲ得ル
[22]七限トス[23](八限以上ヲ設ルモ同シ)
[24]積數ハ元來立積ナリ[25]故ニ底面ヲ以テ三タヒ
除テ其段數齊カルヘク亦底面再自乘ノ數ヲ

16r

以テ積ヲ求ムヘキヿヲ探ル[26]卽是ヲ據トシテ
招差ノ法則ヲ會スル也
[27]各限ノ底面ヲ以テ其段ノ積數ヲ約テ第一定積
トス[28]一限(一个)二限(二个二之一)三限(四个三之二)四限(七个二之一)
五限(一十一个)六限(一十五个六之一)七限(二十个)ヲ得ル[29]定積ヲ
以テ逐段後段ト相減シテ各段ノ定積差トス[30]一
限(一个二之一)二限(二个六之一)三限(二个六之五)四限(三个二之一)五
限(四个六之一)六限(四个六之五)ヲ得ル
[31]其定積差ノ數必底面ヲ以テ約ヘキヿヲ察ト
雖若直ニ底面ヲ以テ約ルトキハ其數參差ト

12v

用ルヿハ唯加減因乘耳[56]可謂無上ノ法則也ト
[57]故ニ今其義ヲ演テ歎美焉
[58]關氏孝和ハ吾師タリ[59]曾立元ノ法ニ據テ變ニ
真假ヲ設テ解伏題ノ法術ヲ立爲セリ[60]是亦神
ナリト謂ヘシ
[61]右立元ノ法則理ニ據テ探リ會スル者トスレハ
大率前說ノ如シト雖必理ニ據テ會ストノ三言
ヘカラス[62]又數ニ據テ會ストノ三モ言ヘカラス
[63]强ニ數理ノ據ヲ得ルニ非サレトモ不測ニ會シ
不識ニ得ノ玄妙有リ[64]卽是據ヲ得テ會スル者ト

13r

全一ニシテ真實ノ至ル時生レ得タル粹質ヨリ
是ヲ會スル者也然ルニ[65]其玄妙ハ立元ノ法耳ニ
[66]非ス淺深難易ヲ不別事事會スルニ於テ悉同一
ナリ[67]如其粹質ヲ不稟者ハ假ヒ筭法ノ限學盡ス
トモ更ニ其真實ヲ識ヘカラス

探約分法　第三

[1]假如有一百六十八分之一百〇五[2]問約之幾何ト○
[3]荅曰八分之五
[4]分母一百六十八ト分子一百〇五トヲ置テ約法
ニヨリ起シテ（[5]約法一ハ原數ナリト雖約ル時ハ分母子トモニ數不動故ニ不用之）

13v

毎ニ一筭ヲ增テ分子ノ數ニ到ル迄逐一ニ約法
トシテ是ヲ約テ驗ルニ或分母子共ニ不整者（[6]約法ニテ分母子ヲ約テ分釐ノ數
ニカカル者ヲ不整トスル也）或分母整テ分子不
整者或分子整テ分母不整者ノ如キハ皆不用之
[7]分母子トモニ整フ者ヲ以テ約法トス[8]然ルニ此
題數ノ如ハ整フ者約法三ト七ト二十一トヲ得
[9]是其三ト七トハ杪ノ數ニシテ兩數相因テ二十
一ノ本數ヲ成者ナルユヘ二十一ノ數一件ヲ取
テ約法トシテ分母子ヲ約ルナリ
[10]如此約法二ヨリ起テ逐上ニ驗テ分母子共ニ

14r

數ノ整フ者ヲ求テ後括術ヲ探ルニ先分子ヲ
以テ分母ヲ除去リ其母餘ヲ以テ却テ分子ヲ
除去リ其子餘ヲ以テ亦母餘ヲ除去リ其餘數
ヲ以テ亦子餘ヲ除去リ是ノ如ク逐テ餘數ヲ
以テ互ニ滿ル者ヲ除キ去テ其分母ト分子ト
餘數相齊キヲ得テ（[11]但最末ニ到テ去盡シテ空トナルヘキ時其所去ノ數
一段ヲ止テ母餘ト子餘ト同數ト爲テ母子各可約ヿヲ見ス也）便約法ト爲ヿ
ヲ會シテ約分互去ノ法術ヲ立ツ
[12]解題本術 [13]置分母一百六十八與分子一百〇五[14]以テ
分子除去リ分母母餘六十三[15]以テ母餘除去リ分子子餘

