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An Arabic Algebraic Compendium of 1000 CE

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Abstract

The manuscript Mashhad Astan Quds 5325, containing the only copy of an anonymous treatise going back to CE 1004/5 (395 H), presents the elements of algebra required in order to solve problems of application. It differs from previous treatises in that it has an elaborate theory on arithmetical operations involving numerical roots (including, in addition to square roots, cube and fourth roots). It also gives geometrical demonstrations of the operations and formulae for solving second-degree equations, and anticipates higher-degree equations, which were to be dealt with geometrically a century later by ʻUmar Khayyām.

I Introduction

I.1 Generalities

The historian and philosopher Ibn Khaldūn wrote in his *Muqaddima* that the first two authors treating algebra (in Islamic times) were al-Khwārizmī (c. 820) and then Abū Kāmil (c. 890).[1] In Khwārizmī's largely accessible (and probably not very original) *Short Account of Algebra* are already found what were to be the three main characteristics of early mediaeval algebra.

First, and unlike in the Greek algebra of Diophantus, there is a *complete absence of symbolism.* Everything, including numbers, is expressed in words. Only a few words, such as those for the powers of the unknown, have a specific meaning in algebra: "thing" (*shay'*) is our x (sometimes also *jidhr*, "root"), "amount" (*māl*) is x^2, "cube" (*kaʻb*) is x^3. In later authors the higher powers are expressed, as were the Greek ones, by combining the words for x^2 and x^3.[2]

A second characteristic of mediaeval algebra is the *recourse to geometrical figures* to illustrate the rules of algebraic reckoning or the solving formulae for equations. In that sense, algebra can be said to have not yet fully gained autonomy; geometrical

[1] Edition of the Arabic text by Quatremère (1858, III, 98); translation by de Slane (1868, III, 136–137).

[2] Since any positive integer $N \geq 2$ may be represented in the form $2n_1 + 3n_2$ (n_1, n_2 not negative integers), any power x^N may be expressed by repeating n_1 times the word for x^2 and n_2 times the word for x^3.

proof was to remain, for centuries in fact, the criterion of mathematical truth in algebraic relations.

A third characteristic, which was of ancient origin and, like the second one, to last until the Renaissance, is the reduction of the (then) algebraically solvable equations *to six specific types* with positive coefficients and at least one positive solution, namely the three equations called "simple" (*mufrada*), which are $ax^2 = bx$, $ax^2 = c$, $bx = c$, and the three equations called "compound" (*muqtarana*), which are $ax^2 + bx = c$, $ax^2 + c = bx$, $ax^2 = bx + c$, the latter mostly found in their reduced forms ($x^2 + px = q$, $x^2 + q = px$, $x^2 = px + q$).

The geometrical figures used to justify the formulae of the compound equations may be different in nature. Khwārizmī's figures are mere illustrations and do not require knowledge of Euclid's *Elements of Geometry*, the basic mathematical tool in ancient and mediaeval times. (By the way, although the use of geometrical figures suggests a Greek influence, Khwārizmī does not mention Euclid at all.) The same holds for his contemporary Ibn Turk. Abū Kāmil has, on the other hand, two kinds of illustration: one is similar to his predecessors' but in the other Euclid is mentioned and reference made to the two theorems *Elements* II.5 and II.6, of which this second kind of illustration is a direct application. That Euclid's name and theorems should appear in Abū Kāmil's *Algebra* but not in Khwārizmī's is, by the way, hardly surprising: Khwārizmī's treatise is elementary and does not suppose any prerequisites in (the then) higher mathematics, whereas Abū Kāmil's *Algebra* is written specifically for mathematicians, that is to say, people trained in the study of Greek mathematics, chiefly Euclid's *Elements*. Note that the demonstrations using *Elements* II.5 and II.6 are also found in a short text by Thābit ibn Qurra (836–901) (Luckey 1941).

The purely illustrative figures, as well as those based on Euclid's theorems II.5 and II.6, are used to explain the general formulae of compound equations; but they do not represent graphically the solution of a specific equation since the length x has been set to begin with. The *Elements* of Euclid, however, serve in addition to actually draw the solution and represent it as a segment of a straight line. To do so, three theorems of the *Elements* are used. The first, auxiliary one is the construction of the root of a given quantity (that is, the root of a given segment of a straight line). Suppose the given length to be a (Fig. 1). We add to it the unit segment and describe the circle with diameter $a + 1$. The height at the extremity of a is then $\sqrt{a}$. This construction, an application of the theorem of the height in a right-angled triangle, is *Elements* II.14.

The other two theorems are *Elements* VI.28–29, which teach one how to construct ("apply," παραβάλλειν) on a *given* segment of a straight line a rectangle (generally, a parallelogram) equal to a *given* rectilineal figure but, relative to the segment of a straight line, in excess or deficit by a square. To use these theorems, the three

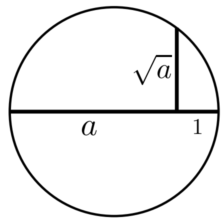

Figure 1: Geometrical construction of the square root of a given segment

equations are considered in the form of products:

$$x^2 + px = q \longrightarrow x(x + p) = q,$$
$$x^2 + q = px \longrightarrow x(p - x) = q,$$
$$x^2 = px + q \longrightarrow x(x - p) = q.$$

Since, in the treatise we shall examine, this construction displays only the segment of a straight line, thus without representing the rectangle and the square, here we have supplied these elements.

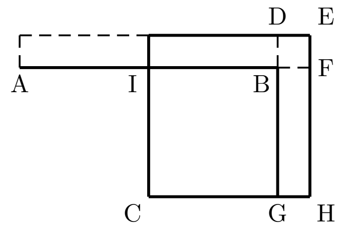

Figure 2: For the equations $x^2 + px = q$ and $x^2 = px + q$

Consider first the equation $x^2+px = q$, with p and q thus given positive quantities. Let us draw AB $= p$, and let I be its midpoint (Fig. 2). So AI $=$ IB $= \frac{p}{2}$, and we construct on IB the square CB $= (\frac{p}{2})^2$.[3] On the base CG of this square, we describe the larger square CE $= (\frac{p}{2})^2 + q$, which we know since we know the quantity $(\frac{p}{2})^2 + q$ and can thus represent it as a segment of a straight line, of which we may then take the root as seen above. The applied rectangle is then AE and the solution of our equation is BD $=$ BF. Indeed, the applied rectangle AE, being $x(x + p)$, equals q, which is also the sum of the areas ID $+$ DF $+$ FG. Furthermore, as we see, this known area exceeds AD, the rectangle on the given straight line AB, by a square

[3] In Greek and Arabic texts, rectangular figures are often designated by the letters at opposite angles.

area, namely BE. We may observe that the given number, q, which is the sum of the areas ID, DF, FG, forms a gnomon around the square $(\frac{p}{2})^2$.[4]

For the equation $x^2 = px + q$, the construction is the same. But this time the solution x is the segment of straight line AF. For, since BF = FE, we have indeed $\text{AF} \cdot \text{BF} = x(x - p) = q$, and the square in excess is BE.

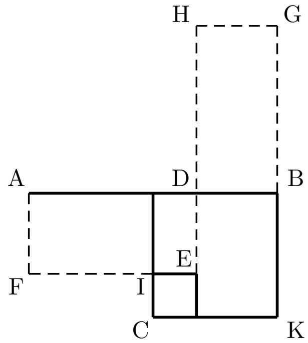

Figure 3: For the equation $x^2 + q = px$

Consider next the equation $x^2 + q = px$. Once again, we put (Fig. 3) AB $= p$ and describe CB, the square on its half. Next, we construct the smaller square CE $= (\frac{p}{2})^2 - q$. The difference between the squares CB and CE is q, which is thus the sum of the two areas ID and DK, or also, since the two rectangles AI and DK are equal, the sum of ID and AI, thus AE. In that case, two applied rectangles $x(p - x) = q$ fulfil the condition: AE, corresponding to the solution DE = DB $= x$; and DG, equal in size to the previous rectangle, corresponding to the solution AD = DH $= x'$. The deficient squares are then, respectively, EB and AH.

I.2 Description and Contents of the Manuscript

The two kinds of figure we have mentioned, as well as the geometrical construction of the solutions, are found in an anonymous treatise written in 1004/5 (395 of the hegira, see the colophon) and extant in a copy made in 1185 (581 of the hegira, see title page), namely MS Mashhad Astan Quds 5325. I was able to examine this manuscript several times when, in the years 1985–1990, my Iranian colleague A. Djafari Naini and myself were visiting the main libraries in Iran. This manuscript, edited here completely, was then used by me in three studies dealing with the treatment of quadratic equations in Arabic algebra.[5]

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[4] A gnomon (γνώμων) is the figure left when from a parallelogram (here a square) a similar figure has been taken from its corner. See *Elements* II, def. 2.

[5] See Sesiano (1999, 83–85; 2002, 193–201; 2009, 79–81).

The manuscript comprises 23 leaves 17.1×6.3 cm in size, with the text taking up 14.8×4.7 cm, and 28 lines on every page except for the first with 4 lines and the last, 25; it reached the library as an endowment (*wafq*) made in 1067 H by a certain Ibn Khātūn. The copy is in excellent condition, in a good hand (*naskhī*); in a few places, however, the paper has been torn and gummed together with (opaque) sticky tape. Headings of chapters or sections are sometimes in bold script, but mostly in red ink; with some being inappropriate, perhaps because an older copy omitted them (altogether, if not partly) since they were supposed to be added later, at the time of the rubrication.[6]

The first leaf of the progenitor has been lost, as mentioned after the title (fol. 1^{r}: "some parts are missing from the beginning of this copy"), and repeated, this time in Persian, by a modern hand, at the top of fol. 2^{r} ("the beginning of this treatise is missing"); fol. 1^{v} is blank (excepting additions by librarians). The title on the actual fol. 1^{r} is a later addition, which is inadequate. First, as pointed out to me by a referee, it alludes to numerical (application) problems, whereas the text itself says explicitly that it will not deal with that topic (see translation, A. 862–863). Second, the (presumed) title is repeated at the end: *Foundations of Algebra and Aspects of the Simple and Compound Equations on which Are Based the Kinds of Numerical Problems Subject to Exact General Procedures* (A. 860–861). The loss of the original first leaf may explain why the name of the author has disappeared. Note that this author does not give himself credit for anything in the text, and we cannot even guess his identity. Whatever the case, he is obviously very competent, and gives us a fine picture of the state of algebra around 1000 CE.

There are no traces left by readers. But it appears that an earlier copy had some marginal remarks, now wholly incorporated into the text. We have bracketed most of them.[7] Two main early readers were particular active. Traces of the first one are numerous in the first pages, with attempts to draw a parallel between operations with powers of the unknown and arithmetical operations with fractions; then this reader calmed down for he realized that our treatise was too advanced for him. Another interpolator intervened in the last chapter, on equations (see note 248); he did not make any interesting comments either.

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[6] See notes 88, 93, 203, 264, below.

[7] See the following lines of the Arabic text (and footnotes in the translation): 7–8/*n.* 24, 10 & 12/*n.* 25, 14–18/*n.* 26, 19–22/*n.* 28, 33–37/*n.* 34, 56, 70, 73–74/*n.* 45, 95–100 & 111-114/*n.* 50, 105–106, 124, 150, 152–153, 234–237/*n.* 101, 240/*n.* 104, 241, 260, 298–299/*n.* 128, 408, 410 & 412/*n.* 170, 421 & 422/*n.* 170, 429, 436 & 441/*n.* 170, 507–511/*n.* 206, 567, 569, 625, 626/*n.* 234, 629–630/*n.* 235, 636/*n.* 238, 639/*n.* 240, 654/*n.* 245, 661/*n.* 248, 665–666/*n.* 251, 699/*n.* 262, 733–734/*n.* 275, 768/*n.* 290, 777/*n.* 294, 791–798/*n.* 298 & (included) 795–797/*n.* 301, 811/*n.* 306, 839/*n.* 316, 850/*n.* 321.

Lacunae are relatively rare, and we have enclosed them in angular brackets.[8] In an earlier copy, rectifications were sometimes indicated in the margin but later copied in the wrong place; see 104–106/*n.* 51, 639/*n.* 240, 777/*n.* 294. Uncorrected places are rare.[9] Finally, the correction of a few mistakes or omissions bears witness to a copyist who carefully verified his copy.[10] But he did not, or not always, follow the computations (e.g. A. 370). A few comments on the Arabic will be found in footnotes.[11]

The manuscript Mashhad Astan Quds 5325 is described in Gulchīn-Maʿānī's eighth volume of the catalogue of the mathematical manuscripts in the Mashhad Shrine Library (*Fihrist* 1971, No. 146). According to an earlier catalogue, this رساله در جبر و مقابله is a copy of Abū Kāmil's Algebra (*Fihrist* 1926, III, No. 98), an attribution already invalidated by the date of its composition. S. Chalhoub, who edited the main part of Abū Kāmil's Algebra (Chalhoub 2004), repeats this and adds a photograph of the first two and last two pages of the Mashhad manuscript (fol. 1^{r} (title) & fol. 2^{r}; fol. 23^{v} & fol. 24^{r}); by some extraordinary oversight he did not notice that the text on these three pages does not correspond to any passage of what he was editing. Chalhoub also provided a German translation of Abū Kāmil's treatise, which in fact reproduces the translation of the *Hebrew* version of Abū Kāmil's Algebra (Abū Kāmil 1935).[12]

Let us now summarize the contents of the treatise. It is divided into four parts, each containing several paragraphs. As said, only one leaf appears to be missing;

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[8] Lines 79, 84, 164, 220, 290, 322, 359, 370, 390, 451, 457, 475, 477, 483, 504, 574, 612, 624, 626, 635, 638, 640, 726–727, 741, 743, 755, 767, 769, 776–777, 779, 786/*n.* 297, 815, 830.

[9] Lines 53/*n.* 37, 104/*n.* 51 (see above, "rectifications"), 424/*n.* 175, 475, 490–491/*n.* 199, 625–630/*n.* 233, 768/*n.* 290 (see above, "incorporated interpolations"), 786/*n.* 294 (see above, "lacunae"), 850/*n.* 321.

[10] E.g. words corrected after erasure (e.g., see MS, *al-niṣf* (*post.*) and *al-maqādīr*, line 18), or corrected above the line (*ll.* 700 & 796) or just added (*ll.* 782, 783, 808, 809); a word originally written twice (*l.* 138) has been crossed out in red, thus at the time of rubrication.

[11] See below, notes 35, 37, 41 & 46, 47, 84, 85, 94–96, 100, 109, 117, 165, 182, 187, 213, 214, 225, 243, 247, 255, 262, 265, 270, 290, 295, 299, 302, 311, 314, 319.

[12] On the works of Abū Kāmil, and the real or alleged mediaeval Latin and Hebrew translations, see our additions to the reprint of A. Anbouba's biography of Abū Kāmil, following the edition of Abū Kāmil's practical geometry (Anbouba 2014). Note that in what follows incidental references to Khwārizmī's Algebra will be to the pages of Rosen's 1831 edition (al-Khwārizmī 1831, translation/text); for Abū Kāmil's *Algebra*, the references are, for the Arabic, to the folio of the manuscript printed in facsimile (Abū Kāmil 1986), with fol. z^{r} of the MS corresponding to p. $2z - 1$ of the facsimile—in Chalhoub's edition (Abū Kāmil 2004), 2/3 means the separation between fol. 1^{v} and fol. 2^{r}), for the Latin translation to the initial line in our edition (Abū Kāmil 1993), for the Hebrew translation to the page of Levey's (not always reliable) edition (Abū Kāmil 1966).

it must have dealt with the first two powers of the unknown, characterized by the proportion $1 : x = x : x^2$. Now, at the beginning of his (later, end of the 11th-century) *Algebra*, ʿUmar K͟hayyām introduces the powers of the unknown as follows:

> It is usual among the algebraists in their art to call the unknown which is to be determined "thing," its product into itself "square" (lit. "amount," *māl*), the product of its square into the thing "cube," the product of its square into itself "square-square," the product of its cube into its square "square-cube," the product of the cube into itself "cube-cube," and so on. It is known from the work of Euclid on the *Elements* that all these powers are in continued proportion, that is, the unit is to the root as the root is to the square and as the square is to the cube; therefore, the number is to the roots as the roots are to the squares, as the squares to the cubes, as the cubes to the square-squares, and so on.[13]

This is an allusion to the definitions 18 and 19 of Book VII of the *Elements* and Proposition 8 of Book IX, where the basic powers, x^2, x^3, are defined and the continued proportion $1 : x = x : x^2 = x^2 : x^3 = \dots$ is set. Now the subject of the missing first paragraph of our treatise, as confirmed by its remaining part, was to define the first two powers of the unknown using the above proportion. Note, finally, that the missing part might be less than the two sides of a leaf: readers' remarks then incorporated into the text may have been numerous (marginal readers' remarks are particularly abundant at the beginning of treatises).

Apart from the first leaf, the extant treatise is complete. In its first part (fol. 2^r – 4^v), the reader is taught the usual Arabic denominations of the first two powers of the unknown: thus (§ 1), as said, *number*, *thing* or *root* (our x) and *square* (x^2); next (§ 2) the *cube* (x^3) as the product of the last two. From the names "square" and "cube" are then formed the next powers: *square-square*, *square-cube*, *cube-cube* (§ 3). This just follows the Greek system as used by Diophantus and defined in the introduction to his *Arithmetica*.[14] After expounding the divisions of these powers among themselves (§ 4), whereby are introduced the inverse powers of the unknown (with the same denominations as the previous ones, but preceded by "part of"), the reader is taught how to multiply these inverse powers (§ 5) and (§ 6) divide them.

The second part (fol. 4^v – 6^v) considers the operations with binomial expressions consisting of a number and some multiple of the unknown (our x). Pairs of such expressions are successively added (§ 1), subtracted (§ 2), multiplied (§ 3), divided (§ 4, with the divisor restricted to a single term). Since the sign before each term may

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[13] Woepcke (1851, 6–7/4). The "number," that is, m instead of 1, thus $m : mx = mx : mx^2 = \dots$. This (for us banal) distinction will also occur in our treatise.

[14] On the Greek system and its adaptation in Arabic texts, see the edition of the Arabic Diophantus (Sesiano 1982, 43–46).

vary, we learn how to deal with positive and negative coefficients and, in the case of multiplication, we are taught the rule of signs. The identity $(u-v)^2 = u^2+v^2-2\,u\cdot v$ is demonstrated geometrically, as will be many identities and formulae subsequently. As observed earlier, this is a characteristic of mediaeval algebra.

Much attention is devoted in the third part (fol. $6^{\text{v}}-16^{\text{r}}$) to computation with numerical roots. First (§ 1) how to take multiples of square, cube, fourth roots, thus how to raise the factor to the appropriate power in order to bring it under the root. Since just the same applies to taking the fraction of a root, the treatment is shorter (§ 2). The addition of roots (§ 3) is then explained for square and cube roots, and the relevant identities, namely

$$\sqrt{u}+\sqrt{v} = \sqrt{u+v+2\sqrt{u\cdot v}} \text{ and}$$
$$\sqrt[3]{u}+\sqrt[3]{v} = \sqrt[3]{u+v+\sqrt[3]{27\,u^2\cdot v}+\sqrt[3]{27\,u\cdot v^2}},$$

are explained and demonstrated geometrically.[15] The same is done for subtraction (§ 4), thus with the corresponding identities

$$\sqrt{u}-\sqrt{v} = \sqrt{u+v-2\sqrt{u\cdot v}} \text{ and}$$
$$\sqrt[3]{u}-\sqrt[3]{v} = \sqrt[3]{(u+\sqrt[3]{27\,u\cdot v^2}\,)-(v+\sqrt[3]{27\,u^2\cdot v}\,)}.$$

(Their descriptions are good instances of verbal algebra, somewhat difficult to follow for a modern reader.) We are then taught the multiplication of square, cube and fourth roots, between them or among them, sometimes with a coefficient, and the basic relation ($\sqrt{u}\cdot\sqrt{v} = \sqrt{u\cdot v}$) is demonstrated geometrically (§ 5). This third part ends with the more restricted case of division (§ 6); division of monomial or polynomial expressions by a single square, cube or fourth root follows a path analogous to that for multiplication; though not so when there is in the divisor a polynomial expression, as pointed out by the author: this is possible, according to him, in just one instance, namely if the divisor is the sum of a number and a square root, whereby we may, using multiplication as a device (*ḥīla*), change the divisor into a rational quantity.

The fourth part (fol. $16^{\text{r}}-23^{\text{v}}$) is entirely devoted to first and second-degree equations. Since, as mentioned above, only positive coefficients and solutions are considered in both ancient and mediaeval times, there are traditionally three forms of simple and compound equations of the first two degrees: equality between two terms in the first case, between one term and the other two in the second.[16] For the "simple" (binomial) ones (§ 1), numerical examples are given. For "compound"

[15] In the second case the multiplicative factor is thus included in the root.

[16] Whence, with positive coefficients throughout, $bx = c$, $ax^2 = c$, $ax^2 = bx$; $ax^2+bx = c$, $ax^2+c = bx$, $ax^2 = bx+c$.

(trinomial) equations, the author gives (§ 2) the solving formulae for the three kinds, applies each of them to a numerical example, then explains each formula, first by an illustration then by constructing the segment of a straight line corresponding to the solution; but in the latter case, as previously said, without the given squares and rectangles being actually drawn.

It is interesting to note that these elements of algebra, as described in Part I and Part II of the present treatise, correspond exactly to the necessary background already described in antiquity by Diophantus in the introduction to his *Arithmetica*. Indeed, after defining the powers of the unknown, he proceeds with their multiplication, then explains the multiplication of inverse powers, either among themselves or with the powers already defined, then the rule of signs, and concludes:

> Since the multiplications of the aforesaid powers have been distinctly explained, their divisions are clear. Now it is appropriate that he who wants to go into that should acquire practice in addition, subtraction and multiplication of the various powers, and know how to add up additive and subtractive powers with different coefficients to others, themselves either additive or also additive and subtractive, and how from a sum of additive and subtractive powers others, either additive or also additive and subtractive, are subtracted.[17]

The purpose of our treatise is clearly to serve thus as an introduction to the use of algebra before solving algebraic problems, just as Diophantus's introduction urged the student to familiarize himself beforehand with algebraic reckoning. Our treatise's subjects differ from those mentioned in Diophantus's introduction in Part III, on operating with numerical roots, which is irrelevant for the *Arithmetica* since there the required quantities must be rational. Diophantus then proceeds to explain how the equation resulting in a problem is changed to one containing either two or three different powers, thereby defining the two operations known in Arabic as restoration (جبر) and reduction (مقابلة).

There is at the end of our treatise (fol. $23^{\mathrm{v}}-24^{\mathrm{r}}$) an allusion to higher-degree equations with either three or four terms.[18] Here, for the first time as it seems, the various types of cubic equations with positive terms are all listed (except for the first, banal case, of x^3 equal to a number). ‘Umar Khayyām thought he was the first to

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[17] Καὶ τῶν πολλαπλασιασμῶν σοι σαφηνισθέντων, φανεροί εἰσιν οἱ μερισμοὶ τῶν προκειμένων εἰδῶν (اجناس). καλῶς οὖν ἔχει ἐναρχόμενον τῆς πραγματείας συνθέσει καὶ ἀφαιρέσει καὶ πολλαπλασιασμοῖς τοῖς περὶ τὰ εἴδη γεγυμνάσθαι, καὶ πῶς εἴδη ὑπάρχοντα καὶ λείποντα μὴ ὁμοπληθῆ προσθῇς ἑτέροις εἴδεσιν, ἤτοι καὶ αὐτοῖς ὑπάρχουσιν, ἢ καὶ ὁμοίως ὑπάρχουσι καὶ λείπουσι, καὶ πῶς ἀπὸ ὑπαρχόντων εἰδῶν καὶ ἑτέρων λειπόντων ὑφέλῃς ἕτερα ἤτοι ὑπάρχοντα, ἢ καὶ ὁμοίως ὑπάρχοντα καὶ λείποντα (Tannery 1893, 14).

[18] We have analyzed this part in a commemorative volume on ‘Umar Khayyām (Sesiano 2002).

have compiled such a list.[19] Our author notes that they do not admit of “numerical procedures” (thus formulae) as do the trinomial second-degree equations, but only of a geometrical solution using conic sections. The Greeks had solved them that way in a few cases, some others were added at about the time of our author, and ‘Umar Khayyām completed these attempts to obtain the positive solutions, using circles, hyperbolas, parabolas; see Fig. 4 below.[20]

1.	$x^3 = c$	x_1 positive, $x_{2,3}$ complex	two parabolas
2.	$x^3 + bx = c$	x_1 positive, $x_{2,3}$ complex	circle and hyperbola
3.	$x^3 + c = bx$	$x_{1,2}$ positive or complex, x_3 negative	parabola and hyperbola
4.	$x^3 = bx + c$	x_1 positive, $x_{2,3}$ negative or complex	parabola and hyperbola
5.	$x^3 + ax^2 = c$	x_1 positive, $x_{2,3}$ negative or complex	parabola and hyperbola
6.	$x^3 + c = ax^2$	$x_{1,2}$ positive or complex, x_3 negative	parabola and hyperbola
7.	$x^3 = ax^2 + c$	x_1 positive, $x_{2,3}$ complex	parabola and hyperbola
8.	$x^3 + ax^2 + bx = c$	x_1 positive, $x_{2,3}$ negative or complex	circle and hyperbola
9.	$x^3 + ax^2 + c = bx$	$x_{1,2}$ positive or complex, x_3 negative	two hyperbolas
10.	$x^3 + bx + c = ax^2$	$x_{1,2}$ positive or complex, x_3 negative	circle and hyperbola
11.	$x^3 = ax^2 + bx + c$	x_1 positive, $x_{2,3}$ negative or complex	two hyperbolas
12.	$x^3 + ax^2 = bx + c$	x_1 positive, $x_{2,3}$ negative or complex	two hyperbolas
13.	$x^3 + bx = ax^2 + c$	x_1 positive, $x_{2,3}$ positive or complex	circle and hyperbola
14.	$x^3 + c = ax^2 + bx$	$x_{1,2}$ positive or complex, x_3 negative	two hyperbolas

Figure 4: Khayyām’s solutions of third-degree equations

The present treatise is in general quite clear, and any reader could benefit from studying it. There are, however, two weak points which make the relevant parts confusing.