10v

會スルノ玄妙神ナリト謂ヘシ
[32]又物若干有ヲ若干人ニ支テ毎人ノ支物幾何ト
問如キハ物數ヲ實ニ置人數ヲ法ニ置テ除法ヲ
用テ毎人ノ支物ヲ得ル者常ノ術ナリ◯[33]新術ニ
依テ實級ヲ空シテ方級ニ一算ヲ置テ假ノ毎人
支物ト名ツケ人數ヲ乘シテ假ノ物數トス[34]眞ノ
物數ヲ用テ相消スルトキハ自然ニ除ヲ用ヘキ
實法二級ノ式ヲ得テ卽商除ノ法ニ依テ毎人ノ
支物ヲ得ル
[35]人數ヲ以テ物數ヲ除テ卽一人ニ就テ支物ヲ

11r

得ハ理ノ當然タリ然ニ[36]立元ノ法ヲ用ル時ハ
不容易ニ似タリト雖除スヘキ理ヲ察ルヿヲ
不用自然ニ可除式ヲ得ヲ奇ナリトス
[37]亦直積若干有テ長平ノ差ヲ云テ長平ヲ問如キ
者ハ古法ニ直積ヲ四因シテ長平ノ差冪ヲ加ヘ
開平方ニシテ長平ノ和ヲ求メ差ヲ以テ和ヲ減
シテ餘折半シテ平ヲ得差ヲ以テ平ニ加テ長ヲ
得ル◯[38]又直積ヲ實ニ置キ長平ノ差ヲ方ニ置キ
一ヲ廉トシ開平方ニシテ平ヲ得ル者是從方ニ
數ヲ帶ルヲ以テ帶從開方ト號ス◯[39]新術ニ依テ

11v

實級ヲ空シテ方級ニ一ヲ置テ假ノ平ト名ツケ
差ヲ以テ假ノ平ニ加テ假ノ長トシ假長ト假平
ト相乘シテ假積トシ眞積ヲ用テ相消スル時ハ
自然ニ方ニ開ヘキ帶從ノ式ヲ得ルナリ
[40]古法ノ如キ直積ヲ四方ニ圍中央ニ長平差冪
ヲ容テ長平ノ和冪ノ圖ヲ造開平方ニシテ先
長平ノ和ヲ求ルヿ此題ニ於テハ其理速也ト
雖總テ如此理ヲ以テ究ントスレハ少キ難題
ニ於テモ心ヲ役シテ猶理ノ察ヘキ所無竟ニ
索數ノ術ヲ成ヿ不能[41]今立元ノ法ハ其理幽ク

12r

隱ル者ト雖速ニ術ヲ得ノ神法タリ[42]然レトモ
不探シテ直ニ得ルニ非ス[43]純因乘加減ノ理ヲ
以テ探リ探テ其度ヲ得者也
[44]解題演段術 [45]立天元一為平。〡[46]用減長平和餘為
長𝍫𝍦〡[47]以平乘之為直積。𝍫𝍦〡[48]寄左[49]置積與寄左
相消得度𝍠𝍰〇𝍫𝍦〡[50]平方開之得平一十二步[51]減和得
長一十五步也[52]本術畧之
[53]理ニ據テ術ヲ索ル者見伏難易ヲ不言[54]立元ノ
法ヲ用ルトキハ其隱微ヲ不索得ト云ヿナシ
[55]乃其法理ヲ探ルヿハ千變萬化ナリト雖事ヲ

08v

然モ假ニ據ヲ以テ會スルノ一端ヲ釋シテ探
索ノ義ヲ呈ス
[6]假如有直積一百八十步[7]長平和二十七步[8]問長平
各幾何◯[9]荅曰平一十二步◯長一十五步
[10]タトヘハ積有テ平方ニ開ク者ハ積ヲ實級ニ置
方級空ニシテ隅級ニ一ヲ置キ三級ニシテ平方
ニ開ク[11]積有テ立方ニ開ク者ハ積ヲ實級ニ置キ
方廉ノ二級皆空ニシテ隅級ニ一ヲ置キ四級ニ
シテ立方ニ開ク[12]積有テ三乘方ニ開ク者ハ積ヲ
實級ニ置キ方廉廉ノ三級皆空ニシテ隅級ニ一

09r

ヲ置キ五級ニシテ三乘方ニ開ク[13]如此其乘數ノ
多キニ從テ階級降リ増ヲ據トス[14]凡開方ノ實級
ノ數ヲ開キ盡者ハ同名ヲ減スルニ非ス[15]皆同加
異減ノ理ナルヿヲ會ス[16]然レハ實級ノ數負ナル
時ハ隅級ハ必正ナリ[17]正負交リ備ルヱヘ自然ニ
開盡ルナリ[18]亦得ル所ノ方面ヲ自乘シテハ平方ノ
積ニ還リ再自乘シテハ立方ノ積ニ還リ三自乘
シテハ三乘方ノ積ニ還ルハ技ノ常ナリ[19]本ヨリ
眞數ハ皆常ニ實級ニ置ヘキヿヲ意テ實級ヲ空
シテ方級ニ一筭ヲ置テ可求方面ニ名ヲ假テ其

09v

假方面ヲ以テ自乘スルトキハ實方ノ二級空ニ
シテ一ノ數隅級ニ降テ總テ三級ト成[20]是平方ノ
假積ナリ[21]又假方面ヲ再自乘スルトキハ實方廉
ノ三級空ニシテ一ノ數隅級ニ降テ總テ四級ト
ナル[22]是立方ノ假積ナリ[23]又假方面ヲ三タヒ自乘
スル時ハ實方廉廉ノ四級空ニシテ一ノ數隅級
ニ降テ總テ五級トナル[24]是三乘方ノ假積ナリ[25]卽
是ニ依テ立ル所ノ方級ノ一筭乘ヲ累ルニ從テ
下降テ隅級ヲ成ヿヲ會ス[26]於此眞ノ積數ヲ以テ
假積ト相激消スルトキハ理ニ於テ當ニ空ト成

10r

ヘシトイヘトモ眞假ノ異ヲ以テスルヱヘ數ニ
於テ空ト成ヿ不能シテ實級ニ積數ノ負ナルヲ
止メ他級皆空ニシテ隅級ニ一ノ數ノ正ナルヲ
止ム[27]（或ハ實ノ數正トスルトキハ隅ハ負ナリ）[28]是自然ニ方ニ開クヘキ
全キ式ヲ成ス者ナリ[29]是ヨリ商ヲ立テ隅級ヨリ
乘シ昇テ實級ニ到テ同加異減シテ開盡ス[30]其得ル
所ノ商ハ便所求ノ方面ナリ
[31]是實級ヲ空シテ方級ニ一筭ヲ置テ可求者ニ
名ツケ眞假ノ同類ヲ求テ相消スルトキハ自
階級ニ隨テ應開方全キ式ヲ成得ルヿヲ探リ

06v

數ヲ實ニ置人數ヲ法ニ置テ初商二斛ナルヲ
察シテ釋九數ノ法ノ辭ヲ誦シテ法ニ因シテ
二六一十二ヲ實ニテ去又次商六斗ナルヲ察
シテ法ニ因シテ六六三十六ヲ實ニテ去盡ス
ヿヲ會シテ商除ノ法ヲ立ツ
[56]亦法實各一ヨリ九ニ到ル數ヲ取テ一[57]但シ法數ハ不取
又實數ノ法數ヨリ者ハ不取ナリ各法ヲ以テ各實ヲ除商ト
及餘トヲ置列ル數ヲ造リ設テ九歸除法ノ辭
トス[58]是ヲ心ニ誦シ諳シテ實數ヲ法數ニ從テ
六一下加四逢六進一十トシテ初商二斛ヲ得

07r

又六三添作五逢六進一十トシテ次商六斗ヲ
得ルヿヲ會シテ九歸除法ヲ立ツ
[59]解題本術[60]置元粟數爲實[61]以人數爲法除[62]之得每人
分粟也
[63]商除及九歸除法ハ不探シテ直ニ得ニ似タリ
ト雖實ハ直ニ得ニ非ス[64]探ヿヲ一旦ニスル也
[65]是首ヨリ其法ヲ會シ難シ[66]碎抹スルヿヲ用テ
後括求ルヿヲ探會シテ法ノ辭ヲ造用ユ
[67]右因乘歸除ノ二法ハ皆碎テ數ヲ求メ理ニ據テ
法則ヲ探會スル者ナリ[68]最簡易ナルユヘ法術ノ