First, there is the author’s insistence on defining the successive powers of the unknown using the continued proportion $1 : x = x : x^2 = x^2 : x^3 = \dots$. It is thus introduced to justify things which are normally self-evident to any (11th- or 21st-century) reader. See notes 208, 230, 236, 320.

Second, the formulae for solving trinomial second-degree equations apply to those with the coefficient of the highest power equal to 1. Thus the question arises as to how to change the given equation to this canonical form. If, in our terms, the given equation is $ax^2 + bx = c$, we shall just multiply each coefficient by $\frac{1}{a}$ and thus get as the required new form $x^2 + \frac{b}{a}x = \frac{c}{a}$; the computation is merely less simple if a is not an integer but contains a fraction. There is one single example where the multiplication by the inverse coefficient is performed (note 253, below). In the other instances the author uses the method of the false position—which often proves to

[19] See the beginning of his *Algebra* (Woepcke 1851, 3/2).

[20] The algebraic formula for a positive solution of one type of the third-degree equation (case 2 above) was first attained towards the very end of the 15th century—curiously enough by a formula of the kind which had been used in ancient school algebra for solving quadratic equations (see Sesiano 2009, 130).

have been a plague in mediaeval mathematics. The idea is that the coefficient a will be changed to unity by either adding or subtracting from it some fraction $\frac{p}{q}$ of it. Take then some false position α—conveniently chosen (normally so as to cancel the denominator)—and calculate first (considering here the addition of a fraction, thus $a < 1$) $(1 + \frac{p}{q})\,\alpha$, then multiply it by a. Dividing then the result, $a\,(1 + \frac{p}{q})\,\alpha$, by the false position α, we shall obtain $a\,(1 + \frac{p}{q})$, in theory equal to 1.[21] The other coefficients must then be multiplied by $1 + \frac{p}{q}$ as well. See below notes 219–222, 226, 228, 232, 233, thus including binomial equations for which such a transformation is even more absurd.

[21] But 2 in one instance (note 220), for the author adds a fraction instead of subtracting it.

II Translation

Prefatory Note

Parentheses are used for our interventions in order to facilitate reading; brackets enclose presumed interpolations, while angular brackets, as stated (note 8, above), designate presumed lacunae.

The division of the text into four "parts" with paragraphs is perfectly adequate. But, for convenience, we have added references to the lines of the edited Arabic text, e.g. (A. 6–13).

As usual with early Arabic algebraic treatises, everything, including numerals, is expressed in words. One of the referees insisted that all the words in Arabic be rendered by words, as is customary for Arabic literary texts. I agree that this would be justified, both for reasons of coherence and conformity. Considering, however, that the readers, if any, will be people with some training in mathematics rather than Arabists or Classicists, we have refrained from adopting a completely literal translation: mathematicians would just stop reading after a few pages, and nobody could blame them for that. In order, however, to account for the two points of view, we have kept a literal translation for the statements of calculations, but adopted, for the subsequent reckoning, numerals, even sometimes algebraic signs. Indeed, the translation *de verbo ad verbum* would be unpleasant for anyone wishing to have an idea of the substance of the text: he will no doubt prefer to read (A. 535–536) "$5+\frac{5}{9}$ minus the square root of $3+\frac{7}{81}$" than "five and five ninths minus the root of three and seven parts of eighty-one parts of one"; likewise, "the ratio 1 : 2" will stand for "the ratio of one to two"; likewise, *māl* for the second power of the unknown will be translated by "square" rather than "amount"; likewise, "fourth root" was preferred to "root of the root," and, finally, "plus" (or even +, as here above) was adopted instead of the insipid "and." For words rendered by a modern mathematical term, we have added the transcription of the Arabic at their first occurrence (the index of Arabic words, in Part IV, giving the other occurrences), or discussed it in a footnote. For those who will find the text indigestible anyway, the footnotes provide a summary.

(1^r) [Treatise on algebra and numerical problems, taken from some earlier scholars, aiming at clarity

The writing of the copy was completed
in the year five hundred and eighty-one

Some parts are missing from the beginning of this copy]

(2^r) [The beginning of this treatise is missing][22]

⟨ First part

On the proportional powers. This is divided into six paragraphs

§ 1. The three proportional quantities ⟩[23]

(A. 6–13) ⟨. . .⟩ according to the same ratio beginning with 1. Thus if the first of the three quantities (*al-maqādir*) is 1, the second will be a root (*jidhr*) and the third, a square (*māl*). [As to the root, it is any number or fraction you wish to multiply by itself, while the square is the result of multiplying the root by itself.][24]

Then if the first of the three quantities is larger than 1—which is what we shall call a number (*ʿadad*)—the second will be roots in the same quantity as the quantity of units in the [first] number, and the third will be squares in the same quantity as well. Likewise, if the first of the three proportional quantities (*al-maqādir al-thalatha al-mutanāsiba*) is a fraction smaller than 1, the second will be a part, or parts, of the root, according to the ratio of the [first] fraction to 1, and the third will likewise be a part or parts of the square, according to the same ratio as well.[25]

(A. 14–23) Example(s).[26] [(*i′*) If we put 2 for the root, the corresponding square will be 4 and the ratio, (which was) 1 : 2, will be the same as the ratio 2 : 4.

(*ii′*) Likewise, if we put 3 for the root, the corresponding square will be 9 and the ratio, (which was) 1 : 3, will be the same as the ratio 3 : 9.

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22 In Persian, thus by a modern hand.

23 Titles conjectured.

24 An early reader considered "root" and "square" to refer to numerical quantities; whence also the subsequent interpolations.

25 Since $1 : x = x : x^2$, then also $m : mx = mx : mx^2$, with m any integer or a fraction ("parts or parts," that is, $\frac{1}{l}$ or $\frac{k}{l}$, here $k < l$). The word "first" (bracketed, twice) probably originates with an early reader (the numerical term is the "first" of the three proportional quantities).

26 Examples *i′*–*iii′* are in line with the above interpolation. The genuine examples *i*, *ii* illustrate the fundamental proportion $m : mx = mx : mx^2$.

(*iii′*) (Now) for the fractions: if we put for the root $\frac{1}{2}$, the corresponding square will be $\frac{1}{4}$ and the ratio, (which was) $1 : \frac{1}{2}$, will be the same as the ratio $\frac{1}{2} : \frac{1}{4}$.]

(*i*) Following the same reasoning,[27] if the first of the three quantities is two units, the second will be two roots [equal to one another whatever their numerical (value)][28] and the third will be two squares [each of them arising from the multiplication of one of the two roots by itself].

(*ii*) Likewise for the fractions: if we put for the first of the three quantities $\frac{1}{2}$, the second will be half a root [(thus) as much as was the fraction of 1] and the third will be half a square [(thus half) of the whole square arising from the multiplication of this root by itself].

Then likewise for whatever (integral) numbers and fractions.

§2. On numerical operations involving the three proportional basic elements

(A. 26–37) Concerning **(2$^{\mathrm{v}}$)** knowledge of the types of treatment (*anwāʿ al-aʿmāl*) involving the three aforesaid proportional basic elements (*al-uṣūl al-t͟halat͟ha al-mutanāsiba*) before (considering) the(ir) equality[29] there are six operations (*aḥwāl*), namely adding, subtracting, taking a multiple, taking a fraction, multiplying and dividing.[30]

As for the first four operations involving them, namely adding, subtracting, taking a multiple and taking a fraction, the treatment for all of them is just like the corresponding treatment for plain numbers (*al-aʿdād al-muṭlaqa*), without any difference. (Indeed,) neither the increment (resulting) from adding and taking a multiple nor the decrement (resulting) from subtracting and taking a (proper) fraction changes the kind (*jins*) (of power), though it changes its coefficient (*kammīya*).

In the case of multiplication, it happens in many situations that the root is multiplied by the square, with both being unknown; then the result is called "cube" (*mukaʿʿab*), and it is the third (term) in the proportion involving (as median terms) root and square.[31] [Indeed, for any four quantities in proportion the multiplication of the first by the fourth equals the multiplication of the second by the third;[32]

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[27] That is, the proportion $1 : x = x : x^2$ remains valid if the first quantity is $m \neq 1$.

[28] This early reader did not fully grasp the meaning of a multiple.

[29] Part IV (A.560–833), on equations. The "three proportional basic elements" are thus: number, root, square.

[30] With the multiplication of the two basic powers (x, x^2) we shall learn the denominations of the subsequent powers and their products (§§ 2–3), and, with the division, the inverse powers (§ 4) and their products and divisions (§§ 5–6).

[31] $1 : x = x^2 : x^3$.

[32] *Elements* VII.19.

and the first of these quantities is, as said, 1,[33] and its multiplication by the fourth will give the fourth (term) itself. For this reason, the result of the multiplication of the root by the square, which are the second and the third, will be the third (term) relative to these two in the proportion—that is, the fourth starting from the first—and this is the aforesaid cube.][34]

(A. 38–47) These three names (*asmā’*), namely root, square and cube, are the simple names by means of which the (first) three proportional powers (*ṭabaqāt*) (of the unknown) are designated. From their mutual multiplications arise other successive powers following the same proportion, the names of which are compounds of the three (basic) names we have indicated.[35]

Thus the square-square (*māl al-māl*), which directly follows the cube in the proportion, results from the multiplication of the root by the cube or from the multiplication of the square by itself. Or else, the square-cube or the cube-square, which directly follows the square-square in the proportion, results from the multiplication of the root by the square-square, or from the multiplication of the square by the cube. Or else, the cube-cube (*muka“ab al-muka“ab*), which directly follows the square-cube in the proportion, results from the multiplication of the root by the square-cube, or from the multiplication **(3ʳ)** of the square by the square-square, or from the multiplication of the cube by itself. Therefore, with this way of proceeding by compounding (being obvious and our) being averse to prolixity, we (considered) refraining from further comment.

§ 3. Multiplication of the proportional powers among themselves and determination of the kind of power resulting

(A. 51–57) If we wish to multiply a square by a cube, we put together the denominations “square” and “cube,” and say that the result of the multiplication is a “square-cube” or a “cube-square.”[36]

If we wish to multiply a root by a cube, we take the number of times[37] the root has been multiplied by itself to give the cube, which is 3, add to it 1, because of the

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33 “As said”: above, A. 6–7.

34 Superfluous.

35 The three “simple” (*mufrada*) names are as given (root, square, cube), and the names of the higher powers are said to be compounds (*mutarakkaba*) of them. As a matter of fact, *more Graecorum*, these higher powers are designated by means of the last two simple names only. The proportion considered will now be extended: $1 : x = x : x^2 = x^2 : x^3 = x^3 : x^4 = x^4 : x^5 = x^5 : x^6$.

36 $x^2 \cdot x^3 = x^5$, thus the fifth power of the unknown. One would expect this to follow the subsequent definition of x^4. But now the exponents are considered.

37 The text has *‘adad al-manzila*, “number of the rank,” which we have, *nolens volens*, changed to *‘adad al-marrāt*. Below, the exponent will be designated simply by *‘idda*, “quantity.”

root, and divide the result, namely 4, into two parts each larger than 1. Such are 2 and 2—there is no other possibility. Then we take "square" for each 2, since, as we have mentioned,[38] the "square" results from (multiplying) a root by [a root] itself; so we shall say that the result of the multiplication is a "square-square."[39]

(A. 58–64) Likewise if we wish to multiply a root by a square-cube: we take 5 for the square-cube—2 for the square and 3 for the cube—add to it 1, because of the root, and divide the result, namely 6, into any two parts, provided that each be (an integer) larger than 1. Say that they are 3 and 3; so we shall take "cube" for each 3, the result being then "cube-cube."[40] Had we divided 6 into two other parts, (thus) 2 and 4, and taken "square" for 2 and "square-square" for 4, and put that together, (giving) "square-square-square," three times, this would be possible; but the expression "cube-cube" is shorter and more concise since there is one repetition in it whereas in "square-square-square" there are two. That is how to proceed.

§4. Division of the proportional powers among themselves and determination of the kind of power resulting

(A. 68–75) If we wish to divide one **(3^v)** of the proportional powers by another and determine the kind of the quotient, then, since dividing is the inverse of multiplying, we shall subtract the quantity[41] of the one closer to the root[42] from the quantity of the one which is farther; the remainder [the quotient] will be (the indication) of the kind of that quantity.[43] If the divided power is the one which is farther from the power of the root, the quotient will (itself) be a power; if the divided power is that which is closer to the power of the root, the quotient will be a part of this (resulting) power.[44] [The part of any power is named by the number of its units.][45] [That is, if the root is two, its part will be a half, that of the square, a fourth, that of the cube, an eighth, that of the square-square, half an eighth.] And so on proceeding likewise.

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[38] At the very beginning (A. 6–7 or missing part).

[39] $x \cdot x^3 = x^{1+3} = x^4 = x^2 \cdot x^2$. Note the denomination of the power, which must not only be a compound of the two words "square" and "cube," but comprise the least number of words, as asserted just below.

[40] $x \cdot x^5 = x^6 = x^3 \cdot x^3$.

[41] Thus, the exponent (the word *ʿidda* used here is hardly appropriate since it will be regularly used for "coefficient" in what follows). Subtracting instead of adding: see, for the latter, § 3.

[42] Thus, the lower exponent.

[43] That is, it will determine the resulting power.

[44] $\frac{x^k}{x^l} = x^{k-l}$, thus a proper power if $k > l$ but an inverse one if $k < l$.

[45] Means that $\frac{k}{x^l}$ is k parts of x^l. Hardly by the author since this is irrelevant here. What follows, by another early reader (probably the one already met several times), is even more so.

(A. 76–85) Example(s). (*i*) If we wish to divide a square-square by a root, we subtract the quantity[46] of the root, namely 1, from the quantity of the square-square, namely 4; the remainder is 3, which is the quantity of the cube; so we shall say that the quotient is a cube. If the divided power is that of the root and the divisor is that of the square-square, the quotient will be a part of a cube.

(*ii*) Likewise if we wish to divide a cube by a square: we subtract the quantity of the square, namely 2, from the quantity of the cube, namely 3; the remainder is 1, which is the quantity of the root; so we shall say that the quotient has the power of the root. If the divided power is that of the square and the divisor is that of the cube, the quotient will be a part of a thing.[47]

(*iii*) If we wish to divide a power by itself, the quotient will be the number 1, because it is a division of like by like. That is how to proceed.

§5. Multiplication of parts of proportional powers among themselves and determination of the resulting part of power

(A. 89–94) If we wish to multiply a part of a power by a part of another power **(4$^{\mathrm{r}}$)** and know of which kind is the power of the resulting part, we multiply the two powers and determine the kind of the product as we did above;[48] (taking) the part of this (resulting) power will give the answer.[49]

Example. We wish to know the result of multiplying a part of a thing, that is, a part of a root, by a part of a square. We multiply a thing by a square, which gives a cube, and take a part of it, thus a part of a cube. So we shall say that the result of multiplying a part of a thing by a part of a square is a part of a cube.

(A. 95–100) [This is analogous to the multiplication of fractions by fractions, for there we multiply the parts by the parts and divide the result by the product of the two denominators (*mukhrajān*). And since here the parts in both the multiplicand and the multiplier are one part, the result of their multiplication will also be one part; and the division of this (unit) by the product of the two denominators, that is (here), of the two powers, will be a part of this result. That is why we multiply together the two powers and take a part of the result, which gives what is required.][50]

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46 The exponent. Again, Arabic *'idda*.

47 Here "thing" is the usual Arabic algebraic denomination for our x (*shay'*), less confusing than "root" since the latter is also used in the arithmetical sense.

48 See § 3.

49 Since $x^k \cdot x^l = x^{k+l}$, so $\frac{1}{x^k} \cdot \frac{1}{x^l} = \frac{1}{x^{k+l}}$.

50 Since $\frac{1}{k} \cdot \frac{1}{l} = \frac{1}{k \cdot l}$ whereas $\frac{1}{x^k} \cdot \frac{1}{x^l} = \frac{1}{x^{k+l}}$, this analogy may not be wholly appropriate. The same kind of analogy occurs at the end of the next paragraph. Both must be interpolations. In any event, they close the sequence of longer interpolations in this first part.

§6. Division of parts of proportional powers among themselves and determination of the power of the quotient

(A. 104–110) If we wish to divide a part of a power by a part of a power and know the kind[51] of power of the quotient, we divide the power of the dividing part by the power of the divided part [and we shall know of which kind is the quotient].[52] If the power of the divided part is that closer to the power of the root, what is sought will be the quotient itself; if the power of the divided part is that farther from the power of the root, what is sought will be a part of this quotient.

Example(s). (*i*) We wish to divide a part of a square by a part of a cube; the quotient will be a thing.

(*ii*) We wish **(4$^{\text{v}}$)** to divide a part of a cube by a part of a square; the quotient will be a part of a thing.

(A. 111–115) [This is also analogous to the division of fractions by fractions. There we multiply each of the two denominators by the parts of the other, using an inverted multiplication, then we divide the resulting dividend by the resulting divisor. (But) since here the (number of) parts, for each of the two terms (*jinsān*), is 1, we divide the two powers, dividend by divisor (*sic*), without needing the inverted multiplication.][53]

That is how to proceed.

Second part

On proportional powers linked together generally. This is divided into four paragraphs.[54]

§1. Adding them

(A. 121–128) If there occur in a problem two expressions (*janbatān*) containing like kinds (*ajnās*) and we are to add them, the coefficient (*ʿidda*) of each kind in one expression is added to the coefficient of its correspondent in the other expression.[55] If the two corresponding (coefficients) are positive (*zāʾidān*), so will the sum be.[56] If they are both negative (*nāqiṣān*), that is, subtractive [from another kind], the

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[51] The manuscript has "part" (*juzʾ*), which is corrected below into *jins*, "kind."

[52] Thus $\frac{1}{x^k} : \frac{1}{x^l}$ reduced to considering $\frac{x^l}{x^k} = x^{l-k}$. As said below, we shall have a power proper if $k < l$ and a part of this power if $k > l$.

[53] $\frac{k_1}{l_1} : \frac{k_2}{l_2}$ gives $\frac{k_1 \cdot l_2}{l_1 \cdot k_2}$, thus here $\frac{l_2}{l_1}$, while $\frac{1}{x^{l_1}} : \frac{1}{x^{l_2}}$ gives $\frac{x^{l_2}}{x^{l_1}} = x^{l_2 - l_1}$. Not very convincing analogy.

[54] Successively: addition, subtraction, multiplication, division involving (except in the last case) two expressions consisting each of a number and some multiple of a thing.

[55] Expressions of the type $a+mx$, $b+nx$. The absolute values of the coefficients of x are considered. The numerical terms a, b are there merely in order to avoid dealing with purely negative quantities.

[56] See below, examples *i* (and *vi*).

sum will be negative, that is, subtractive.[57] If one is positive and the other negative and the coefficient of the positive (kind) is less than the coefficient of the negative (kind), the lesser coefficient is subtracted from the greater and the remainder will be negative, and this will be the (coefficient of the) sum;[58] if the coefficient of the positive (kind) is greater, the lesser coefficient is subtracted from the greater, and the remainder will be positive, and this will be the (coefficient of the) sum.[59]

(A. 129–141) Example(s). (*i*) We wish to add 10 plus a thing to 10 plus a thing; the sum will be 20 plus two things.[60]
(*ii*) Or we add 10 minus a thing to 10 minus a thing; the sum will be 20 minus two things.[61]
(*iii*) Or we add 10 minus a thing to 10 plus a thing; the sum will be 20 altogether.[62]
(*iv*) Or we add 10 plus two things to 10 minus a thing; **(5^r)** the sum will be 20 plus one thing.[63]
(*v*) Or we add 10 minus two things to 10 plus a thing; the sum will be 20 minus one thing.[64]
(*vi*) Or we add 15 plus a thing to a thing minus 10; the sum will be two things plus 5.[65]
(*vii*) Or we add 15 minus two things to a thing minus 10; the sum will be 5 minus one thing.[66]

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[57] See example *ii*.

[58] See examples *v* and *vii* below. *iii* is a particular case.

[59] See examples *iv* and *viii* below.

[60] Example *i*. $(10 + x) + (10 + x) = 20 + 2x$, generally $(a + mx) + (b + nx) = (a + b) + (m + n)x$, with here $a = b$, $m = n$.

[61] Example *ii*. $(10 - x) + (10 - x) = 20 - 2x$, generally $(a - mx) + (b - nx) = (a + b) - (m + n)x$, with here $a = b$, $m = n$.

[62] Example *iii*. $(10 - x) + (10 + x) = 20$, generally $(a - mx) + (b + nx) = (a + b) + (n - m)x$, with here $a = b$, $m = n$.

[63] Example *iv*. $(10 + 2x) + (10 - x) = 20 + x$, generally $(a + mx) + (b - nx) = (a + b) + (m - n)x$, with here $a = b$, $m > n$.

[64] Example *v*. $(10 - 2x) + (10 + x) = 20 - x$, generally $(a - mx) + (b + nx) = (a + b) - (m - n)x$, with here $a = b$, $m > n$.

[65] Example *vi*. $(15 + x) + (x - 10) = 2x + 5$, generally $(a + mx) + (nx - b) = (a - b) + (m + n)x$, with here $a \neq b$, $m = n$.

[66] Example *vii*. $(15 - 2x) + (x - 10) = 5 - x$, generally $(a - mx) + (nx - b) = (a - b) - (m + n)x$, with here $a \neq b$, $m > n$.

(*viii*) Or we add 10 minus a thing to two things minus 15; the sum will be a thing minus 5.[67]

That is how to proceed.

§2. Subtracting them

(A. 144–154) As for subtracting, if there occur in a problem two expressions containing like powers and the (terms) of one of them must be subtracted from those of the other, we subtract the coefficient of each kind in the expression to be subtracted from the coefficient of its correspondent in the expression from which is subtracted.[68] If the two corresponding (coefficients) are positive and the subtracted one is less, the remainder will be positive;[69] if it is greater, the remainder, thus their difference, will be negative, that is, subtractive.[70] If the two corresponding (coefficients) are negative and the subtracted one is less, the remainder will be negative;[71] if it is greater, the remainder, thus the difference between them, will be positive since this (negative) difference is subtracted [from the minuend].[72] If only one of the two corresponding (coefficients), say the subtracted one, is positive, whether less or more than the one from which it is subtracted, and the one from which it is subtracted is negative, the remainder, which is the sum of the two coefficients, will be negative, that is, subtractive [from the aforesaid element; indeed, subtracted from subtracted becomes added in the minuend];[73] if (the subtracted coefficient) is negative, whether less or more than the one from which it is subtracted, and the one from which it is subtracted is positive, the remainder, which is the sum of the two coefficients, will be positive.[74]

(A. 155–165) Example(s). (*i*) We wish to subtract **(5^{v})** 10 plus a thing from 15 plus five things; the remainder is 5 plus four things.[75]

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[67] Example *viii*. $(10-x)+(2x-15) = x-5$, generally $(a-mx)+(nx-b) = (a-b)+(n-m)x$, with here $a \neq b$, $n > m$.

[68] As before, the absolute values of the coefficients are considered. We shall now examine successively $(+)-(+)$; $(-)-(-)$; $(-)-(+)$; $(+)-(-)$.

[69] See example *i* below.

[70] See example *ii*.

[71] See example *iii*.

[72] See example *iv*.

[73] See example *v*.

[74] See example *vi*.

[75] Example *i*. $(15+5x)-(10+x) = 5+4x$, generally $(a+mx)-(b+nx) = (a-b)+(m-n)x$, with $a > b$, $m > n$.

(*ii*) Or we subtract 10 plus five things from 15 plus a thing; the remainder is 5 minus four things.[76]
(*iii*) Or we subtract 10 minus a thing from 20 minus ten things; the remainder is 10 minus nine things.[77]
(*iv*) Or we subtract 10 minus ten things from 20 minus three things; the remainder is 10 plus seven things.[78]
(*v*) Or we subtract 10 plus a thing from 15 minus a thing; the remainder is 5 minus two things.[79]
(*vi*) Or we subtract 10 minus a thing from 15 plus a thing; ⟨the remainder⟩ is 5 plus two things.[80]

That is how to proceed.