07v

儘ニ數ヲ求テ伏タル處無ク其理顯然タリ
[69]凡筭法ハ理ヲ察シ數ヲ求ルニ止ル[70]數ヲ求ル
ヿハ碎抹ニ本ツキ術ヲ施スハ理ヲ察スルヲ
要トス[71]二ツノ者本相因テ法ヲ成ス[72]然レトモ
理ヲ以テ極盡サントシテ不可得[73]必事ニ滯ヿ
有リ[74]數ヲ以テ究盡サントシテ不可得[75]必理ニ
惑ヿ有リ[76]何者如術ノ順逆ヲ不料シテ萬術皆
碎抹シ求ントスレハ筭ノ法則有ルノ功無ク
却テ順ノ術ニ於テ停滯シテ不得有リ[77]又不探
シテ悉理ニ從テ直ニ得ントスレハ逆ノ術ニ

08r

於テ理ノ據無シテ會シ難シ[78]故ニ術ノ順逆ヲ
料リ數理ノ據ヲ詳ニシ形質ニ順テ法ト數ノ
盡ルト不盡ヲ察シテ探ヿシ玄セハ不可會ノ
法無ク不可得ノ數無ランヿ乎

探立元法　第二

[1]立元ノ法ハ何レノ代ニ始ルヿ未知[2]元ノ至元
年中ニ郭守敬カ授時暦ヲ修ルニ此法ヲ用ユ
[3]同代大德年中ニ朱世傑カ所選筭學啓蒙ニ具
ニ其法ヲ說解セリ[4]是素數ノ術ヲ得ルノ神法
タリ[5]此法ヲ會セルノ玄妙說言ヘカラスト雖

04v

シテハ合テ一百六十二錢ナリ[11]又七斛ニシテハ
合テ一百八十九錢也[12]又八斛ニシテハ合テ二百
一十六錢也[13]又九斛ニシテハ合テ二百四十三錢
也[14]又十斛ニシテハ合テ二百七十錢ナリ[15]又十一斛
ニテハ合テ二百九十七錢ナリ[16]其十二斛ニシテ
合テ三百二十四錢ヲ得ル[17]該價銀トス
[18]如此碎累テ[19]（是俗ニ云フ目之子算也）眞數ヲ得テ後括術ヲ
探ルニ一ヨリ九ニ到ル單數ヲ取テ一一ヨリ
九九ニ到テ總テ四十五个ノ合數ヲ求メ爲テ
釋九數ノ法ノ辭トス[20]是ヲ誦シ諳シテ粟一十

05r

二斛ヲ置斛ノ價銀二十七錢ヲ以テ先一十二
因シテ一二如二百錢一七如七十錢トシ次ニ
二斛ニ因シテ二二如四十錢二七一十四錢ト
スル時ハ一般ニシテ該價銀三百二十四錢ヲ
得ルヿヲ會シテ因乘ノ法ヲ立ツ
解題本術[21] [22]置粟數以[23]斛價銀乘之得該價銀也
[24]是不碎累シテ直ニ得ニ似タリト雖實ハ直ニ
得ルニ非ス[25]累ルヿヲ一旦ニスルナリ[26]凡碎抹
スルハ得數ノ本法則ヲ立ルハ施術ノ原ナリ
[27]故ニ算ハ法則ヲ立ルヲ以テ要トス

05v

[28]歸除　[29]假如有粟一十五斛六㪷[30]令六人分之問[31]毎
人分粟幾何○[32]荅曰毎人二斛六㪷
[33]一人ニ分粟一斛トスル時ハ六人ニシテ六斛也
[34]元粟ヨリモ少シ[35]又一人分粟二斛トスルトキハ
六人ニシテ一十二斛ナリ[36]猶元粟ヨリモ少シ[37]又
一人三斛トスルトキハ六人ニシテ一十八斛也
[38]却テ元粟ヨリ多シ[39]故ニ毎人二斛有餘ナルヿヲ
知ル[40]卽其二斛六人ニシテ一十二斛ヲ以テ元粟
ヲ去テ餘三斛六㪷有リ[41]亦一人一㪷トスル時ハ
六人ニシテ六㪷也[42]餘斛ヨリモ少[43]又一人二㪷ト