§3. Multiplying them

(A. 168–174) As for multiplying, if there are two quantities (*miqdārān*) which we wish to multiply by two other quantities, we shall place the multiplicand (*maḍrūb*) in one row and the multiplier (*maḍrūb fīhi*) in another, below, (with corresponding terms) lined up; then we need in that case four multiplications, two diagonally and two vertically. If there are three quantities (to be multiplied) by three quantities, we need in this case nine multiplications, six diagonally and three vertically. And so on by the same reasoning, whatever the (number of) quantities.[81] Moreover, for any two quantities we multiply together which happen to be both positive or both negative, the product will be positive; otherwise it will be negative.

(A. 175–190) Example(s).[82] (*i*) We wish to multiply 10 plus a thing by 10 plus a thing. We place the 10 below the 10, and the thing below the thing; then we multiply 10 by the thing (placed) diagonally to it, which gives ten things; then we

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[76] Example *ii*. $(15+x)-(10+5x)=5-4x$, generally $(a+mx)-(b+nx)=(a-b)-(n-m)x$, with here $a>b$, $m<n$.

[77] Example *iii*. $(20-10x)-(10-x)=10-9x$, generally $(a-mx)-(b-nx)=(a-b)-(m-n)x$, with $a>b$, $m>n$.

[78] Example *iv*. $(20-3x)-(10-10x)=10+7x$, generally $(a-mx)-(b-nx)=(a-b)+(n-m)x$, with here $a>b$, $m<n$.

[79] Example *v*. $(15-x)-(10+x)=5-2x$, generally $(a-mx)-(b+nx)=(a-b)-(m+n)x$, with here $a>b$, $m=n$.

[80] Example *vi*. $(15+x)-(10-x)=5+2x$, generally $(a+mx)-(b-nx)=(a-b)+(m+n)x$, with here $a>b$, $m=n$.

[81] As long as the two expressions contain the same number of terms, say m, there will be m vertical multiplications and m^2-m oblique ones.

[82] Successively, $(+)\cdot(+)$; $(-)\cdot(-)$; $(+)\cdot(-)$. Khwārizmī and Abū Kāmil also have such multiplications of binomials, with geometrical illustrations in Abū Kāmil's treatise (see below).

multiply the other 10 by the other thing, (placed) diagonally to it, which also gives ten things; then we multiply the 10 by the 10 lined up, which gives 100; **(6ʳ)** then we multiply the thing by the thing, also lined up, which gives a square. We add (all) this, which gives 100 plus a square plus twenty things.[83]

(*ii*) Next, (if we wish to multiply 10 minus a thing by 10 minus a thing,) we place the two factors (*maḍrūbān*), (namely) 10 minus a thing by[84] 10 minus a thing, in the (same) place as before; then we multiply 10 by minus a thing[85] placed diagonally to it, which gives ten things, negative, that is, subtractive; then we also multiply the other 10 by minus a thing placed diagonally to it, which also gives ten things, negative; then we multiply 10 by 10, which gives 100, positive, and we multiply minus a thing by minus a thing, which gives a square, positive. We add that, which gives 100 plus a square minus twenty things.[86]

(*iii*) Again, (if we wish to multiply 10 plus a thing by 10 minus a thing), we place the two factors, 10 plus a thing by 10 minus a thing, as in the previous position. We multiply 10 by minus a thing, which gives ten things, negative; then we multiply 10 by a thing, which gives 10 things, positive; then we multiply 10 by 10, which gives 100, positive; and we multiply a thing by minus a thing, which gives a square, negative. We add that, which gives 100 minus a square, for the positive things cancel out the negative things since they are in equal amounts.[87]

(A. 191–200) Reason why the multiplication of negative by negative gives positive.[88] For that, we put line AB, and let it be 10 in number. We construct on it the square ABGD. We subtract from line AB a thing, say BE, and from line AD (a

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[83] $(10+x)(10+x) = 100+x^2+20x$, generally $(a+mx)(b+nx) = ab+mnx^2+(an+bm)x$ with, here and in the two following examples, $a=b$, $m=n$. Same numerical example in Khwārizmī (1831, 24 (trans.), 16–17 (Arabic)) and in Abū Kāmil (1986, fol. 14ᵛ (Arabic); 1966, 61 (Hebrew); 1993, l. 669 (Latin)).

[84] Conveniently, *fī* here instead of (logically) "and" (*wa*) in order to avoid ambiguity (*wa* is also used for +). Same in the next example.

[85] Note the Arabic wording : *fī illā shay'*, with the *illā shay'* considered as a set expression.

[86] $(10-x)(10-x) = 100+x^2-20x$, generally $(a-mx)(b-nx) = ab+mnx^2-(an+bm)x$. Same numerical example in Khwārizmī (1831, 24 (trans.), 17 (Arabic)) and Abū Kāmil (1986, fol. 15ʳ (Arabic); 1966, 61 (Hebrew); 1993, l. 697 (Latin)).

[87] $(10+x)(10-x) = 100-x^2$, generally $(a+mx)(b-nx) = ab-mnx^2-(an-bm)x$, thus here with $a=b$, $m=n$. Same numerical example in Khwārizmī (1831, 25 (trans.), 17 (Arabic)) and Abū Kāmil (1986, fol. 15ᵛ (Arabic); 1966, 63 (Hebrew); 1993, l. 724 (Latin)).

[88] Rather, it proves the identity $(u-v)^2 = u^2+v^2-2\,u\cdot v$ occurring in example *ii*. Here as in other instances, the title is likely to be a reader's addition; see note 6, above.

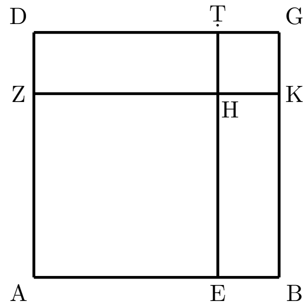

Figure 5: Proof of the identity $(u-v)^2 = u^2 + v^2 - 2\,u \cdot v$

segment) equal to BE, say DZ. We draw line EHṬ perpendicular to AB and line ZHK perpendicular to AD.[89]

Then the rectangle DH results from the multiplication of DZ, thus a thing, by ZH, thus 10 minus a thing, and (therefore) this (rectangle) is ten things minus a square. (Now) the rectangle DH is equal to the rectangle HB. So the two rectangles DH, HB are twenty things minus two squares. And the area GH is a square, for it results from multiplying a thing by itself. Therefore the three areas DH, HG, HB are twenty things minus a square, since the positive square has eliminated **(6$^{\mathrm{v}}$)** one of the two negative squares. (Now) the whole area ABGD arises from the multiplication of 10 by 10, which is 100. When we subtract from the 100 twenty things minus a square, the remainder will be 100 plus a square minus twenty things, and that is equal to the multiplication of AE, which is 10 minus a thing, by itself, that is, the area AH. That is what we wanted to prove.[90]

§4. Dividing them

(A. 203–207) As for dividing, what makes the outcome possible in this general type (of operation) is the division of a polynomial expression (*ajnās muqtirana*), with any number of terms, by a single term (*jins*). (Indeed,) if the divisor consists of more than one term, there is no way to determine the quotient unless it is assumed

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[89] Demonstration also in Abū Kāmil (1986, fol. 16$^{\mathrm{r}}$ (Arabic); 1966, 63 (Hebrew); 1993, l. 732 (Latin)). Here the letters follow the succession of the Greek alphabet (with ه = ε, ح = η, ط = ϑ).

[90] Let AB $= u$ (thus the square ABGD is u^2) and take, on AB, the segment EB $= v$ and, on AD, the segment ZD $=$ EB; from E and Z, draw EHṬ perpendicular to AB and ZHK perpendicular to AD.

Then rectangle DH $=$ ZH $\cdot$ ZD $(= (u-v)\,v = uv - v^2)$. Further, since DH $=$ HB, DH $+$ HB $= 2uv - 2v^2$. But HG $= v^2$, so DH+HB+HG $= 2uv - v^2$. Subtracting this from ABGD $= u^2$, we are left with the square AH, which is therefore AH $=$ AG$-$(DH+HG+HB); that is, $(u-v)^2 = u^2 - (2uv - v^2)$.

The proofs involving two squares (here AG and AH) differing by a gnomon (thus the area ZDGBEHZ here; see note 4, above) will now become recurrent.

(*mafrūḍ*) in the problem; (for) then one uses the multiplication as a device (for verification), (since) indeed for any quantity which is divided by another quantity the quotient when multiplied by the divisor gives again the dividend.[91]

(A. 208–213) Example(s) of the aforesaid case of possibility.[92] (*i*) If we wish to divide 10 plus a thing by 5, we divide 10 by 5, which gives 2, then we divide a thing by 5, which gives a fifth of a thing. We add that, which gives 2 plus a fifth of thing.

(*ii*) If we wish to divide 10 plus five things by a thing, we divide 10 by a thing, which gives ten parts of a thing, and we divide five things by a thing, which gives 5, (thus) a number. We add that, which gives as the (required) quotient 5 plus ten parts of a thing. That is how to proceed.

Third part

On proportional powers when simple and associated.[93] This will comprise six paragraphs.

§1. Taking multiples of them

(A. 219–222) Taking multiples of square roots associated with numbers.[94] If we wish to take the multiple of the square root of a number—the meaning of taking the multiple (*taḍʿīf*) ⟨of a square root⟩ is that one takes it twice, or thrice, or any arbitrary number of times—**(7ʳ)** we multiply the multiple—with its fraction, if any—by itself, then by the number in question (*al-ʿadad al-mansūb*), and take the square root of the product; the result will be what is required.

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[91] If this is the original text, with the multiplication used to verify an assumed quotient, it is quite banal. But, as we shall see at the end of the next part (A. 499–554), in some cases a multiplication might serve to rationalize the denominator and thus make the division possible.

[92] $\frac{10+x}{5} = 2 + \frac{1}{5}x$ and $\frac{10+5x}{x} = \frac{10}{x} + 5$ successively.

[93] The title is, to say the least, misleading, and can hardly be the author's: we shall be taught how to apply the six operations mentioned above (A. 26–27) to *numerical roots*, both square and cube ones, sometimes also fourth roots. Thus powers of an unknown do not intervene. We have already met such an inappropriate title (note 88; see our introduction, note 6).

[94] A "root associated with a number" (*ji<u>dh</u>r mansūb ila ʿadad*, or simply *ji<u>dh</u>r mansūb*) is a numerical root; this must be specified in order to avoid confusion with "root" corresponding to our x. Note too that by "root" (*ji<u>dh</u>r*) applied to numbers our Arabic text means exclusively *square* roots (as we shall specify each time by adding the word "square"). Here a cube root is *kaʿb* (*ḍilaʿ* in Karajī's *Badīʿ*) and a fourth root, *ji<u>dh</u>r ji<u>dh</u>r*. Analogous computations for square roots occur in <u>Kh</u>wārizmī (1831, 27–29 (trans.), 19–20 (Arabic)) and Abū Kāmil (1986, fol. 17ʳ (Arabic); 1966, 67 (Hebrew); 1993, l. 799 (Latin)).

(A. 223–238) (*i*) We shall first take a rational example of that.[95] We wish to double the square root of four.[96] The meaning of this is that we take it twice, which is no different from our statement "two square roots of four, of what quantity (*māl*) is it the square root?" We multiply the number of the multiple, which here is 2, by itself, which gives 4, then by the number in question, namely 4 also, which gives 16. The square root of that, thus 4, is the double of the square root of 4.[97]

(*ii*) Likewise, if we wish to take three times the square root of four—which is once again the same as our statement "three square roots of four, of what quantity is it the square root?" We multiply the number of the multiple, thus 3, by itself, then the result, thus 9, by 4; this gives 36. So the square root of 36, thus 6, is thrice the square root of 4.[98]

(*iii*) Likewise if we wish to take twice and a half times the square root of eight. We multiply the number of the multiple, thus $2 + \frac{1}{2}$, by itself, which gives $6 + \frac{1}{4}$, then by 8, which gives 50. So the square root of this, thus the square root of 50, is equal to the square root of 8 taken twice and a half times.[99]

[Again, we wish (to take) two square roots of nine, that is, take the (square root of nine) twice. We (are to) determine first of what quantity two square roots of nine is the square root. This will follow the previous reasoning: we multiply 2 by itself,[100] because of the "two" square roots, which gives 4, then (this) by 9, which gives 36. Then the square root of 36 will equal two square roots of 9. So our statement is as if we were to double the square root of 36 (*sic*), that is, take it twice.][101]

That is how to proceed.

(A. 239–249) Proof of this.[102] We put, for the reason that we have given,[103] the number of which we want to take a multiple of the square root, the [uniform][104]

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[95] Arabic: *mithāl manṭūq*. The first two examples lead to a rational result, not the third one.

[96] Arabic: *ḍa"afa* = to take a multiple; here *ḍa"afa marratan wāḥidatan* = to double.

[97] $2 \cdot \sqrt{4} = \sqrt{4 \cdot 4} = \sqrt{16} = 4$. Generally, $k \cdot \sqrt{u} = \sqrt{k^2 \cdot u}$.

[98] $3 \cdot \sqrt{4} = \sqrt{9 \cdot 4} = \sqrt{36} = 6$.

[99] $(2 + \frac{1}{2}) \cdot \sqrt{8} = \sqrt{(6 + \frac{1}{4}) \cdot 8} = \sqrt{50}$.

[100] Arabic: *fī mithlihī*: a number is mostly taken grammatically in the singular. But there are exceptions; see e.g. A. 228, 316, 318, 488, 492, 514, 665.

[101] $2 \cdot \sqrt{9} = \sqrt{4 \cdot 9} = \sqrt{36} = 6$. This same example is found in Khwārizmī (1831, 28 (trans.), 20 (Arabic)). But this one cannot be genuine, for, first, we would expect such a simple example to have come before the two previous ones and, second, not only is it superfluous, it is also confused.

[102] That is, generally, that $k \cdot \sqrt{u} = \sqrt{k^2 \cdot u}$. Proof also in Abū Kāmil (1986, fol. 17r (Arabic); 1966, 69 (Hebrew); 1993, l. 817 (Latin)).

[103] Presumably: as in Fig. 5 above, with a square and its side.

[104] Useless specification. But see note 262, below.

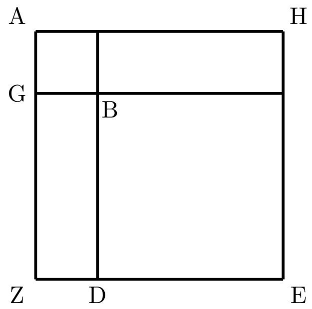

Figure 6: Proof that $k \cdot \sqrt{u} = \sqrt{k^2 \cdot u}$

square AB, and its square root, **(7ᵛ)** line GB.[105] Let the number of the multiple be line BD, and let line BD be perpendicular to BG [at right angles]. We construct on BD the square BE, and we complete the square AZEH.

Then ED is to DZ as the square BE is to the rectangle BZ, for their height is the same;[106] but ZG is equal to ED and AG to ZD, and ZG is to AG as the rectangle BZ is to the square AB; therefore the square BE is to the rectangle BZ as the rectangle BZ is to the square AB. So the rectangle BZ, which is required,[107] is a mean proportional between the two squares AB and BE—the rectangle BZ is called one of the two complements of the two squares AB and BE, and the rectangle BH is the other complement, and they are equal.[108] For this reason, we multiply the number of the multiple, namely BD, by itself, then multiply the result, namely the square BE, by the number of the square root,[109] namely AB, and take the square root of that, which is the rectangle BZ. This is what is required, for it is the product of the multiplication of the square root of AB, thus line BG, by the number of the multiple, thus line BD. This is what we wanted to prove.[110]

(A. 250–262) Multiples of numerical cube roots. There (above) it became evident that multiplying by itself the product of any two numbers equals the product of

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[105] Here and in what follows the quantity under the radical sign, thus the radicand, will be represented by a square of which the side is thus the root considered (“reason”: see above, “of what quantity is it the square root?”).

[106] See *Elements* VI.1.

[107] It represents the multiple of the root.

[108] This last sentence interrupts the reasoning but is a pertinent assertion. This is *Elements* I.43.

[109] Thus the radicand of the square root. Arabic *ʿadad majdhūr*.

[110] To prove that $k \cdot \sqrt{u} = \sqrt{k^2 \cdot u}$, let DB $= k$ (so BE $= k^2$) and GB $= \sqrt{u}$ (so AB $= u$).

Then $\mathrm{BZ} = \mathrm{BH} = \mathrm{DB} \cdot \mathrm{GB}$ $(= k \cdot \sqrt{u})$. Now $\mathrm{DE} : \mathrm{ZD}\,(= \mathrm{DE} \cdot \mathrm{DB} : \mathrm{ZD} \cdot \mathrm{DB}) = \mathrm{BE} : \mathrm{BZ}$; likewise, since DE = ZG and ZD = AG, $\mathrm{DE} : \mathrm{ZD} = \mathrm{ZG} : \mathrm{AG} = \mathrm{BZ} : \mathrm{AB}$. Therefore $\mathrm{BE} : \mathrm{BZ} = \mathrm{BZ} : \mathrm{AB}$, so $\mathrm{BZ}^2 = \mathrm{BE} \cdot \mathrm{AB}$, thus $\mathrm{BZ} = \sqrt{\mathrm{BE} \cdot \mathrm{AB}}$. That is, $k \cdot \sqrt{u} = \sqrt{k^2 \cdot u}$.

the square of one of them by the square of the other.[111] Likewise, if we wish to take a multiple of the cube root of a number, we multiply the multiplicative factor ("number of the multiple": *'adad al-amthāl*) by itself, then the result by the multiplicative factor once again—so that the result becomes a cube—then the result by the number in question, and take the cube root of the result. That will give what is required.[112]

The principle (*al-aṣl*) behind this is (first) that any number equals the square root of its square, the cube root of its cube, the fourth root (*jidhr jidhr*) of its fourth power (*māl māl*);[113] (second, that) for any two numbers the square root of the product of the square of one of them by the square of the other equals the cube root **(8ʳ)** of the product of the cube of one of them by the cube of the other, and this also equals the fourth root of the product of the fourth power of one of them by the fourth power of the other, then (so on) likewise[114]—this for the previous reason that the product of any two numbers, when multiplied by itself, equals the product of the square of one of them by the square of the other.[115] Now since what is required in taking the multiple [of the cube root] of the cube root is multiplying [the cube root of] the cube root (of the number in question) by the multiplicative factor, then, when we raise this product to the cube, it will be as if the number in question has been multiplied by the cube of the multiplicative factor (*aḍ'āf*), and that is why we (then) take the cube root of the (result), which thus gives what is required.[116]

(A. 263–270) Multiples of numerical fourth roots, which are the sides of fourth powers.[117] By the same reasoning, if we wish to take a multiple of the fourth root of a number, that is, take a multiple of the side of a fourth power, we multiply the multiplicative factor by itself, then the result by itself—whereby it becomes a fourth power—then the result by the number in question, and take the fourth root of the result. This gives what is required.

The reason for that is as seen in the two previous examples, namely that what is required is multiplying the side of the fourth power by the multiplicative factor; then if we raise this product to the fourth power, this will become like our multiplying the number in question by the fourth power of the multiplicative factor. That is

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111 $(u \cdot v)^2 = u^2 \cdot v^2$, evident from what precedes (see also below).

112 $k \cdot \sqrt[3]{u} = \sqrt[3]{k^3 \cdot u}$.

113 That is, first, $u = \sqrt{u^2} = \sqrt[3]{u^3} = \ldots = \sqrt[m]{u^m}$, in our case $k = \sqrt[3]{k^3}$; "fourth powers," lit. "square-squares."

114 Then, second, $u \cdot v = \sqrt{u^2 \cdot v^2} = \sqrt[3]{u^3 \cdot v^3} = \ldots = \sqrt[m]{u^m \cdot v^m}$. In our case: $k \cdot \sqrt[3]{u} = \sqrt[3]{k^3 \cdot (\sqrt[3]{u})^3}$, thus $\sqrt[3]{k^3 \cdot u}$.

115 As asserted above (note 111).

116 Thus $k \cdot \sqrt[3]{u} = \sqrt[3]{k^3 \cdot u}$, as asserted above (see notes 112, 114).

117 "side": *ḍila'* (= πλευρά).

why we multiply in this way and take (then) the fourth root of the result, which thus gives what is required.[118]

§2. Taking a fraction of them

(A. 273–276) Taking a fraction of numerical square roots. Taking a fraction just follows the reasoning for taking a multiple. For, if we wish to take the fraction (*tajzi'a*) of the square root of a number—the meaning being that we multiply the square root of this number by a half, or a third, or a fourth, or any of the parts of 1[119]—we multiply this part by itself, then the result by the number in question, and take the square root of the product. This gives what is required.[120] **(8^v)**

(A. 277–282) Example(s). (*i*) If we wish to halve the square root of four—which is like saying "half the square root of four, of what quantity is it the square root?"—we multiply the part, namely a half, by itself, which gives a fourth, then by the number in question, which is four; this gives 1. We take the square root of this, which is 1, and this is what is required.

(*ii*) Likewise if we wish to take a third of the square root of thirty-six—the meaning being "a third of the square root of thirty-six, of what quantity is it the square root?"—we shall multiply a third by a third, which gives a ninth, then by 36, which gives 4. So the square root of this, which is 2, is a third of the square root of thirty-six.

(A. 283–284) Likewise again, taking a fraction of the side of a cube or of the side of a fourth power follows the same reasoning.[121] The reason for that is the previous one, as given in the paragraph on taking multiples.[122]

§ 3. Adding them

(A. 287–289) Addition of numerical square roots. If we wish to add together the square root of a number and the square root of a number, we add the two radicands (*al-ʿadadān al-majdhūrān*) and add to the sum twice the square root of the result of multiplying one of them by the other. Taking the square root of the result will give what is required.[123]

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[118] $k \cdot \sqrt[4]{u} = \sqrt[4]{k^4 \cdot (\sqrt[4]{u})^4} = \sqrt[4]{k^4 \cdot u}$.

[119] Odd restriction to unit fractions. See note 174, below. Maybe because $\frac{p}{q}\, a = p\,(\frac{1}{q}\, a)$, thus reduced to the previous case.

[120] $\frac{1}{k} \cdot \sqrt{u} = \sqrt{(\frac{1}{k})^2 \cdot u}$. Here the two examples are $\frac{1}{2} \cdot \sqrt{4} = 1$, $\frac{1}{3} \cdot \sqrt{36} = 2$. See also Khwārizmī (1831, 28–29 (trans.), 20 (Arabic)).

[121] Thus, taking a fraction of a cube root or a fourth root: $\frac{1}{k} \cdot \sqrt[3]{u} = \sqrt[3]{(\frac{1}{k})^3 \cdot u}$, $\frac{1}{k} \cdot \sqrt[4]{u} = \sqrt[4]{(\frac{1}{k})^4 \cdot u}$.

[122] Above, notes 113, 114.

[123] $\sqrt{u} + \sqrt{v} = \sqrt{u + v + 2\sqrt{u \cdot v}}$.

(A. 290–299) Example(s), (first) for rational square roots (*judhūr munṭaqa*).[124] (*i*) We wish to add the square root of four and the square root of nine. We multiply 4 by 9, which gives 36; we take the square root of that, which is 6; we double it, which gives 12; we add it to the sum of 4 and 9, so the sum is 25. The square root of this, namely 5, is the sum of the square root of four and the square root of nine.[125]

(*ii*) Likewise if we wish to add the square root of three and the square root of five. We multiply 3 by 5, then take the square root of the result, which is the square root of 15. Then we double it, following the previous reasoning in the paragraph on taking multiples of square roots;[126] this gives the square root of 60. We add it to the sum of 3 and 5; the result **(9^r)** is 8 plus the square root of 60. We take the square root of that; this will give what is required.[127]

[The reason for that is the following. If, for any two square numbers, we add to them their two complements, the result will be a square, and, if we subtract these from them, the remainder will be a square.][128]

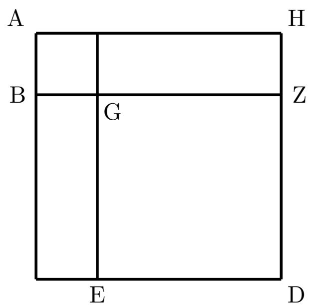

Figure 7: Proof that $\sqrt{u} + \sqrt{v} = \sqrt{u + v + 2\sqrt{u \cdot v}}$

(A. 300–306) To prove that,[129] we draw the two squares AG, GD, the side of the square AG being BG, and the side of the square GD being GZ. We complete the two rectangular complements BE and GH.

We have proved in the paragraph on multiples that each one is a mean proportional between the squares AG and GD.[130] Therefore our multiplying the two

124 $\sqrt{4}$ and $\sqrt{9}$ here, but $\sqrt{3}$ and $\sqrt{5}$ in the next example.

125 $\sqrt{4} + \sqrt{9} = \sqrt{4 + 9 + 2\sqrt{4 \cdot 9}} = \sqrt{13 + 2\sqrt{36}} = \sqrt{25} = 5$. Same example in Abū Kāmil (1986, fol. 20^r (Arabic); 1966, 77 (Hebrew); 1993, l. 975 (Latin)).

126 Above, A. 219–249.

127 $\sqrt{3} + \sqrt{5} = \sqrt{3 + 5 + 2\sqrt{3 \cdot 5}} = \sqrt{3 + 5 + 2\sqrt{15}} = \sqrt{8 + \sqrt{60}}$.

128 $u^2 + v^2 \pm 2\,u \cdot v = (u \pm v)^2$ (*Elements* II.4). This assertion must be an early reader's addition.

129 Proof that $\sqrt{u} + \sqrt{v} = \sqrt{u + v + 2\sqrt{u \cdot v}}$. Also in Abū Kāmil (1986, fol. 20^r (Arabic); 1966, 77 (Hebrew); 1993, l. 980 (Latin)).

130 Above, A. 239–249. Here: $\mathrm{BE} = \mathrm{GH} = \sqrt{\mathrm{AG} \cdot \mathrm{GD}}$.

squares AG, GD will give us the rectangle BE multiplied by itself. We take the square root of that, which gives the rectangle BE. We double it, which gives the sum of the two rectangles BE, GH. We add to that the two squares AG, GD. This completes for us the square AD.[131] We take its square root; this gives AH, which is the sum of the two sides BG and GZ. This is the proof we wanted.[132]

(A. 307–309) The same reasoning applies if we wish to add the square root of a number to some (plain) number: we multiply the plain number (*al-ʿadad al-muṭlaq*) by itself, so it will become a square root (*majdhūr*);[133] that is, it will become of the same kind as the other. Then we proceed with the same treatment as before.[134]

(A. 310–314) Addition of numerical cube roots. If we wish to add the cube root of a number to the cube root of a number, we multiply the square of one of the(se) numbers by the other number, then the result by 27, take the cube root of that and keep it in mind. Then we multiply the square of the other number by the first number, then the result by 27, take the cube root of it, and add it to what we have kept in mind. Next we add the result to the sum of the two cubic numbers in question, and we take the cube root of the result. This will give what is required.[135]

(A. 315–324) Example (9^{v}) for rational cube roots (*kiʿāb munṭaqa*).[136] We wish to add the cube root of eight to the cube root of a hundred and twenty-five. We multiply 8 by itself, then the result by 125, which gives 8000, then by 27, which gives 216 000; we take the cube root of that, namely 60, which we keep in mind. Then we multiply 125 by itself, which gives 15 625, then by 8, which gives 125 000, then by 27, which gives 3 375 000; we take the cube root of that, namely 150. Then we add it to what we have kept in mind, namely 60, which gives 210. ⟨Then⟩ we add that to the sum of the two numbers, thus 8 and 125, which gives 343. The cube root of that, namely 7, is the sum of the cube root of eight and the cube root of a hundred and twenty-five.[137] That is how to proceed.

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131 This might have been the intended place of the interpolation.

132 Let AG $= v$ (so BG $= \sqrt{v}$) and GD $= u$ (so GZ $= \sqrt{u}$). Since BE $=$ GH $= \sqrt{\mathrm{AG}\cdot\mathrm{GD}}$ ($= \sqrt{u\cdot v}$), so AD $= \mathrm{AG}+\mathrm{GD}+2\sqrt{\mathrm{AG}\cdot\mathrm{GD}}$; taking the root of that gives us AH $=$ BZ $= \sqrt{u}+\sqrt{v} = \sqrt{u+v+2\sqrt{u\cdot v}}$.

133 Thus the radicand of a square root, for $v=\sqrt{v^2}$.

134 $\sqrt{u}+v = \sqrt{u}+\sqrt{v^2} = \sqrt{u+v^2+2\sqrt{u\cdot v^2}}$.

135 $\sqrt[3]{u}+\sqrt[3]{v} = \sqrt[3]{u+v+\sqrt[3]{27u^2\cdot v}+\sqrt[3]{27u\cdot v^2}}$. Indeed $(a+b)^3 = a^3+b^3+3\,a^2\cdot b+3\,a\cdot b^2$, with here, e.g., $3\,a^2\cdot b = 3\,(\sqrt[3]{u})^2\cdot\sqrt[3]{v} = \sqrt[3]{27\,u^2\cdot v}$. Here u and v are called first the “numbers,” then the “cubic numbers.”

136 Namely $\sqrt[3]{8}=2$ and $\sqrt[3]{125}=5$, thus giving rational results.

137 Consider thus $\sqrt[3]{8}+\sqrt[3]{125}$. Since $\sqrt[3]{27\cdot 8^2\cdot 125} = \sqrt[3]{216\,000} = 60$ and $\sqrt[3]{27\cdot 8\cdot 125^2} = \sqrt[3]{3\,375\,000} = 150$, so $\sqrt[3]{8}+\sqrt[3]{125} = \sqrt[3]{133+60+150} = \sqrt[3]{343} = 7$.

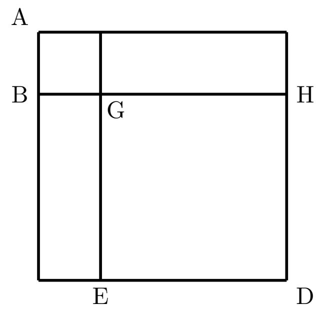

Figure 8: Proof that $\sqrt[3]{u}+\sqrt[3]{v}=\sqrt[3]{u+v+\sqrt[3]{27\,u^2\cdot v}+\sqrt[3]{27\,u\cdot v^2}}$

(A. 325–348) To prove that,[138] we imagine two different cubes with the square bases AG and GD, and let the diagonal of base AG be in the prolongation of the diagonal of base GD, and let the shorter one be AG. We imagine that the square AD is the base of the cube enclosing the two cubes considered, that is, comprising them. It is known (by considering the figure) that this larger cube exceeds the two cubes considered by two equal parallelepipeds with their bases equal to the rectangle BE and their height equal to line BH, and also by two further, (this time) different, parallelepipeds having, respectively, as base the square AG and as height EG, and the square GD and the height BG.[139]

Now the product of line BG by itself, then of the result by GE, when added to the product of GE by itself, then of the result by BG,[140] is equal to the product of BG by GE, **(10$^{\text{r}}$)** then the result by the sum of BG and GH; that is, (the result will be) the parallelepiped with base BE and height BH.[141] Then the sum of the two parallelepipeds with base AG and height GE, respectively base GD and

[138] Namely that $\sqrt[3]{u}+\sqrt[3]{v}=\sqrt[3]{u+v+\sqrt[3]{27\,u^2\cdot v}+\sqrt[3]{27\,u\cdot v^2}}$.

[139] Consider the cube BH^3 on the square base AD and the two smaller cubes $\mathrm{BG}^3=v$, on the square base AG, and $\mathrm{GH}^3=u$, on the square base GD.

Considering the whole cube, we see that $\mathrm{BH}^3=\mathrm{BG}^3+\mathrm{GH}^3+$ four complementary parallelepipeds, namely (*i*, upper left) $\mathrm{BG}^2\cdot\mathrm{GH}$; (*ii*, upper right) $\mathrm{BG}\cdot\mathrm{GH}\cdot\mathrm{BH}=\mathrm{BG}\cdot\mathrm{GH}\,(\mathrm{BG}+\mathrm{GH})=\mathrm{BG}^2\cdot\mathrm{GH}+\mathrm{BG}\cdot\mathrm{GH}^2$; (*iii*, lower left) $\mathrm{BG}\cdot\mathrm{GE}\cdot\mathrm{BH}=\mathrm{BG}\cdot\mathrm{GH}\cdot\mathrm{BH}=\mathrm{BG}^2\cdot\mathrm{GH}+\mathrm{BG}\cdot\mathrm{GH}^2$ (same as *ii*); (*iv*, lower right) $\mathrm{GH}^2\cdot\mathrm{BG}$. We thus obtain, for these four parallelepipeds altogether, $i+ii+iii+iv=3\cdot\mathrm{BG}^2\cdot\mathrm{GH}+3\cdot\mathrm{BG}\cdot\mathrm{GH}^2$, and this is the excess of the cube on AD over the two cubes with bases AG and GD.

[140] Thus the above complements *i* and *iv*.

[141] Since GH = GE, so $\mathrm{BG}^2\cdot\mathrm{GH}+\mathrm{GH}^2\cdot\mathrm{BG}=\mathrm{BG}\cdot\mathrm{GH}\cdot(\mathrm{BG}+\mathrm{GH})=\mathrm{BG}\cdot\mathrm{GH}\cdot\mathrm{BH}$. This is our *ii* or *iii*, thus indeed a parallelepiped (one of the two lateral ones). The excesses *i* and *iv* taken together are therefore equal to *ii* (or *iii*). The inference will be that the whole excess must equal thrice one of the two lateral parallelepipeds.

height BG,[142] is a third of the excess of the larger cube with base AD over the sum of the two cubes with bases AG and GD.[143] So if we multiply each of these two parallelepipeds by 3, the result will equal the whole of this excess. (Now) we know that the parallelepiped with base AG and height GE arises from multiplying BG by itself and the result by GE; we (also) know that multiplying the square of BG (first) by GE, then by 3, next raising the result to the cube, will equal (the result of) multiplying the cube with side BG by itself and then the result by the cube with side GE, (the result being) multiplied by 27.[144] Because of this we multiply the cube with base AG by itself, then by the cube with base GD, then multiply (all that) by 27, and the cube root of that is taken; the (result) will equal (three times) the parallelepiped with square base AG and height GE.[145] Then we also multiply the cube with base GD by itself, then by the cube with base AG, (then all that) by 27, and take the cube root of that; this will equal (three times) the parallelepiped with square base GD and height BG.[146] (But) we have shown that (three times) these two parallelepipeds is the excess of the larger cube over the two smaller cubes considered. Therefore we shall add (three times) these two cubes (*sic*) to the sum of the two cubic numbers,[147] in order for us to complete the larger cube, and take the cube root of that; the result **(10$^{\mathrm{v}}$)** will be equal to the required sum of the two cube roots. This is what we wanted to prove.[148]

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[142] Thus, and again, the above complements *i* and *iv*.

[143] The excess thus consists of three equal parts: $i + iv$ (the two parallelepipeds on the two given cubes), the lateral parallelepiped *ii*, the (equal) lateral parallepiped *iii*.

[144] $3 \cdot \mathrm{AG} \cdot \mathrm{GE} = 3 \cdot \mathrm{BG}^2 \cdot \mathrm{GE}$, so $(3 \cdot \mathrm{AG} \cdot \mathrm{GE})^3 = (3 \cdot \mathrm{BG}^2 \cdot \mathrm{GE})^3 = 27 \cdot (\mathrm{BG}^3)^2 \cdot \mathrm{GE}^3$.

[145] $\sqrt[3]{27 \cdot (\mathrm{BG}^3)^2 \cdot \mathrm{GE}^3} = \langle 3 \rangle\, \mathrm{BG}^2 \cdot \mathrm{GE}$. This is our complement *i* (but taken three times).

[146] Likewise, $\sqrt[3]{27 \cdot (\mathrm{GH}^3)^2 \cdot \mathrm{BG}^3} = \langle 3 \rangle\, \mathrm{GH}^2 \cdot \mathrm{BG}$. This is our complement *iv* (but taken three times).

[147] That is, the given numbers of which we wish to add the cube roots.

[148] Since this whole reasoning is rather abstruse, let us repeat it. We have, for the whole cube, $\mathrm{BH}^3 = \mathrm{GH}^3 + \mathrm{BG}^3 + 3\,\mathrm{BG} \cdot \mathrm{GH}^2 + 3\,\mathrm{BG}^2 \cdot \mathrm{GH}$, so that $\mathrm{BH} = \sqrt[3]{\mathrm{GH}^3 + \mathrm{BG}^3 + 3\,\mathrm{BG} \cdot \mathrm{GH}^2 + 3\,\mathrm{BG}^2 \cdot \mathrm{GH}}$. Now since $\mathrm{BG} = \sqrt[3]{v}$, $\mathrm{GH} = \sqrt[3]{u}$, thus $\mathrm{BH} = \sqrt[3]{u} + \sqrt[3]{v}$, and $\mathrm{GH}^3 = u$, $\mathrm{BG}^3 = v$, $3\,\mathrm{BG} \cdot \mathrm{GH}^2 = 3\,\sqrt[3]{v} \cdot (\sqrt[3]{u})^2 = \sqrt[3]{27\,u^2 \cdot v}$, $3\,\mathrm{BG}^2 \cdot \mathrm{GH} = 3\,(\sqrt[3]{v})^2 \cdot \sqrt[3]{u} = \sqrt[3]{27\,u \cdot v^2}$, this indeed means that $\sqrt[3]{u} + \sqrt[3]{v} = \sqrt[3]{u + v + \sqrt[3]{27\,u^2 \cdot v} + \sqrt[3]{27\,u \cdot v^2}}$. This also proves the identity $(a + b)^3 = a^3 + 3\,a^2 \cdot b + 3\,a \cdot b^2 + b^3$, to be used later on.

§4. Subtracting them

(A. 351–353) Subtraction of numerical square roots. For that, if we wish to subtract the square root of a number from the square root of a number, we add the two radicands (*al-'adadān al-majdhūrān*), subtract from the result the double of the square root of their product, and take the square root of the result. This will give what is required.[149]

(A. 354–357) Example. We wish to subtract the square root of four from the square root of nine. We multiply 4 by 9, which gives 36, and take the square root of that, which is 6. Then we double it, which gives 12. We subtract it from the sum of 4 and 9, thus 13, which leaves 1, and take the square root of it, which is 1. Such is the remainder of the subtraction of the square root of four from the square root of nine.[150]

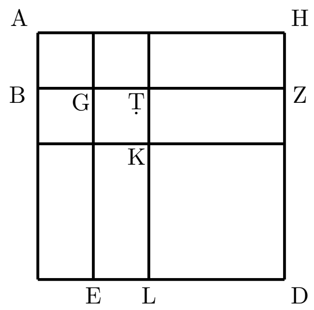

Figure 9: Proof that $\sqrt{u} - \sqrt{v} = \sqrt{u + v - 2\sqrt{u \cdot v}}$

(A. 358–363) To prove this,[151] we imagine the square AG smaller than the square GD, and, from the latter's side GZ, we subtract (a segment) equal to the side BG (of AG); let it be GṬ. We draw line ṬKL parallel to GE ⟨and line ṬZ parallel to LD⟩.

Now since the rectangle BE is equal to the rectangle EṬ,[152] the remaining rectangle KZ will be equal to the rectangle EṬ minus the square GK. But the square GK is equal to the square AG. So the two rectangles EṬ and KZ plus the square AG are equal to the two complements BE and GH. Therefore we shall subtract that from the (sum of the) two squares AG and GD, this leaving the square KD. We take

149 $\sqrt{u} - \sqrt{v} = \sqrt{u + v - 2\sqrt{u \cdot v}}$.

150 $\sqrt{9} - \sqrt{4} = \sqrt{9 + 4 - 2\sqrt{4 \cdot 9}} = 1$. Same example in Abū Kāmil (1986, fol. 20^{v} (Arabic); 1966, 79 (Hebrew); 1993, l. 993 (Latin)).

151 Namely that $\sqrt{u} - \sqrt{v} = \sqrt{u + v - 2\sqrt{u \cdot v}}$. Also in Abū Kāmil (1986, fol. 20^{v} (Arabic); 1966, 81 (Hebrew); 1993, l. 998 (Latin)).

152 And also to the rectangle GH.

its square root, which is LD, that is, ṬZ. Then the result (thus ṬZ) is the excess of GZ over BG. This is what we wanted to prove.[153]

(A. 364–369) Subtraction of numerical cube roots. If we wish to subtract the cube root of a number from the cube root of a number, we multiply the square of the lesser number by the greater number, then the result by 27, take the cube root of it, add it to the greater number and keep the result in mind. Then we multiply **(11^r)** the square of the greater number by the lesser number, then (the result) by 27, take the cube root of that, then add this to the lesser number. We subtract the result from what we have kept in mind, and take the cube root of the remainder. The result will be what is required.[154]

(A. 370–379) Example. If we wish to subtract the cube root of eight from the cube root of a hundred and twenty-five, we multiply the square of 8, thus 64, by 125, which gives 8000, then by 27, which gives 216 000; we take the cube root of that, which is 60, then add it to 125 and keep the result, thus 185, in mind. Then we multiply the square of 125, thus 15 625, by 8, which gives 125 000, then by 27, which gives 3 375 000; we take the cube root of that, which is 150, then add it to 8, which gives 158. We subtract that from what we have kept in mind, namely 185, which leaves 27. The cube root of that, namely 3, is the remainder of the subtraction of the cube root of eight from the cube root of a hundred and twenty-five.[155] That is how to proceed.

(A. 380–394) Proof of that.[156] The reason for that is (partly based on) the proof already given, in the paragraph on addition, namely that, if we multiply the cube with base AG by itself, then the result by the cube with base GZ, this being mul-

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[153] Let the two squares GD $= u$, with side GZ $= \sqrt{u}$, and AG $= v$, with side BG $= \sqrt{v}$ $(v < u)$. Take, on GZ, GṬ = BG, then draw ṬKL parallel to GE and ṬZ parallel to LD (extended in our drawing). Required GZ − GṬ = ṬZ = LD $= \sqrt{u} - \sqrt{v}$, which is the side of the square KD.

Now KD = GD + AG − (EṬ + GK + KZ) (with the square GK = AG thus occurring twice in the subtracted part), and EṬ+GK+KZ = BE+GH, so KD = GD+AG−(BE+GH). Taking the root of this last equality, we find, since $\sqrt{\text{KD}} = \text{LD}$, $\sqrt{u} - \sqrt{v} = \sqrt{u + v - 2\sqrt{u}\cdot\sqrt{v}} = \sqrt{u + v - 2\sqrt{u \cdot v}}$.

[154] $\sqrt[3]{u} - \sqrt[3]{v} = \sqrt[3]{(u + \sqrt[3]{27\,u \cdot v^2}\,) - (v + \sqrt[3]{27\,u^2 \cdot v}\,)}$.

[155] $\sqrt[3]{125} - \sqrt[3]{8} = \sqrt[3]{(125 + \sqrt[3]{27 \cdot 125 \cdot 8^2}\,) - (8 + \sqrt[3]{27 \cdot 125^2 \cdot 8}\,)}$, which therefore will take the form $\sqrt[3]{(125 + \sqrt[3]{216\,000}\,) - (8 + \sqrt[3]{3\,375\,000}\,)} = \sqrt[3]{(125 + 60) - (8 + 150)} = \sqrt[3]{185 - 158} = \sqrt[3]{27}$, and thus $\sqrt[3]{125} - \sqrt[3]{8} = 3$.

[156] Thus, that $\sqrt[3]{u} - \sqrt[3]{v} = \sqrt[3]{(u + \sqrt[3]{27\,u \cdot v^2}\,) - (v + \sqrt[3]{27\,u^2 \cdot v}\,)}$. Let $\sqrt[3]{u}$ = GH = GE, $\sqrt[3]{v}$ = BG (thus we have, for the two squares, GZ $= (\sqrt[3]{u}\,)^2$, AG $= (\sqrt[3]{v}\,)^2$); required GH − BG.

Take, on BH, GD = BG. By the known expansion of $(u - v)^3$, we know that $(\text{GH} - \text{BG})^3 = \text{GH}^3 - 3\,\text{GH}^2 \cdot \text{BG} + 3\,\text{GH} \cdot \text{BG}^2 - \text{BG}^3 = (\text{GH}^3 + 3\,\text{GH} \cdot \text{BG}^2) - (\text{BG}^3 + 3\,\text{GH}^2 \cdot \text{BG})$. These two expressions will be considered successively.

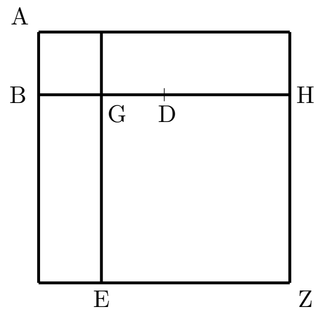

Figure 10: Proof of the subtractive case: $\sqrt[3]{u} - \sqrt[3]{v} = \sqrt[3]{(u + \sqrt[3]{27\,u \cdot v^2}\,) - (v + \sqrt[3]{27\,u^2 \cdot v}\,)}$

tiplied by 27, and take the cube root of the result, this will equal the product of base AG by the side GH, then the result by 3.[157] Then we separate from side GH (a segment) equal to the side BG, say GD.

According to what precedes, if we add the square of BG, or the equal square of GD, multiplied **(11ᵛ)** by GH, then the result by 3 [158]—which is the cube arising from side GD, taken three times, with the product of the square of GD by DH taken three times—to the cube arising from side GH [159]—that is, the two cubes with sides GD, DH, plus the product of the square of GD by DH taken three times, plus the product of the square of DH by GD taken three times—this is altogether equal to the cube of GD, taken four times, plus the cube of DH, once, plus the product of the square of GD by DH, taken six times, plus the product of the square of DH by GD, taken three times.[160]

This will be subtracted from: the multiplication of the square of GH by GD, then by 3 [161]—and this is equal to the multiplication of each of the two squares of GD and DH by GD, taken three times, and the multiplication of the area GD by DH, then by GD taken twice, then by 3, (this last term) being equal to the multiplication of the square of GD by DH, taken six times—this (sum) being added to the cube

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157 $\sqrt[3]{(\mathrm{BG}^3)^2 \cdot \mathrm{GH}^3 \cdot 27} = \left(\sqrt[3]{v^2 \cdot u \cdot 27} = \right) \mathrm{BG}^2 \cdot \mathrm{GH} \cdot 3$—this is simply $\sqrt[3]{u^3 \cdot v^3 \cdot w^3} = u \cdot v \cdot w$; see note 114, above.

158 Thus $3\,\mathrm{GH} \cdot \mathrm{BG}^2$, now to be transformed.

159 Thus GH^3, now to be transformed.

160 Consider first the above additive expression $\mathrm{GH}^3 + 3\,\mathrm{GH} \cdot \mathrm{BG}^2$ (which is $u + \sqrt[3]{27\,u \cdot v^2}\,$). On the one hand, $3\,\mathrm{GH} \cdot \mathrm{BG}^2 = 3\,(\mathrm{GD} + \mathrm{DH}) \cdot \mathrm{GD}^2 = 3\,\mathrm{GD}^3 + 3\,\mathrm{GD}^2 \cdot \mathrm{DH}$; on the other, $\mathrm{GH}^3 = \mathrm{GD}^3 + \mathrm{DH}^3 + 3\,\mathrm{GD}^2 \cdot \mathrm{DH} + 3\,\mathrm{GD} \cdot \mathrm{DH}^2$. Thus altogether, for the additive expression: $4\,\mathrm{GD}^3 + \mathrm{DH}^3 + 6\,\mathrm{GD}^2 \cdot \mathrm{DH} + 3\,\mathrm{GD} \cdot \mathrm{DH}^2$.

161 $3\,\mathrm{GH}^2 \cdot \mathrm{BG} = 3\,\mathrm{GH}^2 \cdot \mathrm{GD}$, now to be transformed and then added to $\mathrm{GD}^3 = \mathrm{BG}^3$.

arising from GD.[162] (After the subtraction,) there will remain the cube of DH.[163] That is why we take the cube root of it, which gives DH, which is the remainder of the subtraction of the cube root of BG from the cube root of GH. This is what we wanted to prove.

§5. Multiplying them

(A. 397–399) Multiplication of numerical square roots. If we wish to multiply the square root of a number by the square root of a number, we multiply the two radicands and take the square root of the result; this will give what is required.[164]

(A. 400–402) Example for rational square roots.[165] If we wish to multiply the square root of four by the square root of nine, we multiply 4 by 9, which gives 36, take the square root of that, which is 6. This is the product of the square root of four by the square root of nine.[166]

(A. 403–408) For the proof of that,[167] we replace the two radicands by the two squares DB and BE. We wish to multiply their square roots. Let the side of the square DB be AB **(12^r)** and the side of the square BE be BG. We complete the square DE.

Then the rectangle AG is that enclosed by the two square roots, and it is, as we have shown in the paragraph on multiples,[168] a mean proportional between the two squares DB and BE. Therefore we multiply together these two quantities (*mālān*), that is, the square DB and the square BE, and take the square root of the result. This will give what is required [it is the result, which is the rectangle AG].[169]

(A. 409–414) Following the reasoning we have (just) explained, when we wish to multiply two square roots of 9 by three square roots of 4, we shall determine first

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[162] Consider now the above subtractive expression $\mathrm{BG}^3 + 3\,\mathrm{GH}^2 \cdot \mathrm{BG}$ (which is $v + \sqrt[3]{27\,u^2 \cdot v}$). With BG = GD, this expression becomes $\mathrm{GD}^3 + 3\,\mathrm{GH}^2 \cdot \mathrm{GD}$. Since GH = GD + DH, its second term becomes $3\,(\mathrm{GD}^2 + \mathrm{DH}^2 + 2\,\mathrm{GD} \cdot \mathrm{DH}) \cdot \mathrm{GD} = 3\,\mathrm{GD}^3 + 3\,\mathrm{GD} \cdot \mathrm{DH}^2 + 6\,\mathrm{GD}^2 \cdot \mathrm{DH}$. So, altogether, this second expression takes the form $4\,\mathrm{GD}^3 + 3\,\mathrm{GD} \cdot \mathrm{DH}^2 + 6\,\mathrm{GD}^2 \cdot \mathrm{DH}$.

[163] Subtracting now the second expression from the first gives $(4\,\mathrm{GD}^3 + \mathrm{DH}^3 + 6\,\mathrm{GD}^2 \cdot \mathrm{DH} + 3\,\mathrm{GD} \cdot \mathrm{DH}^2) - (4\,\mathrm{GD}^3 + 3\,\mathrm{GD} \cdot \mathrm{DH}^2 + 6\,\mathrm{GD}^2 \cdot \mathrm{DH}) = \mathrm{DH}^3$. Since this is indeed the cube with side $\mathrm{DH} = \mathrm{GH} - \mathrm{GD} = \sqrt[3]{u} - \sqrt[3]{v}$, we have proved the identity.