06r

スル時ハ六人ニシテ一斛二㪷ナリ[44]餘斛ヨリモ
少シ[45]又一人三㪷トスルトキハ六人ニシテ一斛
八㪷也[46]餘斛ヨリモ少シ[47]又一人四㪷トスル時ハ
六人ニシテ二斛四㪷也[48]餘斛ヨリモ少シ[49]又一人
五㪷トスル時ハ六人ニシテ三斛也[50]餘斛ヨリモ
猶少シ[51]又一人六㪷トスル時ハ六人ニシテ三斛
六㪷ナリ[52]餘斛ト合ス[53]故ニ六㪷ニ整フヿヲ知ル
[54]卽其六㪷六人ニシテ三斛六㪷ヲ以テ餘斛ヲ去
盡シテ二斛六㪷ヲ以テ毎人ノ分粟トス
[55]如此碎探テ眞數ヲ得テ後括術ヲ探ニ元粟ノ

02v

03r

目録

採法則 四條
乘除 據理採法 第一
立元 據理採法 第二
約分 據數採法 第三
招差 據數採法 第四
採術理 四條
織工 據理採術 第五
直堡 據理採術 第六
算脫 據數採術 第七

03v

球面 據數採術 第八
採員數 四條
碎抹 據理採數 第九
開方 據理採數 第十
圓數 據數採數 第十一
弧數 據數採數 第十二
自質說 一條

04r

綴術算經
法則 四條
[1]因乘 [2]採乘除法 第一
假如有粟一十二斛[3]每斛價銀二十七錢[4]問
該價銀幾何◯[5]荅曰該價銀三百二十四錢
[6]每一斛ノ價銀二十七錢ナルユヘ二斛ニシテハ
二ツノ價ヲ合テ銀五十四錢也[7]又三斛ニシテハ
三ツノ價ヲ合テ銀八十一錢也[8]又四斛ニシテハ
四ツノ價ヲ合テ銀一百八錢也[9]又五斛ニシテハ
五ツノ價ヲ合テ銀一百三十五錢ナリ[10]又六斛ニ

01r

綴術算經自序
[1]綴術ハ綴テ探索テ術理ヲ會シ得ル者也[2]凡探索
ノ方理ニ據ル者有リ[3]又數ニ據ル者有リ[4]探ルヿ
一件ニシテ術理ヲ不會ハ二件ニシテ探ル[5]二件
ニシテ不會ハ三件ニシテ探ル[6]若術理深ク潜伏
[illegible]トモ探ルヿヲ數般ニスル時ハ熟スル期到テ
覺ニ不探會ト云ヿナシ[7]然ルニ其潜伏スル者ト
雖一旦ニシテ即探リ得ル有リ[8]又簡易ナル者ト
雖數般ニシテ徐探リ得ル有リ[9]蓋人質純粹ナル
者ノ有ヿ無シ[10]稟ニ敏魯有テ共ニ皆常ナルヿ不能

01v

[11]以是時トシテ屈伸有テ伸トキハ通シ屈トキハ
滯ル[12]故ニ會ルニ遲速遲利ノ異ヲ爲ス耳[13]夫筭ハ
法則ヲ立術理ヲ究員數ヲ計ヲ以テ事トス[14]其事
タル也理ヲ察シテ術ヲ施シ術ニ依テ數ヲ得ル
者ハ順也[15]數ニ從テ術ヲ課リ術ニ滯テ理ヲ索ル
者ハ逆也[16]其逆順皆綴術ニ貫セリ[17]故ニ探テ法則
ヲ可立探テ術理ヲ可察探テ員數ヲ可計[18]仍テ法
術數ノ三等ヲ立テ數理ノ據ヲ別チ一十二條ノ
術例ヲ擧テ探索ノ大意ヲ述綴術ノ證トス[19]且吾
生得質分ノ偏駁ハ自然ニシテ實ニ不可變ヿヲ

02r

説テ此書ヲ爲レル所以ヲ著ス[20]隋史ヲ按スルニ
祖冲之所著之書名爲綴術[21]學官莫能究其深奧[22]是
故廢而不理ト云リ[23]吾適ニ彼綴ノ一字ヲ採用ニ
到テ熟思ニ冲之ハ是上古ノ達人ト謂ヘシ[24]蓋其
玄妙ノ眞實ハ聽テ不可識思テ不可得者乎
[25]享保七歲次壬寅孟春七日
[26]武陽　江城　匾耆士不休書

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