[164] $\sqrt{u} \cdot \sqrt{v} = \sqrt{u \cdot v}$.

[165] Arabic: each of $\sqrt{4}$ and $\sqrt{9}$ is a *ji<u>dh</u>r maftūḥ* (syn. *ji<u>dh</u>r munṭaq*, see A. 290).

[166] $\sqrt{4} \cdot \sqrt{9} = \sqrt{36} = 6$. Same example in <u>Kh</u>wārizmī (1831, 30 (trans.), 21 (Arabic)) and Abū Kāmil (1986, fol. 18^r (Arabic); 1966, 71 (Hebrew); 1993, l. 858 (Latin)).

[167] That $\sqrt{u} \cdot \sqrt{v} = \sqrt{u \cdot v}$. Proof with separate segments of a straight line in Abū Kāmil.

[168] Above, A. 239–249.

[169] Let $\mathrm{AB} = \sqrt{u}$, $\mathrm{BG} = \sqrt{v}$, and consider the rectangle AG. Since $\mathrm{AG} = \mathrm{AB} \cdot \mathrm{BG}\ (= \sqrt{u} \cdot \sqrt{v})$ and $\mathrm{AG}^2 = \mathrm{AB}^2 \cdot \mathrm{BG}^2 = \mathrm{DB} \cdot \mathrm{BE}$, so $\mathrm{AG} = \mathrm{AB} \cdot \mathrm{BG} = \sqrt{\mathrm{DB} \cdot \mathrm{BE}}$, that is, $\sqrt{u} \cdot \sqrt{v} = \sqrt{u \cdot v}$.

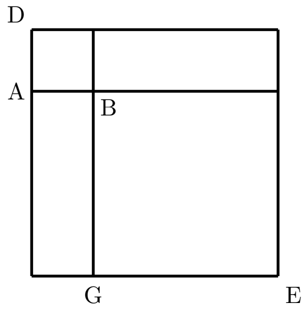

Figure 11: Proof that $\sqrt{u} \cdot \sqrt{v} = \sqrt{u \cdot v}$

of what quantity two square roots of 9 is the square root by the previous reasoning; it is [the square root of] 36,[170] which we keep in mind. Then we also determine of what quantity three square roots of 4 is the square root; it is also [the square root of] 36. Thus it is like (saying): we wish to multiply the square root of 36 by the square root of 36. So we multiply 36 by 36 and take the square root of the result; it is 36, and this is the product of two square roots of 9 by three square roots of 4.[171]

(A. 415–426) Multiplying fractions of numerical square roots. If we wish to multiply a fraction of the square root of a number by a fraction of the square root of a number, we multiply each of the two fractions by itself, then by the corresponding number, then multiply the two results and take the square root of that. The result will be what is required.[172]

Example. We wish to multiply two thirds of the square root of 9 by three fifths of the square root of 25. We first determine of what quantity two thirds of the square root of 9 is the square root according to the previous reasoning in the paragraph on taking fractions;[173] it is [the square root of] 4. Then we also determine of what quantity **(12ᵛ)** three fifths of the square root of 25 is the square root; it is [the square root of] 9. So our proposition is as if we were to multiply the square root of 4 by the square root of 9.[174] That is how to proceed.

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[170] Same supplement in what follows (A. 412, 421, 422, 436, 441). Unlikely to be by the author.

[171] $(2\sqrt{9}) \cdot (3\sqrt{4}) = \sqrt{36} \cdot \sqrt{36} = 36$. Same example in Khwārizmī (1831, 30–31 (trans.), 21 (Arabic)).

[172] $\frac{p_1}{q_1}\sqrt{u} \cdot \frac{p_2}{q_2}\sqrt{v} = \sqrt{(\frac{p_1}{q_1})^2 u} \cdot \sqrt{(\frac{p_2}{q_2})^2 v} = \sqrt{(\frac{p_1}{q_1})^2 u \cdot (\frac{p_2}{q_2})^2 v}$.

[173] Reference perhaps added by a reader; we just met this procedure before. See also subsequent additions.

[174] $\frac{2}{3}\sqrt{9} \cdot \frac{3}{5}\sqrt{25} = \sqrt{\frac{4}{9} \cdot 9} \cdot \sqrt{\frac{9}{25} \cdot 25} = \sqrt{4} \cdot \sqrt{9} = 6$ (note 166, above). Thus here no unit fractions (see note 119).

The reason for that is the following: what is required is the rectangle enclosed by the two square roots,[175] which is the mean proportional between their two squares. Therefore we shall determine the square of each one, multiply together these two quantities and take the square root of the product; the result will be what is required.

(A. 427–429) By the same reasoning, if we wish to multiply the square root of a number by a fraction of the square root of a number, we shall determine of what quantity that fraction is the square root, then multiply the resulting quantity[176] by the number of the square root and take the square root of that. This will be what is required.[177] [This for the reason already given.]

(A. 430–432) Multiplication of numerical cube roots. If, for all what we have explained involving square roots in this paragraph, there are cube roots instead, we shall raise to the cube here what we have squared before, and take the cube root here instead of taking the square root as before, without any (other) modification.

(A. 433–443) Multiplication of numerical square roots by numerical cube roots. Likewise if we wish to multiply the square root of a number by the cube root of a number.[178] As if we were to multiply the square root of four by the cube root of eight. We make the square root of 4 a cube, namely by multiplying it by itself, which gives 4, then by the square root of 4, which gives four square roots of 4; next we determine to what [square root of a] quantity correspond four square roots of 4, according to the previous reasoning;[179] this is the square root of 64. That is the cube arising from the square root of 4, and its cube root is the cube root of the square root of 64. Our question (stated above) is then as if we were to multiply the cube root of 8 by the cube root of the square root of 64. In accordance with the previous reasoning, we multiply one of the two cubes, here 8, by the cube of the other, thus (by) the square root of 64; this gives **(13$^{\mathrm{r}}$)** eight square roots of 64. We again need to determine of what quantity eight square roots of 64 is the square root; this is [the square root of] 4096; we take the cube root of the square root of it.[180] This gives 4, and such is the product of the cube root of eight by the square root of four.[181] That is how to proceed.

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[175] Manuscript: *juz'ān* instead of *jidhrān*, a copyist's confusion. The square roots are supposed to include the multiplicative fractions.

[176] Here appropriate (see note 170).

[177] Since $\frac{p}{q}\sqrt{v} = \sqrt{\frac{p^2}{q^2}\,v}$, so $\sqrt{u} \cdot \frac{p}{q}\sqrt{v} = \sqrt{u \cdot \frac{p^2}{q^2}\,v}$.

[178] $\sqrt{u} \cdot \sqrt[3]{v} = \sqrt[3]{\sqrt{u^3}} \cdot \sqrt[3]{v} = \sqrt[3]{\sqrt{u^3}} \cdot \sqrt[3]{\sqrt{v^2}} = \sqrt[3]{\sqrt{u^3 \cdot v^2}}$.

[179] Above, A. 219–222.

[180] Thus the sixth root of 4096.

[181] Thus (shorter): $\sqrt{4} \cdot \sqrt[3]{8} = \sqrt[3]{\sqrt{4^3}} \cdot \sqrt[3]{\sqrt{8^2}} = \sqrt[3]{\sqrt{64}} \cdot \sqrt[3]{\sqrt{64}} = \sqrt[3]{\sqrt{64} \cdot \sqrt{64}} = \sqrt[3]{64} = 4$.

(A. 444–448) Multiplication of numerical fourth roots, which are the sides of fourth powers.[182] If we wish to multiply the fourth root of a number by the fourth root of a number, we shall multiply together the two numbers, for they are of the same kind (*mutajānisān*), and take the fourth root of the product. The result will be what is required.[183]

The reason for that has been given, namely that for any two numbers the square root of the product of their squares equals the fourth root of the product of their fourth powers.[184]

(A. 449–451) We shall proceed likewise if we wish to multiply the fourth root of a number by the square root of a number. We shall multiply the radicand of the square root (*al-ʿadad al-majdhūr*) once by itself, so that it becomes of the same kind as the other, then multiply together the two quantities and take the ⟨fourth⟩ root of the product. The result will be what is required.[185]

§6. Dividing them

(A. 454–460) Division of numerical square roots. If we wish to divide the square root of a number by the square root of a number, we divide the quantity of the dividend (*māl al-maqsūm*) by the quantity of the divisor. Taking the square root of the quotient will give the answer.[186]

Example of that for rational roots.[187] If we wish to divide the square root of 36 by the square root of 4, we divide 36 by 4; the result is 9. So the square root of 9, thus 3, is the quotient of the square root of 36 divided by the square root of 4.[188]

The reason for that we have given elsewhere, (namely) that the division is the inverse of the multiplication.[189] **(13$^{\mathrm{v}}$)**

(A. 461–464) Division of fractions of numerical square roots. If we wish to divide a fraction of the square root of a number by a fraction of the square root of a number, we multiply the (first) fraction by itself, then the result by its associated number, and we do the same with the other fraction; then we divide the result for

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[182] Literally: "sides of square-squares" (see note 113).

[183] $\sqrt[4]{u} \cdot \sqrt[4]{v} = \sqrt[4]{u \cdot v}$.

[184] $\sqrt{u^2 \cdot v^2} = \sqrt[4]{u^4 \cdot v^4} = u \cdot v$ (above, A. 250–262, note 114).

[185] $\sqrt[4]{u} \cdot \sqrt{v} = \sqrt[4]{u} \cdot \sqrt[4]{v^2} = \sqrt[4]{u \cdot v^2}$.

[186] $\frac{\sqrt{u}}{\sqrt{v}} = \sqrt{\frac{u}{v}}$. See Khwārizmī (1831, 29–30 (trans.), 20–21 (Arabic)); Abū Kāmil (1986, fol. 19$^{\mathrm{v}}$ (Arabic); 1966, 75 (Hebrew); 1993, l. 931 (Latin)).

[187] Arabic: *al-judhūr al-maftūḥa*, see note 165.

[188] $\frac{\sqrt{36}}{\sqrt{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$.

[189] A. 69, applied here to the previous § 5 (dividing now instead of multiplying as before).

the dividend by the result for the divisor, and take the square root of the quotient. The result will be the answer.[190]

(A. 465–468) We shall proceed likewise if we wish to divide the square root of a number by a fraction of the square root of a number: we shall determine the quantity corresponding to that (latter) square root, namely by multiplying the fraction by itself and the result by its associated number, then we shall divide the radicand of the dividend (*‘adad al-maqsūm*) by the (radicand of the) newly-formed divisor and take the square root of the quotient. The result will be the answer.[191]

(A. 469–471) Division of numerical cube roots. If there are, instead of the square roots mentioned in this paragraph, cube roots, the procedure remains the same, except that we shall raise to the cube here instead of raising to the square there, and take the cube root here instead of taking the square root there, without any (other) difference.[192]

(A. 472–478) Division of numerical cube roots and square roots. We wish to divide the cube root of a number by the square root of a number.[193] As if we were to divide the cube root of eight by the square root of four. We make the square root of 4 a cube, and we proceed just as we did in the paragraph on multiplication;[194] so the treatment will end up with our having to divide the cube root of 8 by ⟨the cube root of⟩ the square root of 64. So we need here to multiply 8 by itself in order to make it of the same kind as the other, then divide the result, namely 64, by the divisor, which is also 64; this gives 1. We take ⟨the cube root of⟩ its square root, which is also 1, and that will be the answer.[195]

(A. 479–492) Likewise if we wish to divide the square root of a number by the cube root **(14^r)** of a number.[196] As if we were to divide the square root of sixty-four by the cube root of eight. We reduce the square root of 64 to the kind of the cube, namely by multiplying the square root of 64 by itself, which gives 64, then by the square root of 64, which gives 64 times the square root of 64. Then we shall determine of what quantity 64 times the square root of 64 is the square root;[197] the

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190 $\frac{\frac{p_1}{q_1}\cdot\sqrt{u}}{\frac{p_2}{q_2}\cdot\sqrt{v}} = \frac{\sqrt{(\frac{p_1}{q_1})^2\cdot u}}{\sqrt{(\frac{p_2}{q_2})^2\cdot v}} = \sqrt{\frac{(\frac{p_1}{q_1})^2\cdot u}{(\frac{p_2}{q_2})^2\cdot v}}\ \left(= \sqrt{(\frac{p_1 q_2}{q_1 p_2})^2\cdot\frac{u}{v}}\right).$

191 $\frac{\sqrt{u}}{\frac{p_2}{q_2}\cdot\sqrt{v}} = \sqrt{\frac{u}{(\frac{p_2}{q_2})^2\cdot v}}.$

192 $\frac{\sqrt[3]{u}}{\sqrt[3]{v}} = \sqrt[3]{\frac{u}{v}}$ (and note 112, if there are multiplicative factors).

193 $\frac{\sqrt[3]{u}}{\sqrt{v}} = \frac{\sqrt[3]{u}}{\sqrt[3]{(\sqrt{v})^3}} = \sqrt[3]{\frac{u}{\sqrt{v^3}}} = \sqrt[3]{\frac{\sqrt{u^2}}{\sqrt{v^3}}} = \sqrt[3]{\sqrt{\frac{u^2}{v^3}}}.$

194 Above, A. 433–443 (note 178).

195 $\frac{\sqrt[3]{8}}{\sqrt{4}} = \frac{\sqrt[3]{8}}{\sqrt[3]{(\sqrt{4})^3}} = \frac{\sqrt[3]{8}}{\sqrt[3]{\sqrt{64}}} = \sqrt[3]{\frac{\sqrt{8^2}}{\sqrt{64}}} = \sqrt[3]{\frac{\sqrt{64}}{\sqrt{64}}} = \sqrt[3]{\sqrt{\frac{64}{64}}}.$

196 $\frac{\sqrt{u}}{\sqrt[3]{v}} = \frac{\sqrt[3]{(\sqrt{u})^3}}{\sqrt[3]{v}} = \sqrt[3]{\frac{\sqrt{u^3}}{v}} = \sqrt[3]{\frac{\sqrt{u^3}}{\sqrt{v^2}}} = \sqrt[3]{\sqrt{\frac{u^3}{v^2}}}.$

197 Calculating thus $64\cdot\sqrt{64} = \sqrt{64^3} = \sqrt{262\,144}$.

corresponding method is the previous one, namely that we multiply the number of the multiple, thus 64, by itself, then by the number in question, which is also 64; this gives 262 144; the square root of the cube root of this is the square root of 64 reduced to the kind of the cube.[198] So it is as if we were to divide the square root of the cube root of 262 144 by the cube root of 8. According to the previous procedure, we make the cube root of 8 a square root, whereby it will be of the same form as the dividend; that is, we multiply 8 by itself, which gives 64. Then we divide 262 144 by 64, which gives the quotient 4096. Such is the result of dividing the square root of sixty-four by the cube root of eight.[199] Since the cube root of 4096 is 16 and the square root of that, 4, this is the answer.[200]

(A. 493–495) Division of numerical fourth roots, which are the sides of fourth powers. If we wish to divide the fourth root of a number by the fourth root of a number, we divide the number of the dividend by the number of the divisor and take the fourth root of the quotient. The result will be the answer.[201] **(14$^{\text{v}}$)**

(A. 496–498) We shall proceed likewise if we wish to divide the fourth root of a number by the square root of a number: we multiply the radicand of the square root once by itself, so that it will have the same form as the other, then we divide the dividend by the newly-formed divisor and take the fourth root of the quotient. The result will be the answer.[202]

(A. 499–511) Division of numerical general powers (*ṭabaqāt muṭlaqa mansūba*), whether one or several.[203] In such a type of division, that which can be treated successfully is the division of a set of terms (*ajnās muqtarana*), however many, by a single term (*jins wāḥid*), whatever it is, general (*muṭlaq*) or numerical (*mansūb*). (But) if the divisor (contains) more than one term, the division is hardly possible, except by (resorting to) devices which the (usual) approach does not require; this is the case when the divisor does not (contain) more than two terms, one of which is known and the other a numerical ⟨square root⟩.[204] If the divisor (consists of) two terms one of which is general—that is, a thing, a cube or something like that—or if

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198 $\sqrt{\sqrt[3]{262\,144}}$, instead of $\sqrt[3]{\sqrt{262\,144}}$. See notes 193, 196.

199 This sentence should be either deleted or placed after the next one.

200 $\frac{\sqrt{64}}{\sqrt[3]{8}} = \frac{\sqrt[3]{(\sqrt{64})^3}}{\sqrt[3]{8}} = \frac{\sqrt[3]{64\cdot\sqrt{64}}}{\sqrt[3]{8}} = \frac{\sqrt[3]{\sqrt{262\,144}}}{\sqrt[3]{8}} = \frac{\sqrt[3]{\sqrt{262\,144}}}{\sqrt[3]{\sqrt{8^2}}} = \sqrt[3]{\frac{\sqrt{262\,144}}{\sqrt{64}}} = \sqrt[3]{\sqrt{4096}} = \sqrt[3]{64} = 4.$
Or, as in the text, $\sqrt{\frac{\sqrt[3]{262\,144}}{\sqrt[3]{64}}} = \sqrt{\sqrt[3]{4096}} = \sqrt{16} = 4.$

201 $\frac{\sqrt[4]{u}}{\sqrt[4]{v}} = \sqrt[4]{\frac{u}{v}}.$

202 $\frac{\sqrt[4]{u}}{\sqrt{v}} = \frac{\sqrt[4]{u}}{\sqrt[4]{v^2}} = \sqrt[4]{\frac{u}{v^2}}.$

203 Rather: Division of expressions involving numerical roots, each with one or several terms. (Such divisions are considered feasible if there is no compound expression left in the denominator.)

204 See last two examples below, where the divisor is the sum of an integer and a square root. But the same treatment could be applied if there were two square roots.

there are more than two terms, of whatever kind, then there is no way to determine the quotient.[205]

[Example of that when the divisor is monomial (*mufrad*). We wish to divide ten plus the square root of fifteen by a thing. We divide 10 by a thing, so the quotient is ten parts of a thing. Then we divide the square root of 15 by a thing, that is, we multiply the thing by itself, whence a square, then by 15, which gives fifteen squares, and we take the square root of that, which is the square root of fifteen squares. We add (all) this, which gives ten parts of a thing plus the square root of fifteen squares. That is how to proceed.][206]

(A. 512–517) Example of that when the divisor is a single numerical (term). We wish to divide ten plus the square root of twenty by the square root of four. We divide 10 by the square root of 4, that is, we multiply 10 by itself, which gives 100, then we divide 100 **(15^r^)** by 4, which gives 25, and take the square root of that, which is 5; we keep it in mind. Then we divide the square root of 20 by the square root of 4 in the manner seen before; the quotient will be the square root of 5. Adding it to what we have kept in mind gives 5 plus the square root of 5.[207] That is how to proceed.

(A. 518–543) Example with the dividend simple and the divisor compound (*maqrūn*). If we wish to divide fifty by ten plus the square root of ten, we use for that a multiplication as a device: we subtract the square root of 10 from 10, which leaves 10 minus the square root of 10, and then multiply 10 minus the square root of 10 by 10 plus the square root of 10; this gives 90.[208]

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205 This needs to be clarified. Thus, first, a very complete treatment of numerical square and fourth roots is found in Chapter A-IX of Johannes Hispalensis' *Liber Mahameleth*, written ca. 1150, where Problems A.320–322, in particular, rationalize trinomial divisors, consisting of either one number and two square roots or three square roots (Sesiano 2014, 1334–1336). Second, and more generally, the divisor may well contain more terms and roots with higher indices, but then the treatment for rationalizing it becomes more complicated than in the simpler case of two terms with square roots.

206 Erroneous and out of place, thus most probably interpolated. Erroneous, for in fact $\frac{10+\sqrt{15}}{x} = \frac{10}{x} + \frac{\sqrt{15}}{x} = \frac{10}{x} + \sqrt{\frac{15}{x^2}}$; out of place, because of the presence of a "thing" (doubtless inspired by the "thing" mentioned above)—indeed, Part III deals only with numerical roots. Note that division by a "thing" occurred in A. 208–213.

207 $\frac{10+\sqrt{20}}{\sqrt{4}} = \sqrt{\frac{100}{4}} + \sqrt{\frac{20}{4}} = 5 + \sqrt{5}$. The reference is to A. 454–460 (note 186).

208 We are to calculate $\frac{50}{10+\sqrt{10}}$, and for that shall multiply dividend and divisor by $10-\sqrt{10}$ in order to rationalize the denominator. But the modern reader will omit the next paragraph, which is yet another justification using the theory of proportions. Thus, here, since $(10+\sqrt{10})(10-\sqrt{10}) = 90$, and so $\frac{90}{10+\sqrt{10}} = 10 - \sqrt{10}$, while we wish to calculate $\frac{50}{10+\sqrt{10}} = t$, so $50 : t = 90 : (10-\sqrt{10})$, whence $t = \frac{50\,(10-\sqrt{10})}{90}$.

Since 10 plus the square root of 10 has been multiplied by 10 minus the square root of 10 with the result 90, if we divide 90 by 10 plus the square root of 10 the quotient will be 10 minus the square root of 10; indeed, for any two numbers when the first is multiplied by the other and the result is divided by one of them, this will give the other.[209] But if we divide 50, which is the dividend, by 10 plus the square root of 10, our result will be a number such that the ratio of 50 to this number equals the ratio of 90 to 10 minus the square root of 10; for both 90 and 50 are divided by the same number, namely 10 plus the square root of 10. So the ratio of the first dividend to its quotient will equal the ratio of the other dividend to its quotient. These four numbers are then proportional, three of them being known and one unknown.

So we (now) multiply 50 by 10 minus the square root of 10; the multiplication gives 500 minus fifty square roots of 10, and we divide that by 90. The quotient of 500 divided by 90 is 5 plus $\frac{5}{9}$, that of fifty square roots of 10, negative, thus the square root of 25 000, negative, by 90 **(15^v^)** is the square root of $3+\frac{7}{81}$, negative. Adding that gives $5+\frac{5}{9}$ minus the square root of $3+\frac{7}{81}$.[210] That is how to proceed.

The reason for that is (the following).[211] We have subtracted the square root of 10 from 10, then multiplied the remainder by 10 plus the square root of 10 in order to obtain a rational number (*ʿadad munṭaq*); indeed, any binomial (*ʿadad d̲h̲ū al-ismayn*) when multiplied by its apotome (*munfaṣil*) gives a rational result.[212] A general binomial (*d̲h̲ū al-ismayn muṭlaq*) is any number composed of (either) two numbers rational in power or one rational in length (*munṭaq fī'l-ṭūl*) and the other rational in power (*munṭaq fī'l-qūwa*); such are the square root of 10 plus the square root of 3, or 10 plus the square root of 10 and the like.[213] The apotome is a binomial with its smaller part subtracted from the larger one; then the remainder is said to be an apotome in general.[214]

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[209] $v \cdot \frac{u}{v} = u$. Mentioned in K̲h̲wārizmī (1831, 50 (trans.), 36 (Arabic)) and Abū Kāmil (1986, fol. 19^r (Arabic); 1966, 75 (Hebrew); 1993, l. 915 (Latin)).

[210] $\frac{50}{10+\sqrt{10}} = \frac{50\,(10-\sqrt{10})}{(10+\sqrt{10})\,(10-\sqrt{10})} = \frac{500-50\sqrt{10}}{90} = \frac{500-\sqrt{25\,000}}{90} = 5+\frac{5}{9}-\sqrt{3+\frac{7}{81}}$.

[211] Explains why the denominator has thus been rationalized, with reference to the *Elements*.

[212] "Binomial" here in Euclid's sense. See *Elements*, X.36, X.73, X.114. Whence it could be inferred that the above multiplication procedure will also work if the denominator contains two square roots (see note 205).

[213] Here the binomial is called "general" (*muṭlaq*) because there are six kinds of it. See *Elements*, X, def. II (following prop. 47). As for the expressions "rational in length" and "rational in power," they are clear from the context: the first is rational, the second becomes rational when squared.

[214] Again "general" because there are six kinds of apotome. See *Elements*, X, def. III (following prop. 84). Root extraction of binomials and apotomes is taught in the *Liber Mahameleth* (A–IX, Sesiano 2014); see also *Elements* X.54–59, X.91–96. Algebraic extraction of roots of binomials and

(A. 544–554) Example when both the dividend and the divisor are compound.[215] We wish to divide fifty plus the square root of two hundred by ten plus the square root of ten. We first divide 50 by 10 plus the square root of 10 just as we did in the previous example; this gives us the same result, thus $5+\frac{5}{9}$ minus the square root of $3+\frac{7}{81}$. Then we divide the square root of 200 by 10 plus the square root of 10, just as in the previous treatment; that is, we multiply the square root of 200 by 10 minus the square root of 10, which gives ten square roots of 200, thus the square root of 20 000, minus the square root of 2000, and we divide that by 90; this gives the square root of $2+\frac{38}{81}$, minus the square root of $\frac{20}{81}$. Adding (all) that gives $5+\frac{5}{9}$ minus the square root of $3+\frac{7}{81}$, plus the square root of $2+\frac{38}{81}$, **(16r)** minus the square root of $\frac{20}{81}$.[216] That is how to proceed.

Fourth part

On simple and compound equations involving proportional powers. This will comprise two paragraphs.

§1. Simple equation

(A. 560–567) A simple equation (*mu‘ādala mufrada*) is that in which one of the proportional kinds (*anwā‘ mutanāsiba*) mentioned (previously) is equated to another one, that is, is equal to it. There are three such simple equations, namely those occurring between (two of) the first three proportional kinds, thus number, root and square. They are the basis (*uṣūl*) for the other simple equations, since the latter will reduce to them and (their degrees) lowered (*munḥaṭṭa*) to theirs in such a way that (each term) becomes of a certain (lower) kind if none of the two equated (terms already) has the power of a number.[217] In some of the treatments we shall see, for all these (equations), the coefficient (*‘idda*) of the higher (in degree) of the two equated terms (*al-nau‘ān al-mu‘ādilān*) may be more or less than the unit: then we

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apotomes is also found in Karajī's *Badī‘* (al-Karajī 1964). Note that we have already met with a piece of theory occurring within the text: see note 108.

215 Both are binomials, consisting thus (here) of a number and a square root.

216 $\frac{50+\sqrt{200}}{10+\sqrt{10}}=\frac{50}{10+\sqrt{10}}+\frac{\sqrt{200}}{10+\sqrt{10}}$. Then, since, as seen, $\frac{50}{10+\sqrt{10}}=5+\frac{5}{9}-\sqrt{3+\frac{7}{81}}$ while $\frac{\sqrt{200}}{10+\sqrt{10}}=\frac{\sqrt{200}\,(10-\sqrt{10})}{(10+\sqrt{10})(10-\sqrt{10})}=\frac{10\sqrt{200}-\sqrt{10\cdot 200}}{90}=\frac{\sqrt{20\,000}-\sqrt{2000}}{90}=\sqrt{2+\frac{38}{81}}-\sqrt{\frac{20}{81}}$, so, altogether, $\frac{50+\sqrt{200}}{10+\sqrt{10}}=5+\frac{5}{9}-\sqrt{3+\frac{7}{81}}+\sqrt{2+\frac{38}{81}}-\sqrt{\frac{20}{81}}$.

217 We are thus to divide the two terms by the lower power, so that we shall be left with one power equal to a number. This is exactly the instruction given by Diophantus for simple equations in the part of the *Arithmetica* extant in Arabic, which involves higher powers (Sesiano 1982, 88, 179; Tannery 1893, 60.20 (Greek)). Since at that time $x=0$ as a solution was not considered, such a reduction was obvious. (The first "simple equation" $ax^2=bx$ nevertheless remained still used as such.) Here it is implicit that the two powers of the given equation of higher degree must not differ by more than two degrees. See also A. 637–642 for such reductions.

need to change this (coefficient) to the integer 1, (namely) by making up for what lacks or removing what is in excess, the same treatment being (then) applied to the kind lesser (in degree) of the equation [whereby they will conform to (the terms of) the elementary ratio].[218]

(A. 568–578) The treatment for reducing these (higher) equated kinds to a single one if there is no fraction(s) is simple, involving little work and trouble. If there are fractions [or there are fractions with both of them], we need then to apply (one of) two treatments, one to add to a given number a given fraction of itself and the other to remove from it a given fraction of itself.[219]

The case of addition is when we wish to add to a given number a given fraction of itself. We then put the denominator of this fraction in two places, add to one of them the fraction of itself, multiply the result by the (given) number **(16ᵛ)** and divide the result ⟨by⟩ the denominator in the second place; the quotient will be the number increased by its fraction.

The case of subtraction is when we wish to subtract from a given number a given fraction of itself. We shall just do as we did for the addition, only subtracting from one of the two places its fraction instead of adding it there; the resulting quotient will be what is required.

(A. 579–588) Example for the addition. We wish to add to $1+\frac{2}{3}$ its fifth.[220] We put the denominator of a fifth, thus 5, in two places. We add to one of the two places its fifth, thus 1, which gives 6. Then we multiply 6 by the (given) number, thus $1+\frac{2}{3}$, which gives 10. We then divide 10 by the denominator in the other place, thus 5; the quotient is 2, and this is $1+\frac{2}{3}$ increased by its fifth. That is how to proceed.

The reason for that is (the following): when we have placed 5 in two places and added to one of them its fraction, namely a fifth, whence 6, the ratio of 5 to 6 will be the same as the ratio of the (given) number, namely $1+\frac{2}{3}$, to what is required, for what is required must be equal to $1+\frac{2}{3}$ with its fifth. This is why we multiply

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218 The elementary ratio is that involving the terms 1, x, x^2. Note that, here, taking the coefficient of the higher power equal to 1 in simple (binomial) equations is merely a formal requirement. For trinomial equations, that is in keeping with the usual canonical forms; indeed, the solving formulae are taught for a coefficient of the highest power equal to 1, as seen below.

219 If the coefficient of the higher power is the integer a, the whole will be multiplied by $\frac{1}{a}$; if it is $a+\frac{p}{q}=\frac{aq+p}{q}$, we shall analogously multiply the whole by $\frac{q}{aq+p}$. Here the text is quite confusing, and its use of a false position (see our introduction) quite inappropriate. Note that Kh̲wārizmī already has the simple, and correct, procedure (see, e.g., 1831, 39 (trans.), 27 (Arabic)).

220 Adding to the quantity considered, $\frac{5}{3}$, its fifth will change it to $\frac{6}{3}$. We do not obtain the coefficient 1, though this was the requirement. We should have been told to subtract from the given quantity its $\frac{2}{5}$, but the author, for whatever reason, keeps to the use of unit fractions (see note 119).

6 by $1+\frac{2}{3}$, thus the second (proportional term) by the third, and divide that by 5, which is the first, the result being what is required, namely the fourth.[221]

(A. 589–592) Example for the subtraction. We wish to subtract from $1+\frac{1}{3}$ its fourth.[222] We put the denominator of a fourth, thus 4, in two places, and we subtract from one of them its fourth, which is 1, leaving 3, then we multiply 3 by $1+\frac{1}{3}$, giving 4; we divide this by the denominator in the second place, which is also 4; the result is 1, and that is the answer.

The three equations involving a number, roots and squares[223]

(A. 594–602) The first **(17^r^)** is "roots equal to a number."

(*i*) This is like our statement "a root equals three." So the root is 3 and the corresponding square, 9.[224]

(*ii*) And like our statement "four roots equal twelve." The coefficient of the higher (in degree; *ab'ad*) of the two equated kinds, namely the quantity of the roots, is larger than 1, for it is 4.[225] So, in order to reduce it to 1, we are to subtract from the whole we have, (thus) roots and number, their three fourths.[226] We shall end up with a root equal to 3, so the root is 3 and the corresponding square, 9.

(*iii*) And like our statement "half a root is equal to $1+\frac{1}{2}$." Since here the coefficient of the roots is less than 1 (namely a half), what we need, to bring it to the integer 1, is to add to what we have the same. So we shall have a root equal to 3; so the root is 3 and the corresponding square, 9.

(A. 603–611) The second equation is "squares equal to a number."

(*i*) This is like our saying "a square equals nine." So the square is 9 and the corresponding root, 3.[227]

(*ii*) And like our statement "three and a third squares equal 30." Since the coefficient of the squares is larger than 1 and there is a fraction with it, we shall convert (*basaṭa*)

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[221] With 5 becoming 6, we must have $\frac{5}{6}=\frac{1+\frac{2}{3}}{z}$, with $z=\frac{5}{3}\cdot\frac{6}{5}$ thus the required quantity. But a simple multiplication by $\frac{3}{5}$ would spare the reader these needless explanations.

[222] Removing from the quantity considered, $\frac{4}{3}$, its fourth will change it to $\frac{3}{3}$.

[223] Thus involving, as "simple" equations, two of these terms.

[224] Same example in Khwārizmī (1831, 7 (trans.), 4 (Arabic)). Here the two following examples end up with the same equation.

[225] "The quantity of *the* roots," as in the text; the article has its importance in verbal algebra, for it indicates that the quantity referred to has already been mentioned.

[226] Here the absurdity of the transformation becomes patent.

[227] Same example in Khwārizmī (1831, 7 (trans.), 4 (Arabic)). Here again the two following examples reduce to the same equation.

that to the kind (*jins*) of the fraction, (namely) thirds, which gives $\frac{10}{3}$. So what we have to subtract from that, in order to reduce the (higher power) to one square, is $\frac{7}{3}$, thus its $\frac{7}{10}$;[228] and we subtract from 30 its $\frac{7}{10}$ as well, which is 21. After that, we shall have a square equal to 9.

(*iii*) And like our statement "two thirds of a square equal 6." We need here to add to all that we have its half. After that, we shall have a square equal to 9.

(A. 612–636) The third equation is "squares equal to roots."

(*i*) This is like our statement "a square equals three roots." Since we have shown[229] that the ratio of the square to the root is the same as the ratio of the root to 1, then the ratio of the square to the three roots will be the same as the ratio of the root to three units.[230] So the root of the square is 3 and the corresponding **(17ᵛ)** square is 9, which (indeed) equals three of its roots.

(*ii*) And like our statement "two and a third squares equal seven roots."[231] Since the coefficient of the squares is larger than 1 and there is a fraction with it, we shall reduce that to the kind of the fraction (thus) thirds; this gives $\frac{7}{3}$, which are $\frac{4}{7}$ of it;[232] if we subtract from the seven roots their four sevenths, thus 4 (roots), there will remain three roots, and these will be the roots equal to one square; so we shall say that the square is equal to three roots, and so the root equals 3. (Thus) the root is 3 and the square, 9. Twice and a third times this square is 21, and this (indeed) equals seven of its roots.

(*iii*) And like our statement "two thirds of a fifth of a square is equal to a seventh of its root." Since the quantity of the squares is less than 1 and its denominator (consists) of 3 by 5, thus of 15, so its $\frac{2}{3}$ of $\frac{1}{5}$ is $\frac{2}{15}$. So we need for reducing it to one integral square to add to it thirteen ⟨times its half⟩, thus $6 + \frac{1}{2}$ times itself.[233]

As to the multiplication of it and what it is equal to [by $7 + \frac{1}{2}$], we shall follow in that the method of the reasoning presented before, namely to put 1 in two places [because of the same],[234] and add to ⟨one of⟩ the two places $6 + \frac{1}{2}$ times itself; this gives $7 + \frac{1}{2}$. We then multiply this by the coefficient of the root, thus $\frac{1}{7}$, which gives $1 + \frac{1}{2}\frac{1}{7}$. We divide that by the second place, which is 1. [The result of the division is $1 + \frac{1}{2}\frac{1}{7}$ since anything multiplied or divided by 1 remains unchanged.][235] Then

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228 For if $\frac{10}{3}\,(1 - \frac{p}{q}) = 1$, so $1 - \frac{p}{q} = \frac{3}{10}$, thus $\frac{p}{q} = \frac{7}{10}$.

229 A. 6–7 and missing part.

230 Meaningless use of the ratio instead of the equality.

231 Reducible to the same equation as before.

232 $\frac{7}{3}(1 - \frac{p}{q}) = 1$, so $\frac{p}{q} = \frac{4}{7}$.

233 $\frac{2}{15}\,(1 + \frac{p}{q}) = 1$, so $\frac{p}{q} = \frac{15}{2}\,(1 - \frac{2}{15}) = \frac{13}{2}$. The subsequent text is partly corrupt.

234 Or: because of the (one) time. Anyway, *obscurum per obscurius*.

235 Seems superfluous. We have now found the new coefficient of the roots.

we shall say that the roots equal to one square are a root plus half a seventh of a root. According to the proportionality (*tanāsub*) seen before,[236] the root will be $1 + \frac{1}{2}\frac{1}{7}$ and the corresponding square, $1 + \frac{1}{7} + \frac{1}{196}$.[237] If we convert the square to the kind of the fraction, thus to 196ths, the result will be **(18^r^)** 225, $\frac{2}{3}\frac{1}{5}$ of which is 30, which we keep in mind; next we reduce the root to the kind of this fraction, ⟨thus to⟩ 196ths, which will give 210; a seventh of this is 30, which equals what was kept in mind. [The fraction has been taken correctly.][238] That is how to proceed.

(A. 637–642) The equality may involve any pair of powers among the other proportional powers we have named,[239] except that the rule for that is, if one of the two does not ⟨have⟩ the power of a number, to lower (*ḥaṭṭa*) each of the two powers by one degree (*manzila*), or several, so that the one which is lower in degree turns [into the power of the number] into the kind of the number.[240] Thus, if there are cubes equal to ⟨fourth⟩ powers, we lower that by three degrees, so the cubes become a number and the fourth powers, roots. And, if fourth powers are equal to squares, we lower them by two degrees, so the squares become a number and the fourth powers, squares. That is how to proceed.[241]

§2. Compound equation

(A. 645–650) Among the compound equations those which can be treated successfully in the science of algebra are those in which occur the (first) three of the proportional elements (*uṣūl*) mentioned by us before.[242] So there are (basically) three such compound equations, which are those involving the first three elements.[243] The first is: squares and roots are equal to a number. The second is: squares and a number are equal to roots. The third is: roots and a number are equal to squares. They are the basis for the other compound equations, for these are reducible to them and (their degrees) lowered to theirs so as to take their form, as we have explained this for the simple equations.[244]

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[236] Above, example *i*; as useless as before (note 230).

[237] That is, $x = \frac{15}{14}$ $(= \frac{210}{196})$ and $x^2 = \frac{225}{196} = 1 + \frac{29}{196} = 1 + \frac{28+1}{196} = 1 + \frac{1}{7} + \frac{1}{196}$. A numerical proof of the result (considering the numerators) now follows.

[238] That was just checking the answer.

[239] A. 38–47.

[240] One of the two is a correction (both "power" and "kind" are found in this context).

[241] Implicit: the two powers involved are either consecutive or differ by two degrees.

[242] See A. 6–7.

[243] It is understood that the powers in these trinomial equations must have consecutive degrees and positive coefficients (no equality to 0). In the manuscript the equality is mostly expressed by the singular of *ʿadala*.

[244] Above, A. 637–642.

(A. 651–655) It may happen, in some of their treatments we shall make known, that, in the equation, the coefficient of the power (*manzila*) with the highest (degree) is more or less than the unit; in that case we are to reduce it to the integer 1 by completing what is lacking **(18$^{\text{v}}$)** or removing what exceeds, and the same treatment (will be applied) to the (coefficients of the) other two equated powers [so that the three will conform to the initial ratio, whatever they were].[245] The treatment for reducing these (quantities of the highest) kinds to a single one is (just) as we have explained for the simple equations, without any difference.[246]

(A. 656–692) Determining the side of the square in the first compound equation.[247]

You must know that the unknown (*majhūl*) which we are to calculate and determine in each of these three compound equations is the side of the square mentioned in them. And what must be used for its determination in the first compound equation, namely when squares and roots are equal to a number—after reduction of the coefficient of the squares to the integer 1 if it is less or more, and the same (transformation being applied) to the accompanying roots and number—is (the following): we shall multiply half the quantity of the roots by itself [that is, the number of the multiple of the quantity of half the roots],[248] the result being (then) added to the number in the equation (*al-ʻadad al-muʻādil*), the root of the result being taken and the quantity of half the roots then being subtracted from this. The remainder will be the side of the unknown square.[249]

Example(s) of that. (*i*) A square and ten roots are equal to thirty-nine.[250] We halve the quantity of the roots, namely 10, so its half is 5, which we multiply by itself; this gives 25 [which is a number since we have multiplied a number, equal to the number of half the (quantity of) roots, and we did not multiply roots];[251] then we add this to the number, which is 39; this gives 64, of which we take the square root, which is 8; then we subtract from it half the quantity of the roots, thus 5, which leaves 3. This is the root of the square, and the square is 9. Ten roots of it are 30, and their sum is 39.

245 Thus involve only the three powers obeying the proportion $1 : x = x : x^2$ (irrelevant).

246 In changing the coefficient of the term of highest degree to 1 the author will (again) proceed with his addition or subtraction of some fraction of it.

247 Arabic for "compound equation": *muqtiran*, abbreviation of *muʻādala muqtirana*.

248 Early reader's correction or clarification.

249 For $x^2 + px = q$, the only positive solution is $x = \sqrt{\left(\frac{p}{2}\right)^2 + q} - \frac{p}{2}$. Before that, better: "half the quantity of the roots."

250 $x^2 + 10x = 39$ is the classic example. See Khwārizmī (1831, 8 (trans.), 5 (Arabic)) and second demonstration (1831 15–16 (trans.), 10–11 (Arabic)); Abū Kāmil (1986, fol. 4$^{\text{r}}$ (Arabic); 1966, 31 (Hebrew); 1993, l. 62 (Latin)); also (later) ʻUmar Khayyām (*Algebra*, Woepcke 1851, 17/11 & 19*n*).

251 Same interpolator as just before.

(*ii*) And like our statement: Two and a third squares and seven roots are equal to forty-two.[252] Since the quantity of the squares is more than 1, we shall change it to 1. That is, we reduce it (first) to the kind of the fraction, (thus) thirds, which gives $\frac{7}{3}$; (then,) since we are to subtract from this, and (also) from what we have as roots and number, **(19^r)** its $\frac{4}{7}$, we follow the reasoning of the procedure seen above and multiply both the roots and the number by 3 and divide the result by 7.[253] The result of the two (operations)[254] is that the (coefficient of the) square is equal to the integer 1 and, having done that (for the two other terms), we shall obtain a square and three roots equal to eighteen. Then we multiply half the quantity of the roots, namely $1 + \frac{1}{2}$, by itself, whence $2 + \frac{1}{4}$, add this to the number, namely 18, whence $20 + \frac{1}{4}$, and take the root of that, which is $4 + \frac{1}{2}$. From that, we subtract half the quantity of the roots, namely $1 + \frac{1}{2}$, which leaves 3. This is the root of the square, and the square is 9. Doubling it and adding to the result a third of the square gives 21, and adding to that the value of seven of its roots, which is also 21, (indeed) gives the result 42.

(*iii*) And like our statement: A half and a third of a square and two and a third roots are equal to fourteen and a half units.[255] We need here to complete the square, that is, to add to it, and to what there is as roots and number, its fifth.[256] Following the reasoning of the procedure seen previously, we put the denominator of the fifth (thus 5) in two places, add to the (first) one its fifth, thus 1, which gives 6, then multiply both (the quantity of) roots and the number by 6, and divide the result(s) by (the number in) the second place, thus 5; the result of the two (operations) will correspond to the equation involving one integral square. Having done that (reduction for the other terms), we shall obtain a square and two and four fifths roots equal to the number seventeen and two fifths. Then we multiply half the quantity of the roots, namely $1 + \frac{2}{5}$, by itself, and add the result,[257] namely $1 + \frac{4}{5} + \frac{4}{5}\frac{1}{5}$, to the number, namely $17 + \frac{2}{5}$, which gives $19 + \frac{1}{5}$ **(19^v)** $+ \frac{4}{5}\frac{1}{5}$, and take the root of that, which is $4 + \frac{2}{5}$. We subtract from it half the quantity of the roots, thus $1 + \frac{2}{5}$, which leaves 3. Such is the root of the square, and the square is 9.

That is how to proceed for all there is and arises in that kind, God Almighty willing.

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[252] $(2 + \frac{1}{3})\, x^2 + 7x = 42$, reduced to $x^2 + 3x = 18$.

[253] See above, A. 616–621, same reduction (equation $\frac{7}{3}\, x^2 = 7x$).

[254] The two transformations (multiplication and subsequent division).

[255] $(\frac{1}{2} + \frac{1}{3})\, x^2 + (2 + \frac{1}{3})\, x = 14 + \frac{1}{2}$, reduced to $x^2 + (2 + \frac{4}{5})x = 17 + \frac{2}{5}$. Arabic: only occurrence in this text of the (elsewhere common) *dirham* for "unit."

[256] Since $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

[257] $(\frac{7}{5})^2 = \frac{49}{25} = \frac{25+20+4}{25}$.

(A. 693–708) Illustration of this treatment.[258] Since the sum of the number and the square of half the quantity of the roots is (taken to be) a square number, we know that the representation of the number is a gnomon (*ʿalam*) around the square of half the (quantity of the) roots.[259] (Now) any gnomon is equal to a square and two complements (*mutammimān*);[260] so the number is equal to a square and two complements. But it is equal to a square and ten roots;[261] therefore each of the two complements is five roots, for each one is a rectangle the sides of which are a root, since it is a side of (the) square, and half the quantity of the roots, thus the number 5.

Consider (therefore) that we represent the unknown square by a square [equilateral and equiangular],[262] say the square ABGD. We extend its side AB in a straight line to the point E, putting BE equal to half the quantity of the roots, and we construct on AE the square AEZH. We extend the sides BG, DG in a straight line towards the sides HZ, EZ.

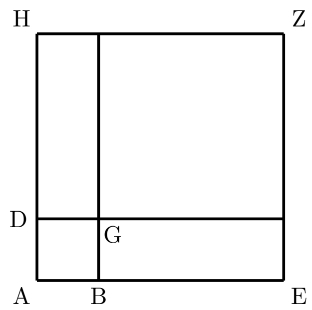

Figure 12: Illustration of the formula for $x^2 + px = q$

Since each of the sides BG, GD is a root and each of the sides DH, BE is 5, each of the rectangles HG, GE will be five roots, and they and the square ABGD altogether, that is, the gnomon, (will represent) a square and ten roots, and this equals 39. Since the square GZ is 25, the whole (square) area AZ is 64, and its

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[258] Remember that the equation considered is $x^2 + px = q$, with $x = \sqrt{\left(\frac{p}{2}\right)^2 + q} - \frac{p}{2}$, and the squares $\left(\frac{p}{2}\right)^2$ and $\left(\frac{p}{2}\right)^2 + q$—thus differing by q.

[259] As seen several times before (above, note 90).

[260] As seen before. Here it consists of the square AG and the two rectangular complements GH and GE.

[261] Equation $x^2 + px = q$, $p = 10$, $q = 39$ (note 250).

[262] The word for "square," *murabbaʿ*, is sometimes used to mean any four-sided figure; but hardly in our text, where the term has already occurred several times with the meaning of "square." It is obvious that this latest interpolator began directly with the part on compound equations. But see note 104.

root AE is 8. So if, from this, we subtract BE, which is half the quantity of the roots, **(20ʳ)** thus 5, there will remain 3 for AB, which is the root of the required ("unknown": *majhūl*) square. This is what we wanted to prove.[263]

(A. 709–719) Treatment of this problem by geometry, proof of it and of the reason for halving the roots there, using segments of a straight line.[264]

We wish to determine (geometrically) the side of the unknown square. We make line AB equal to the quantity of the roots and apply (*aḍāfa*) on it a rectangle[265] equal to the given number (and) exceeding at its end by a square, as made clear in the twenty-ninth (proposition) of the sixth Book of the *Elements*. Let the rectangle be AG by GB, and BG the side of the exceeding square. I say that BG is the side of the required ("unknown": *majhūl*) square.[266]

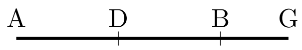

Figure 13: Construction of the solution of $x^2 + px = q$

Proof. We halve line AB at D. Thus line AB is divided into two halves at D and has an additional (segment), namely BG. So[267] the multiplication of AG by GB, plus the square of DB, will equal the square of DG. But the multiplication of AG by BG is known (*maʿlūm*) since it is equal to the given (*maʿlūm*) number.[268] Therefore, by

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[263] Let the square ABGD represent x^2. Extend AB by BE $= \frac{p}{2}$ and complete the whole square AZ, which then comprises the squares AG and GZ and the rectangles GH and GE.

By construction, GE $=$ GH $= \frac{p}{2}\, x$; furthermore, by the equation, GH $+$ GE $+$ AG $= q$. Adding to this gnomon the square GZ $= (\frac{p}{2})^2$ gives the whole square AZ, thus equal to $(\frac{p}{2})^2 + q$, but also, by construction, to $(x + \frac{p}{2})^2$. This illustrates the formula. Same proof and figure (with the equation $x^2 + 10x = 24$) in Ibn Turk (Sayılı 1985, 162–163 (trans.), 145–146 (Arabic)). Similar proof in Khwārizmī (second proof, 1831, 15–16 (trans.), 10–11 (Arabic)) and in Abū Kāmil's *Algebra* (second proof, 1986, fol. 6ʳ (Arabic); 1966, 35 (Hebrew); 1993, l. 151 (Latin)).

[264] This heading is rather misleading, as are the two similar ones later on. As indicated by the first sentence below, this will be the geometrical construction of the solution, now taking into account its size (see our introduction).

[265] Arabic: *saṭḥ qā'im al-zawāyā*, specified here (normally in this text *saṭḥ* alone means "rectangle"). The reason may be that, in the proposition referred to below, Euclid applies a parallelogram.

[266] See our introduction. Here the rectangle has base AG, height equal to BG, and it exceeds line AB on which it is applied by the square on BG. This construction is indeed explained in *Elements* VI.29.

[267] According to *Elements* II.6 (see our introduction).

[268] Construction of a rectangle of given area ($= q$) on a segment of a straight line of given length ($= p$), its height being determined by the fact that it exceeds or falls short (in this case: exceeds) by a square area (particular case of *Elements* VI.29).

adding the square of half the (given) quantity of the roots, thus the square of DB, to the given number, which is the rectangle AG by GB, we shall know the square of DG. We then take its root, which is DG, and subtract from it half the (quantity of the) roots, which is DB; this leaves BG, (thus) known, which is the side of the (required) square. This is what we wanted to prove.[269]

(A. 720–746) Determining the side of the square in the second compound equation.

The treatment for determining the side of the square in the second compound equation, which is "squares and a number are equal to roots"—after reducing the squares to a single one if there are fewer or more—is (the following): we halve the quantity of the roots, multiply this half quantity by itself, subtract from the result the given number, take the square root of the remainder and subtract this from, or add this to, the quantity of half the roots;[270] the result will be the root of the required square.[271]

As we said, we "subtract" or "add" the root of the remainder; for ⟨among the treatments for such⟩ algebraic problems some are solved (*k̲haraja*), ⟨as⟩ in **(20ᵛ)** this compound equation, by both addition and subtraction, in others solely by subtraction or only by addition. Then we must verify for all the problems which reduce to this compound equation, with each of the aforesaid aspects,[272] that they fall into the solvable domain (*ḥadd al-jawāb*); (for) in no case may, in this compound equation, the square of half the quantity of the roots ever be less than the number which is with the square: should that be the case, such a problem will be impossible; if they are equal, then the root of the required square is equal to half the quantity of the roots.

Example(s) of that.[273] A square and the number twenty-one are equal to ten roots.[274] This means that a square when increased by the number 21 gives a result equal to ten of its roots.[275] We halve the quantity of the roots and multiply this half quantity,

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[269] With AD = DB = u, BG = v (added segment), we have AG = $2u+v$, and, since $(2u+v)\,v+u^2 = (u+v)^2$ (*Elements* II.6), AG · BG + DB^2 = DG^2. The sum of the square DB^2 ($= (\frac{p}{2})^2$) and the rectangle AG · BG ($= q$) being known, and equal to the square DG^2, DG is known, so also BG = DG − DB, that is, our x.

[270] *Sic* (see note 249). Also found below.

[271] For $x^2 + q = px$, there are (if the discriminant is positive) two positive solutions, namely $x = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$.

[272] When it is not yet in the canonical form.

[273] Successively: case with two distinct solutions; case with one solution.

[274] $x^2 + 21 = 10x$. Another classic example, found in K̲hwārizmī (1831, 11 (trans.), 7 (Arabic)), Ibn Turk (Sayılı 1985, 163–165 (trans.), 146–149 (Arabic)) and Abū Kāmil (1986, fol. 6ᵛ (Arabic); 1966, 39 (Hebrew); 1993, l. 200 (Latin)).

[275] This useless "clarification" may well be an addition.

thus 5, by itself, which gives 25, and subtract from it the number, namely 21, which leaves 4; we take its square root, which is 2, then subtract it from half the quantity of the roots, thus from 5, and this leaves 3, which is the root of the square, the corresponding square being 9; or we add it to it, which makes 7, which is the root of the square, the corresponding square being 49. When we add to any of these two squares the number 21, the result will equal 10 roots of it.

As to (the case) in which the square of half ⟨the quantity of⟩ the roots is equal to the number which is with the square, it is like our statement "a square and the number twenty-five are equal to ten roots."[276] (Indeed) if we multiply half the quantity of the roots by itself, it gives 25, just like the number. So we shall say that the root of the square is equal to half the quantity of the roots, thus 5, with the corresponding square being 25. When we add to this square the (given) number 25, the sum is 50, which is equal to ten times its root.

That is how to proceed for what arises in this **(21$^{\mathrm{r}}$)** kind, God Almighty willing.

(A. 747–765) Illustration of this treatment.[277] When the square of half the quantity of the roots exceeds the given number by a rational quantity (*ʿadad majd͟hūr*),[278] we know[279] that the number is represented as a gnomon around the square of half the quantity of the roots.[280] But the given number plus the unknown square is equal to ten roots.[281] So half the gnomon plus half the (unknown) square is equal to five roots.[282] This (sum) is then (equal to) a rectangle comprised by two segments of a straight line, one of which is the root of the unknown square and the other, a segment equal to half the quantity of the roots.[283]

Consider (therefore) that we make line AB equal to half the quantity of the roots, on which we construct the square AG, and we put AD as the side of the unknown

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[276] $x^2 + 25 = 10x$. Classic example. See Ibn Turk (Sayılı 1985, 165–166 (trans.), 149–150 (Arabic)) and Abū Kāmil (1986, fol. 9$^{\mathrm{r}}$ (Arabic); 1966, 45 (Hebrew); 1993, l. 345 (Latin)).

[277] Illustration of the general case. The particular one, just seen, will follow (A. 766–774). Other demonstrations for the general case: K͟hwārizmī (1831, 16–18 (trans.), 11–13 (Arabic)); Abū Kāmil (1986, fols. 7$^{\mathrm{v}}$, 8$^{\mathrm{r}}$ (Arabic); 1966, 39 *seqq.* (Hebrew); 1993, ll. 257, 292 (Latin)); but in our text these two possibilities are represented by a single figure (whence the occurrence of D and E twice in the figure).

[278] Rather: when $(\frac{p}{2})^2 - q$ is positive (the solution is in the present case $x = \frac{p}{2} \pm \sqrt{(\frac{p}{2})^2 - q}$).

[279] As seen in the illustration of the previous equation.

[280] The two squares $(\frac{p}{2})^2$ and $(\frac{p}{2})^2 - q$, when placed with a common corner and aligned, differ by the gnomon q.

[281] $x^2 + q = px$.

[282] $\frac{1}{2}(x^2 + q) = \frac{p}{2} \cdot x$.

[283] The quantity $x^2 + q$ will then appear as the sum of two equal rectangles (AB · AD and AD · AZ, with two positions of D in the figure below).

square; it will be either smaller than AB, in the (case of) subtraction, or larger than AB, in the (case of) addition. In both cases we construct on AD the square AE, and we extend (*akhraja*) the lines of the figure. Since line AB is 5 and line AD (represents), in both cases, the root ⟨of the required square⟩, the gnomon MNS plus the square AE will be, in both cases, equal to the product of AB by AD taken twice, thus ten roots. In the case of subtraction, this is clear.[284]

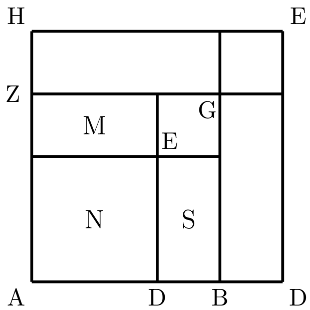

Figure 14: Solution of $x^2 + q = px$, general case

In the case of addition, the gnomon MNS plus the larger square AE will be equal to the (sum of the) two rectangles HB and ZD (the larger), for the gnomon MNS plus the larger square AE is equal to twice the gnomon MNS, plus twice the square EG, plus twice the rectangle GH.[285] This altogether equals twice the rectangle HB, that is, the (sum of the) equal rectangles HB and ZD (the larger). (Now) each of them is five roots.[286] So the given number plus the required square AE equals ten roots.

Then MNS equals (in both situations) the given number. For this reason we subtract it from the square AG. This leaves the square EG, (thus) known.[287] So

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[284] Indeed, consider the case AD < AB. Since AB $= \frac{p}{2}$, thus AG $= (\frac{p}{2})^2$, and AD $= x$, then N + S $= \frac{p}{2}\,x =$ M + N, so N + (M + N + S) $= px$.

It will be inferred below that, since here N $= x^2$ while $x^2 + q = px$, we must have M + N + S $= q$. So $\sqrt{(\frac{p}{2})^2 - q} = \sqrt{\text{AG} - (\text{M} + \text{N} + \text{S})} = \sqrt{\text{EG}} = \text{DB}$, and $x = \text{AD} = \text{AB} - \text{DB} = \frac{p}{2} - \sqrt{(\frac{p}{2})^2 - q}$.

[285] Indeed, the larger square AE, or x^2, is made up of the gnomon M + N + S, plus the equal squares EG and GE, plus the equal rectangles HG and GD. Furthermore, with the terms grouped, this equals, as we are about to be told, the sum of the two rectangles ZD and HB. For, considering the larger square, we see that AE = M + N + S + EG + GE + HG + GD, so M + N + S + AE = (M + N + S + EG + HG) + (M + N + S + GE + GD) = HB + ZD.

[286] AB is 5, and the larger AD is x.

[287] EG is $(\frac{p}{2})^2 - q$, thus the square of the discriminant, the same in both cases.

we take its root, which is DB, subtract it from, or add it to, AB, which gives as a remainder **(21^{v})** or as a sum AD, which is the root of the required square.[288]

(A. 766–774) As to its illustration in the (case of) equality, namely when the number is equal to the square of half the quantity of the roots,[289] we know that the representation of the number ⟨is a square⟩. Since, furthermore, the sum of the number and the (required) square is equal to a known (quantity of) roots, we know that the representation of this sum is a rectangle comprised by two segments of a straight line, one of which is the root of the (required) square[290] and the other, a number equal to the quantity of the roots. Now this rectangle is divided into two square halves, one of which is the (required) square and the other, the (given) number, and the sum of their two sides equals the quantity of the roots.[291] Therefore half the quantity of the roots is the root of the required square.

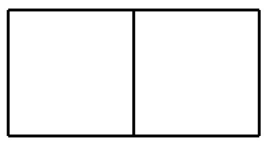

Figure 15: Solution of $x^2 + q = px$, case of equality

Consider that we put line AB equal to the given quantity of the roots. We halve it at G and construct on each of AG and GB the squares AD and BD. Then either of the lines AG and GB will be the root of the required square, and it equals half the quantity of the roots.[292]

(A. 775–790) Treatment of this problem by geometry, proof of it and of the reason for halving the roots there, using segments of a straight line.[293] If we wish to determine the side of the unknown square, we put line AB ⟨equal to the quantity of the roots,

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[288] Thus we have, in both cases $2 \cdot \mathrm{AB} \cdot \mathrm{AD} = \mathrm{M}+\mathrm{N}+\mathrm{S}+\mathrm{AE}$, where $2 \cdot \mathrm{AB} \cdot \mathrm{AD} = px$ and $\mathrm{AE} = x^2$, hence, since $x^2+q = px$, $\mathrm{M}+\mathrm{N}+\mathrm{S} = q$. Thus, considering the two squares AE and their respective root $\mathrm{AD} = x$, and the equality of the two squares EG, we have $x = \mathrm{AD} = \mathrm{AB} \pm \mathrm{BD} = \frac{p}{2} \pm \sqrt{(\frac{p}{2})^2 - q}$, which illustrates the formula.

[289] Equation $x^2+q = px$, with $q = (\frac{p}{2})^2$. See Abū Kāmil (1986, fol. 9^{r} (Arabic); 1966, 45 (Hebrew); 1993, l. 341 (Latin)).

[290] Arabic *al-māl* (MS: *lā muḥāl*).

[291] Since $px = x^2 + q = x^2 + (\frac{p}{2})^2$, with two (necessarily) equal square parts, the obvious inference will be that $x = \frac{p}{2}$.

[292] Since $\mathrm{AB} = px$ and G is its midpoint, $\mathrm{AD} = \mathrm{BD} = (\frac{p}{2})^2 = x^2$, and $\mathrm{AG} = \mathrm{GB} = \frac{p}{2} = x$.

[293] Geometrical construction of the solution x knowing $\mathrm{AB} = p$ and the area q of the rectangle applied on AB. The MS has the two figures, but the text refers to the right-hand one.

and place points on it⟩ in three places for (respectively) the (cases of) addition, subtraction, equality [equal to the quantity of the roots].[294] We apply on it a rectangle[295] equal to the given number, (but) falling short of it at its end by a square, as made clear from the twenty-eighth (proposition) ⟨from the sixth Book⟩ of the *Elements*. Let the applied rectangle (*al-saṭḥ al-muḍāf*) be the rectangle AG by GB and the side of the square falling short, GB. Then I say that GB is the side of the required square.

Figure 16: Construction of the solution of $x^2 + q = px$

Proof. We halve line AB at the point D. Then, (considering) the two (upper) figures, point D will fall in (the case of) subtraction between points A and G on line AG, in the other figure, for the addition, between G and B **(22^r^)** on line GB, and in the third figure, for (the case of) equality, on point G itself. Since (in the first two cases) line AB is divided into halves at D and into unequal parts at G, the product of AG by GB ⟨plus the square of DG will equal the square of DB.[296] But the product of AG by GB⟩ is known, for it is equal to the given number;[297] the square of BD is (also) known since BD equals half the quantity of the roots. For this reason we subtract the given number, thus the rectangle AG by GB, from the square of half the quantity of the roots, that is, (from) the square of BD, and take the root of the remainder, which is DG; then we subtract it from, or add it to, half the quantity of the roots. The remainder or the sum will be the side of the required square, thus BG.[298]

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294 (Partial) correction to the above lacuna.

295 Again (see note 265), *saṭḥ qā'im al-zawāyā*; specified because Euclid speaks about a parallelogram.

296 By *Elements* II.5, $\mathrm{AG} \cdot \mathrm{GB} + \mathrm{DG}^2 = \mathrm{BD}^2$, two terms of which are known, namely $\mathrm{AG} \cdot \mathrm{GB} = q$ and $\mathrm{BD}^2 = (\frac{p}{2})^2$, whence $\mathrm{DG} = \sqrt{\mathrm{BD}^2 - \mathrm{AG} \cdot \mathrm{GB}} = \sqrt{(\frac{p}{2})^2 - q}$.

297 No early reader's remark here, so the lacuna (obviously by homoeoteleuton) might be by our copyist himself.

298 $\mathrm{BG} = x = \mathrm{DB} \pm \mathrm{DG} = \frac{p}{2} \pm \sqrt{(\frac{p}{2})^2 - q}$. We would now expect the case of equality to be treated. But the text here is obviously corrupt. It is supposed to treat the case of "impossibility" (namely $(\frac{p}{2})^2 < q$); in fact, it partly repeats what has already been said, then ends with stating the case of impossibility. Using modern terms, the reasoning should be as follows. Consider the segment of a straight line with length p (thus AB) and the part x of it (thus BG). We know that $x(p - x)$

(A. 791–798) [For the (case of) impossibility, we put line AB equal to the quantity of the roots and take BG from it as the side of the unknown square; then the product of AB by BG will be equal to the quantity of the roots.[299] But the quantity of the roots is equal to the given number plus the unknown square, for the rectangle AB by BG is equal to the product of AG by GB plus the square of GB. (Now) we had put GB as the side of the required square; so the rectangle AG by GB is equal to the given number.[300] [Then if we halve line AB (at D), the midpoint will fall either on the part AG or on the part GB of line AB. In both situations the product of AG by GB plus the square of GD will be equal to the square of DB.][301] But the product of AG by GB, which is the (given) number, has been put[302] larger than the square of DB, and this is not possible.]

(A. 799–807) Determining the side of the (unknown) square in the third compound equation.

The treatment for determining the side of the (unknown) square in the third compound equation, which is "roots and a number are equal to squares"—after reduction of the squares to a single one if there are fewer or more (than one)—consists in multiplying half the quantity of the roots by itself, adding that to the number, taking the square root of the result **(22$^{\text{v}}$)** and adding it to the quantity of half the roots; the sum will be the root of the required square.[303]

Example. Three roots and the number four are equal to a square.[304] If we wish to determine the side of the square, we multiply half the quantity of the roots, thus $1 + \frac{1}{2}$, by itself, whence $2 + \frac{1}{4}$, add that to the number, thus 4, whence $6 + \frac{1}{4}$, take the square root of it, which is $2 + \frac{1}{2}$, then add it to the quantity of half the roots, which is $1 + \frac{1}{2}$; the result is 4, which is the root of the required square.

(A. 808–821) Illustration of this treatment. Since the sum of the number and the square of half the quantity of the roots is a square, we know that the number is represented by a gnomon around the square of half the quantity of the roots. The

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represents the given number q. Now the area $x(p-x)$ is maximal for $x = \frac{p}{2}$, that is, when this area is a square, the given number q being then $(\frac{p}{2})^2$; this is the case of equality. So supposing $q > (\frac{p}{2})^2$ is not possible, as indeed stated in the final sentence. See also Ibn Turk (Sayılı 1985, 166–167 (trans.), 150–152 (Arabic)).

299 Equation $px = x^2 + q$, with $AB = p$, $BG = x$. "Quantity of the roots" (*ʿidda al-ajdhār*) used here both for our p and our px.

300 Since $px = AB \cdot BG = AG \cdot BG + BG^2$ and $BG^2 = x^2$, so $AG \cdot BG$ must be q.

301 This is just the previous situation and has nothing to do with the case of equality or impossibility.

302 *wuḍiʿa*, perhaps for *wuqiʿa*, "has become."

303 $x^2 = px + q$. The only positive solution is $x = \frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + q}$.

304 $x^2 = 3x + 4$. Same example in Khwārizmī (1831, 12 (trans.), 8 (Arabic)); Abū Kāmil (1986, fol. 10$^{\text{v}}$ (Arabic); 1966, 49 (Hebrew); 1993, l. 424 (Latin)).

side of this whole square exceeding half the quantity of the roots, it will be the root of the required square.[305]

Consider that we represent the unknown square by the [equilateral and equiangular][306] square ABGD, and let BE on the side AB be equal to the quantity of the roots. We halve EB at the point Z, and construct on AZ the square AZHṬ. We extend lines ZH and HṬ in a straight line to meet the sides DG and BG, and construct on AE the square AEKL. We extend lines EK and LK in a straight line to meet the sides ṬH, ZH.

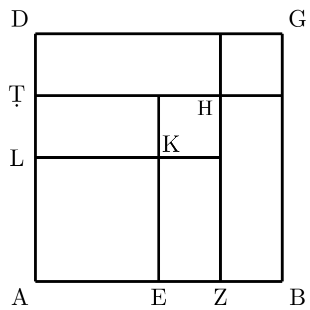

Figure 17: Illustration of the formula for $x^2 = px + q$

Now since DG is the root of the (required) ⟨square⟩ and DṬ (= ZB) is one and a half, for it is equal to half the quantity of the roots, the rectangle ṬG is one and a half (times) the root. Likewise, we shall show that the rectangle GZ is also one and a half (times) the root. So (the sum of) the rectangles DH, once, GH, twice, and HB, once, is equal to three roots.[307] But the area KH equals the area HG. This leaves the (sum of the) areas ṬK, KA, KZ, thus the gnomon (around KH), equal to the given number.[308] Therefore we add (this) number **(23^r)** to the square of half

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[305] Part of the reasoning must be missing. There are three squares (in Fig. 17, KH, AH, AG):

$$(\frac{p}{2})^2 < (\frac{p}{2})^2 + q < \left(\frac{p}{2} + \sqrt{(\frac{p}{2})^2 + q}\right)^2,$$

and the side of the largest will be shown to be the required x. Demonstrations of this case also in Khwārizmī (1831, 19–20 (trans.), 13–15 (Arabic)); Ibn Turk (Sayılı 1985, 168–169 (trans.), 152–153 (Arabic)), equation $x^2 = 4x + 5$); Abū Kāmil (1986, fols. 10^v, 11^r (Arabic); 1966, 49, 51 (Hebrew); 1993, ll. 446, 483 (Latin).

[306] See note 262.

[307] Since $ṬG = ZG = \frac{p}{2}x$, so $DH + 2 \cdot HG + HB = px$.

[308] Since, by the equation, $x^2 - px = q$ while, in the figure, $x^2 = AG$ and $px = ṬG + GZ = ṬG + HG + HB = ṬG + KH + HB$, the gnomon left in the square AH after removing the square KH must be q.

the quantity of the roots so as to obtain the square AH.[309] We take its root, namely AZ, and then add it to half the quantity of the roots, namely ZB; the sum is AB, which is the root of the required square.[310] This is what we wanted to prove.

(A. 822–833) Treatment of this problem using geometry, proof of it and of the reason for halving the roots there, by means of segments of a straight line.

If we wish to determine in this problem the side of the unknown square, we put line AB equal to the known quantity of the roots, and apply on it a rectangle[311] equal to the number (and) exceeding the (segment) at its end by a square. Let the applied rectangle be the rectangle AG by GB, and (let) BG be the side of the square in excess. So I say that AG is the side of the unknown square.

Figure 18: Construction of the solution of $x^2 = px + q$

Proof. We halve the known line AB at the point D. Then line AB is divided into halves at the point D and has an additional (segment), namely BG. So the product of AG by GB plus the square of DB equals the square of DG.[312] But the product of AG by GB is known since it equals the given number, and the square of DB is known since DB equals ⟨half⟩ the quantity of the roots; so the square of DG is known. Therefore the square of half the (quantity of the) roots, thus the square of DB, is added to the given number, thus the rectangle AG by GB, and we take the square root of the sum; this is the root of the square of DG, that is, DG. Then we add to it the quantity of half the roots, thus AD; the sum is AG, which is the root of the square.[313] This is what we wanted to prove.

(A. 834–839) It has appeared clearly from the foregoing that the construction leading to (determining) the sides of the unknown squares in each of these three compound equations is the construction set forth by Euclid towards the end of the sixth Book of his *Elements*; namely: the application on a given **(23^v^)** segment of a straight line of a (given) parallelogram[314] which exceeds (this segment) at its end,

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[309] Since $\mathrm{KH} = (\frac{p}{2})^2$, so square $\mathrm{AH} = (\frac{p}{2})^2 + q$.

[310] Thus $\mathrm{AB} = \mathrm{ZB} + \mathrm{AZ} = \frac{p}{2} + \sqrt{(\frac{p}{2})^2 + q}$.

[311] Again (see note 295), *saṭḥ qā'im al-zawāyā*.

[312] Let $\mathrm{AB} = p$ be halved at D and extended by BG. Then (*Elements* II.6) $\mathrm{AG} \cdot \mathrm{GB} + \mathrm{DB}^2 = \mathrm{DG}^2 = q + (\frac{p}{2})^2$.

[313] Since we know DG^2, with $\mathrm{DG} = \sqrt{q + (\frac{p}{2})^2}$, and $\mathrm{AD} = \frac{p}{2}$, so we also know $\mathrm{AG} = x = \frac{p}{2} + \sqrt{q + (\frac{p}{2})^2}$.

[314] *saṭḥ mutawāzī al-aḍlā'*. Understand: "rectangle"; see notes 265, 295, 311.

or falls short of it, by a square.[315] That is to say, the side of the square in excess is the side of the unknown square in the first compound equation; in the second compound equation, it is the side of the square falling short; in the third compound equation, it is the sum of the line on which the rectangle is applied and the side of the square in excess. This is what we wanted to prove.[316]

(A. 840–846) Case of compound equations involving three elements not in (continued) proportion, then of more, either in (continued) proportion or not.

This is the case for the two possible trinomial categories (*ḥayyiz thulāthī*), namely, first, cubes, squares and a number, and, second, cubes, roots and a number, consisting of six compound equations (altogether).[317] Or the single possible quadrinomial category (*ḥayyiz rubāʿī*), namely cubes, squares, roots and a number, consisting of seven compound equations (altogether).[318] Or others involving higher powers. Now (all) these do not admit of numerical (exact) procedures (*qiyāsāt ʿadadīya*) as above but only of some sort of estimation, using conic sections (*quṭūʿ makhrūṭīya*).[319]

(A. 847–855) Case of the two trinomial categories mentioned above. In the three kinds they each comprise, the situation of continued proportion is not encountered.[320] For the ratio of the cube to the square is not equal to the ratio of the square to the number since there exists one power between the square and the number, namely that of the root. Neither is the ratio of the cube to the root equal to the ratio of the root to the number[321] since there exists one power between the cube and the root, namely that of the square. (Thus) each of their six forms (*qarāʾin*) is not solvable by way of our above discourse concerning numerical procedures. Indeed, the unknown which we must calculate and determine in each of these compound

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[315] *Elements* VI.28–29. See A. 709–719, A. 775–790, A. 822–833 and the discussion in our introduction.

[316] This last sentence might be an addition.

[317] Remember that the terms may occur only with the positive signs on either side of the equation. Hence there are three for each kind, namely, by analogy to the second-degree equations, first (with cubes, squares, number) $ax^3 + bx^2 = d$, $ax^3 + d = bx^2$, $ax^3 = bx^2 + d$, and second (with cubes, roots, number) $ax^3 + cx = d$, $ax^3 + d = cx$, $ax^3 = cx + d$. Our text omits the first (indeed banal) binomial case of $ax^3 = d$, thus reduced to the extraction of a cube root.

[318] Namely $ax^3 + bx^2 + cx = d$, $ax^3 + bx^2 + d = cx$, $ax^3 + cx + d = bx^2$, $ax^3 = bx^2 + cx + d$, $ax^3 + bx^2 = cx + d$, $ax^3 + cx = bx^2 + d$, $ax^3 + d = bx^2 + cx$.

[319] "Numerical procedures," that is, solving formulae, like those seen above for second-degree equations.

[320] Still the obsession with proportions! Remember that the first step towards the general algebraic solution of the third-degree equation in the 16th century was *precisely* to remove one of the intermediate terms.

[321] This is "square" in the text, thus left uncorrected by our copyist (see note 9).

equations is the side of the aforesaid cube, and the corresponding analysis (*taḥlīl*) leads to the application on a given **(24ʳ)** segment of a straight line of a given (right) parallelepiped (*mujassam mutawāzī al-suṭūḥ*) which exceeds this segment, or falls short of it, by a cube.[322] Now this can be carried out only by means of conic sections.

(A. 856–859) Quadrinomial category, thus with an additional term relative to the (previous) three. Even if the situation of continued proportion is encountered, its seven forms do not meet the requirement for general (numerical) procedures. For the unknown which we must determine is the side of the aforesaid cube. Now it cannot be expressed using the above numerical procedures but only by the aforesaid conic sections.

(A. 860–870) Such are the foundations of algebra and the aspects of the simple and compound equations on which are based the kinds of numerical problems subject to exact general procedures. We have expounded them with a clear explanation and a correct demonstration, and have treated exhaustively their elements by classifying, ordering, revising and clarifying them. As to the (practical) problems connected with them, we have not considered mentioning things extraneous to our purpose and intention: these (rather) belong to the kinds of branches which rely upon the foundations described by us.[323] So let us put an end to our discourse. Praise be to God, Lord of everything created, blessed be the esteemed Muḥammad and his family.

Made in the year 395 of the hegira

The copy was completed on Friday, the twelfth of Rabīʿ II in the year 581

May God be merciful towards its writer and its readers

God alone and his guidance will suffice us

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[322] Three-dimensional correspondents to the three types of quadratic equation seen above, thus, in reduced form (with p the given segment of a straight line, q the given parallelepiped, x^3 the cube), first $x^3 + px^2 = q$ or $x^2(x + p) = q$; second, $x^3 + q = px^2$ or $x^2(p - x) = q$; third $x^3 = px^2 + q$ or $x^2(x - p) = q$.

[323] Unlike the usual algebra-books, our text omits applications and only expounds the elements of algebraic reckoning.

III Arabic Text

(1r) [رسالة فى الجبر والمقابلة والمسائل الحسابية
تأليف بعض قدماء علماء الى تبيين

وفارغ كتابة النسخة سنة احدى وثمانين وخمسمائة

وقد سقط من اوّل هذه النسخة اجزاء]

(2r) [اول اين رساله افتاده است]

على النسبة الواحدة التى اوّلها واحد. فاذا كان اوّل المقادير الثلثة واحدًا كان الثانى جذرًا والثالث مالًا. [امّا الجذر فكلّ عدد او كسر اردت ان تضربه فى مثله والمال ما اجتمع من ضرب الجذر فى مثله.]

فاذا كان اوّل المقادير الثلثة اكثر من الواحد وسمّيناه عددًا كان الثانى جذورًا عدّتها مثل عدّة آحاد العدد [الاوّل] وكان الثالث اموالًا على مثل تلك العدّة ايضًا. وعلى هذا القياس اذا كان اوّل المقادير الثلثة المتناسبة كسرًا اقلّ من الواحد كان الثانى جزءًا من الجذر او اجزاء على نسبة الكسر [الاوّل] الى الواحد وكان الثالث كذلك جزءًا من المال او اجزاء على مثل تلك النسبة ايضًا.

مثال ذلك. [اذا فرضنا الجذر اثنين كان المال الذى يكون منه اربعة وكانت نسبة الواحد الى الاثنين كنسبة الاثنين الى الاربعة. وكذلك اذا فرضنا الجذر ثلثة كان المال الذى يكون منه تسعة وكانت نسبة الواحد الى الثلثة كنسبة الثلثة الى التسعة. وفى الكسور اذا فرضنا الجذر نصفًا كان المال الذى يكون منه رُبعًا وكانت نسبة الواحد الى النصف كنسبة النصف الى الرُبع.] وعلى هذا القياس اذا كان اوّل المقادير الثلثة احدين كان الثانى جذرين [كم كانا من العدد متساويين] وكان الثالث مالين [كلّ واحد منهما من ضرب احد الجذرين فى مثله]. وكذلك فى الكسور اذا فرضنا الاوّل من المقادير الثلثة نصفًا كان الثانى نصف جذر [كم كان من كسر الواحد] والثالث نصف مال [من جميع المال الذى من ضرب ذلك الجذر فى مثله].

ثمّ على هذا القياس بالغة ما بلغت الاعداد والكسور.

الباب الثانى

فيما يعرض للاصول الثلثة المتناسبة من الاحوال المعدودة

وقد يعرض (2v) لما ذكرنا من الاصول الثلثة المتناسبة عند تعريفها فى انواع الاعمال قبل المعادلة احوال ستّ وهى جمع ونقصان وتضعيف وتجزئة وضرب وقسمة.

امّا ما يعرض لها فى الاحوال الاربعة الاوَل التى هى الجمع والنقصان والتضعيف والتجزئة فان العمل فى جميعها مثل العمل فيما يعرض من ذلك فى الاعداد المطلقة سواء لا تختلف. امّا المبلغ فى الجمع والتضعيف والباقى فى النقصان والتجزئة فانّه لا يتغيّر عن جنسه وان كان يتغيّر فى كمّيته.

وامّا الضرب فقد يعرض فى كثير من المواضع ان يُضرب الجذر فى المال وهما مجهولان فيسمّى المبلغ مكعّبًا ويكون ثالثًا فى النسبة للجذر والمال. [وذلك ان كلّ اربعة مقادير متناسبة فضرب الاوّل فى الرابع مثل ضرب الثانى فى الثالث واوّل هذه المقادير كما قلنا واحد وضربه فى الرابع هو الرابع بعينه فلهذه العلّة كان المبلغ من ضرب الجذر فى المال وهما الثانى فى الثالث هو الثالث لهما فى النسبة اعنى الرابع من الاوّل وهو المكعّب الذى ذكرنا.]

فهذه الاسماء الثلثة التى هى الجذر والمال والمكعّب هى الاسماء المفردة التى يتسمّى بها الطبقات الثلثة المتناسبة وقد ينشأ من ضرب بعضها فى بعض طبقات اخر متوالية على نسبتها ويكون اساميها متركّبة من اسمائها الثلثة التى سمّيناها.

وذلك مثل مال المال الذى هو تالى المكعّب فى النسبة فان مبلغه من ضرب الجذر فى المكعّب او من ضرب المال فى مثله. ومثل مال المكعّب او مكعّب المال الذى هو تالى مال المال فى النسبة فان منشأه من ضرب الجذر فى مال المال او من ضرب المال فى المكعّب. ومثل مكعّب المكعّب الذى هو تالى مال المكعّب فى النسبة فان منشأه من ضرب الجذر فى مال المكعّب او من ضرب (3r) المال فى مال المال او من ضرب المكعّب فى مثله ولذلك لاجل المناسبة المتّصلة تركنا شرحها كراهة امتداد الكلمة.

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(26) تعريفها : تصريفها | (39) ينشأ : ينشؤ | (40) من : عن | (42) مال المكعّب : مال مكعب | مكعّب المال : مكعب مال | (46) ولذلك : وذلك.

الباب الثالث
فى ضرب الطبقات المتناسبة بعضها فى بعض
ومعرفة جنس المبلغ من ایّ طبقة هو

اذا اردنا ان نضرب مالًا فى مكعّب جمعنا لفظة المال والمكعّب وقلنا ان المبلغ من الضرب هو مال مكعّب او مكعّب مال.

واذا اردنا ان نضرب جذرًا فى مكعّب أخذنا عدد المرّات التى ضُرب به الجذر فى مثله حتّى كان منه المكعّب وهو ثلثة وجمعنا الى ذلك واحدًا لاجل الجذر وقسمنا المبلغ وهو اربعة بقسمين يكون كلّ واحد منهما اكثر من الواحد وهما اثنان اثنان اذ لا يتهيّأ فيه غير ذلك فنأخذ لكلّ اثنين مالًا لان المال كما بيّنّا هو من جذر فى [جذر] مثله ونقول ان المبلغ من الضرب هو مال مال.

وعلى هذا القياس اذا اردنا ان نضرب جذرًا فى مال مكعّب أخذنا لمال المكعّب خمسة للمال اثنين وللمكعّب ثلثة وجمعنا الى ذلك واحدًا لاجل الجذر وقسمنا المبلغ وهو ستّة بقسمين كيف ما كانا بعد ان يكون كلّ واحد منهما اكثر من واحد وليكونا ثلثة وثلثة فنأخذ لكلّ ثلثة مكعّبًا فيكون المبلغ مكعّب مكعّب. ولو كنّا قسمنا الستّة بقسمين آخرين احدهما اثنان والآخر اربعة وأخذنا للاثنين مالًا وللاربعة مال مال وجمعنا ذلك وهو مال مال مال ثلث مرّات كان ذلك جائزًا لكن لفظة مكعّب المكعّب اخصر واوجز لان التكرير فيه مرّتين وهو فى مال مال المال ثلثة. وهذا قياس ذلك.

الباب الرابع
في قسمة الطبقات المتناسبة بعضها على بعض
ومعرفة جنس الخارج بالقسم من ایّ طبقة هو

اذا اردنا ان نقسم طبقة (3^v) من الطبقات المتناسبة على طبقة اخرى منها ونعلم جنس الخارج بالقسم فمن اجل ان القسمة هى عكس الضرب فانّا ننقّص عدّة اقربهما من طبقة الجذر من عدّة ابعدهما عنها فما بقى [فالخارج بالقسم] من جنس تلك

(53) المرّات : المنزلة.

العدّة. فان كانت الطبقة المقسومة ابعدهما من طبقة الجذر فان الخارج بالقسم يكون طبقة وان كانت الطبقة المقسومة اقربهما من طبقة الجذر فان الخارج بالقسم يكون جزءًا من تلك الطبقة. [وجزء كلّ طبقة هو المسمّى لعدد آحادها.] [اعنى اذا كان الجذر اثنين كان جزؤه نصفًا وجزء المال رُبعًا وجزء المكعّب ثُمنًا وجزء مال المال نصف ثُمن.] ثمّ على هذا القياس.

مثال ذلك. اذا اردنا ان نقسم مال مال على جذر نقّصنا عدّة الجذر وهى واحد من عدّة مال المال وهى اربعة فيبقى ثلثة وهى عدّة المكعّب فنقول ان الخارج بالقسم هو مكعّب. واذا كان الطبقة المقسومة هى طبقة الجذر والمقسومة عليها هى طبقة مال المال كان الخارج ⟨بالقسم⟩ جزء مكعّب.

وكذلك اذا اردنا ان نقسم مكعّبًا على مال نقّصنا عدّة المال وهى اثنان من عدّة المكعّب وهى ثلثة فيبقى واحد وهو عدّة الجذر فنقول ان الخارج بالقسم هو طبقة الجذر. وان كانت الطبقة المقسومة هى طبقة المال والمقسومة عليها طبقة المكعّب كان الخارج بالقسم جزء شىء.

واذا اردنا ان نقسم طبقة على مثلها فان الخارج ⟨بالقسم⟩ يكون واحدًا بالعدد لانّه من قسمة مثل على مثل. وذلك قياسه.

الباب الخامس
في ضرب اجزاء الطبقات المتناسبة بعضها فى بعض ومعرفة الجزء المجتمع من اىّ طبقة هو

اذا اردنا ان نضرب جزء طبقة فى جزء طبقة اخرى (4^r) ونعلم طبقة الجزء المجتمع من اىّ جنس هى ضربنا الطبقتين احديهما فى الاخرى وعلمنا جنس المبلغ على مثال ما قدّمنا فما كان فجزء تلك الطبقة وهو الجواب.

مثال ذلك. اذا اردنا ان نعلم ما يجتمع من ضرب جزء شىء اعنى جزء جذر فى جزء مال ضربنا شيئًا فى مال فكان مكعّبًا وأخذنا جزءه وهو جزء مكعّب فقلنا ان المبلغ من ضرب جزء شىء فى جزء مال هو جزء مكعّب.

(73) المسمّى : السمّى | (76) وهى : وهو | (77) (*1um*) وهى : وهو | (78) والمقسومة : والمقسوم | (82) والمقسومة : والمقسوم | (85) لانّه من : لانها | (89) فى : ما فى (*ut vid. del.* ما) | (90) هى : هو.

[وذلك على قياس ضرب الكسور فى الكسور لانّا نضرب هناك الاجزاء فى الاجزاء ونقسم المبلغ على مضروب المخرجين احدهما فى الآخر. ولمّا كانت الاجزاء هاهنا فى كلّ واحد من المضروب والمضروب فيه جزءًا واحدًا كان مبلغ ضرب احدهما فى الآخر هو ايضًا جزء واحد وقسمة ذلك على مضروب المخرجين اعنى الطبقتين بعضهما فى بعض هو جزء من ذلك المبلغ. فلهذه العلّة نضرب الطبقتين بعضهما فى بعض ونأخذ جزء المبلغ فيكون المطلوب.]

الباب السادس

في قسمة اجزاء الطبقات المتناسبة بعضها على بعض ومعرفة الخارج بالقسم من اىّ طبقة هو

اذا اردنا ان نقسم جزء طبقة على جزء طبقة ونعلم جنس طبقة الخارج بالقسم قسمنا الطبقة المقسوم جزءها على الطبقة المقسوم على جزءها [وعلمنا جنس الخارج بالقسم]. فان كانت الطبقة المقسوم جزءها اقرب الى طبقة الجذر كان المطلوب هو ذلك الخارج بالقسم بعينه. وان كانت الطبقة المقسوم جزءها ابعد من طبقة الجذر فان المطلوب يكون جزءًا من ذلك الخارج بالقسم.

مثال ذلك. اذا اردنا ان نقسم جزء مال على جزء مكعّب كان الخارج بالقسم شيئًا.

واذا اردنا (4^{v}) ان نقسم جزء مكعّب على جزء مال كان الخارج بالقسم جزء شىء.

[وذلك ايضًا على قياس قسمة الكسور على الكسور فانّا نضرب هناك كلّ واحد من المخرجين فى اجزاء المخرج الآخر ضربًا موشّحًا ثمّ نقسم من المبلغين المقسوم على المقسوم عليه. ولمّا كانت الاجزاء هنا فى كلّ واحد من الجنسين جزءًا واحدًا قسمنا الطبقتين المقسومة على المقسومة عليها واستغنينا عن الضرب الموشّح.]

وذلك قياسه.

(104) جنس : جزء | (106) المقسوم : المقسومة | (113) كلّ واحد من الجنسين : كلّ واحدة من الجنسين (*sc.* الجنبتين).

النوع الثانى

فيما يعرض للطبقات المتناسبة اذا كانت مطلقة مقرونة وهو يشتمل على اربعة ابواب

الباب الاوّل

في جمع بعضها الى بعض

اذا اتّفق ان يكون فى المسئلة جنبتان وكان ما فى احديهما من الاجناس نظير ما فى الاخرى واحتيج الى زيادة ما فى احدى الجنبتين على ما فى الاخرى فانّه يزاد عدّة كلّ جنس من احدى الجنبتين على عدّة نظيره من الجنبة الاخرى. فان كان النظيران زائدين معًا فالمبلغ زائد. وان كانا ناقصين معًا اى مستثنيين [من جنس آخر] فالمبلغ ناقص اى مستثنًى. وان كان احدهما زائدًا والآخر ناقصًا وكانت عدّة الزائد اقلّ من عدّة الناقص فانّه يُنقّص اقلّ العدّتين من اكثرهما فما بقى فهو ناقص وهو المبلغ. وان كانت عدّة الزائد اكثر فانّه يُنقّص اقلّ العدّتين من اكثرهما فما بقى فهو زائد وهو المبلغ.

مثال ذلك. اذا اردنا ان نجمع عشرة وشيئًا الى عشرة وشىء كان المبلغ عشرين وشيئين اثنين.

او نجمع عشرة الّا شيئًا الى عشرة الّا شيئًا فان المبلغ يكون عشرين الّا شيئين اثنين.

او نجمع عشرة الّا شيئًا الى عشرة وشىء كان المبلغ عشرين كامله.

او نجمع عشرة وشيئين الى عشرة الّا شيئًا (5^r) كان المبلغ عشرين وشيئًا واحدًا.

او نجمع عشرة الّا شيئين الى عشرة وشىء فان المبلغ يكون عشرين الّا شيئًا واحدًا.

او نجمع خمسة عشر وشيئًا الى شىء الّا عشرة كان المبلغ شيئين اثنين وخمسة.

او نجمع خمسة عشر الّا شيئين اثنين الى شىء الّا عشرة فان المبلغ خمسة الّا شيئًا واحدًا.

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(121) نظير : نظاير | (138) عشر *(postea rubro col. eras.) iter.*

او نجمع عشرة الا شيئا الى شيئين اثنين الا خمسة عشر فان المبلغ شىء الا خمسة. وذلك قياسه.

الباب الثانى

في نقصان بعضها عن بعض

وامّا النقصان فانّه اذا اتّفق ان يكون فى المسئلة جنبتان وكان ما فى احديهما من الاجناس نظير ما فى الاخرى واحتيج الى نقصان ما فى احدى الجنبتين عمّا فى الجنبة الاخرى فانّا ننقّص عدّة كلّ جنس من الجنبة المنقوصة عن عدّة نظيره من الجنبة المنقوصة عنها. فان كان النظيران زائدين وكان المنقوص اقلّ فالباقى زائد. وان كان اكثر فالباقى وهو فضل ما بينهما ناقص اى مستثنىً. وان كان النظيران ناقصين وكان المنقوص اقلّ فالباقى ناقص. وان كان اكثر فالباقى وهو فضل ما بينهما زائد لان ذلك الفضل يكون استثناء [من المستثنى عنه]. وان كان احد النظيرين وليكن المنقوص زائدًا فقط وكان اقلّ او اكثر من المنقوص عنه وكان المنقوص عنه ناقصًا كان الباقى وهو مبلغ العدّتين ناقصًا اى مستثنىً [عن الاصل الذى قدّمنا وذلك ان الاستثناء من الاستثناء زيادة فى المستثنى عنه]. وان كان ناقصًا وكان اقلّ او اكثر من المنقوص عنه وكان المنقوص عنه زائدًا كان الباقى وهو مبلغ العدّتين زائدًا.

مثال ذلك. اذا اردنا ان ننقّص (5^v) عشرة وشيئًا من خمسة عشر وخمسة اشياء كان الباقى خمسة واربعة اشياء.

او ننقّص عشرة وخمسة اشياء من خمسة عشر وشىء كان الباقى خمسة الّا اربعة اشياء.

او ننقّص عشرة الّا شيئًا من عشرين الّا عشرة اشياء كان الباقى عشرة الّا تسعة اشياء.

او ننقّص عشرة الّا عشرة اشياء من عشرين الّا ثلثة اشياء كان الباقى عشرة وسبعة اشياء.

او ننقّص عشرة وشيئًا من خمسة عشر الّا شيئًا كان الباقى خمسة الّا شيئين اثنين.

او ننقّص عشرة الّا شيئًا من خمسة عشر وشىء كان ⟨الباقى⟩ خمسة وشيئين.

وذلك قياسه.

(145) نظير : نظاير | (146) جنس : جنبتين | (147) المنقوص : المنقوصة | (153) (2^{um}) عنه : منه.

الباب الثالث

في ضرب بعضها فى بعض

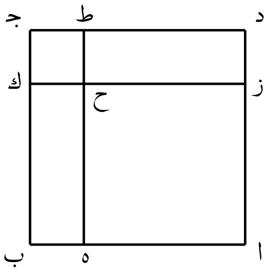
وامّا الضرب فهو انّه اذا كان مقداران واردنا ان نضربهما فى مقدارين آخرين فانّا نضع المضروب فى سطر والمضروب فيه فى سطر آخر تحته بحذائه. ثمّ نحتاج عند ذلك الى اربع ضربات ضربتين متقاطرتين وضربتين قائمتين. واذا كانت ثلثة مقادير فى ثلثة مقادير كان ما نحتاج فيه الى تسع ضربات ستّ ضربات متقاطرة وثلث ضربات قائمة. ثمّ على هذا القياس بالغة ما بلغت المقادير. وايضًا فكلّ مقدارين نضرب احدهما فى الآخر ويكونان متّفقين فى زيادتهما او نقصانهما فالمبلغ من الضرب زائد وان اختلفا فهو ناقص.

مثال ذلك. اذا اردنا ان نضرب عشرة وشيئًا فى عشرة وشىء وضعنا العشرة تحت العشرة والشىء تحت الشىء ثمّ ضربنا العشرة فى الشىء الذى يقاطرها فيكون عشرة اشياء ثمّ ضربنا العشرة الاخرى فى الشىء الآخر الذى يقاطرها فيكون المبلغ ايضًا عشرة اشياء ثمّ ضربنا العشرة فى العشرة وهو القائم فيكون المبلغ مائة (6^r) ثمّ ضربنا الشىء فى الشىء وهو ايضًا قائم فيكون مالًا. ونجمع ذلك فيكون مائة ومالًا وعشرين شيئًا.

ثمّ نضع المضروبين عشرة الّا شيئًا فى عشرة الّا شيئًا على الوضع المتقدّم. ثمّ نضرب عشرة فى الّا شىءٍ المقاطر لها فيكون عشرة اشياء ناقصة اى مستثناة ثمّ نضرب ايضًا العشرة الاخرى فى الّا شىء المقاطر لها فيكون ايضًا عشرة اشياء ناقصة ثمّ نضرب عشرة فى عشرة فيكون مائة زائدة ونضرب الّا شيئًا فى الّا شىء فيكون المبلغ مالًا زائدًا. ونجمع ذلك فيكون مائة ومالًا الّا عشرين شيئًا.

وايضًا فانّا نضع المضروبين عشرة وشيئًا فى عشرة الّا شيئًا على الوضع الاوّل فنضرب عشرة فى الّا شىء فيكون عشرة اشياء ناقصة ثمّ نضرب عشرة فى شىء فيكون عشرة اشياء زائدة ثمّ نضرب عشرة فى عشرة فيكون مائة زائدة ونضرب شيئًا فى الّا شىء فيكون مالًا ناقصًا. ونجمع ذلك فيكون مائة الّا مالًا لان الاشياء الزائدة ذهبت بالاشياء الناقصة لانّهما متساويان فى العدّة.

(188) فيكون (2^{um}) : يكون.



فامّا علّة كون ضرب الناقص فى الناقص زائدًا. فانّا نضع لذلك خطّ اب وليكن عشرة من العدد ونعمل عليه مربّع ابجد ونستثنى من خطّ اب شيئًا وليكن به ومن خطّ اد مثل به وهو دز ونخرج خطّ هحط قائمًا على اب وخطّ زحك قائمًا على اد.

فسطح دح هو من ضرب دز وهو شىء فى زح وهو عشرة الّا شيئًا وذلك عشرة اشياء الّا مالًا وسطح دح مثل سطح حب فكلا سطحى دح حب عشرون شيئًا الّا مالين وسطح حج مال لانّه من ضرب شىء فى مثله فسطوح دح حج حب الثلثة اذن عشرون شيئًا الّا مالًا لان المال الزائد ذهب (6v) بأحد المالين الناقصين وسطح ابجد كلّه هو من ضرب عشرة فى عشرة وهو مائة ومتى ما نقصنا من المائة عشرين شيئًا الّا مالًا كان الباقى مائة ومالًا الّا عشرين شيئًا وذلك مثل ضرب اه وهو عشرة الّا شيئًا فى مثله اعنى سطح اح. وذلك ما اردنا ان نبيّن.

الباب الرابع
فى قسمة بعضها على بعض

وامّا القسمة فالتى يتهيّأ اطّرادها فى هذا النوع المطلق هى قسمة الاجناس المقترنة كم كانت على جنس واحد. فامّا اذا كان المقسوم عليه اكثر من جنس واحد فلا سبيل الى معرفة الخارج بالقسم الّا ان يكون مفروضًا فى المسئلة فهناك يستعمل ضرب من الحيلة وهو ان كلّ مقدار قُسم على مقدار آخر فان الخارج بالقسم اذا ضُرب فى المقسوم عليه عاد المقسوم.

ومثال ما ذكرنا من الوجه الجائز. اذا اردنا ان نقسم عشرة وشيئًا على خمسة فانّا نقسم عشرة على خمسة فيخرج اثنان ثمّ نقسم شيئًا على خمسة فيخرج خُمس شىء.

(193) هحط : هح *pr. scr.*, ط *add. supra, postea* هحط *scr. in marg.* | (198) متى ما (*scripsi*) : متيما | (202) *haec iter. in marg.* | (205) بالقسم : بالقسمة.

ونجمع ذلك فيكون اثنين وخُمس شىء.

واذا اردنا ان نقسم عشرة وخمسة اشياء على شىء فانّا نقسم عشرة على شىء فيكون عشرة اجزاء شىء ونقسم خمسة اشياء على شىء فيكون خمسة من العدد. ونجمع ذلك فيكون الخارج بالقسم خمسة وعشرة اجزاء شىء. وذلك قياسه.

النوع الثالث
فيما يعرض للطبقات المتناسبة اذا كانت مفردة منسوبة
وهو يشتمل على ستّة ابواب

الباب الاوّل
في تضعيفها

تضعيف الجذور المنسوبة. اذا اردنا ان نضعّف جذرًا منسوبًا الى عدد ومعنى التضعيف ⟨للجذر⟩ ان نجعله مثليه او ثلثة امثاله او ما شئنا من الامثال (7^r) فانّا نضرب الامثال وما معها من الكسر ان كان فى مثلها ثمّ فى العدد المنسوب اليه ونأخذ جذر المبلغ فما كان فهو المطلوب.

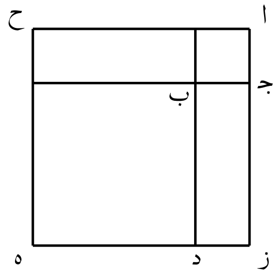
ونجعل المثال فى ذلك اوّلًا منطوقًا به وهو اذا اردنا ان نضعّف جذر اربعة مرّة واحدة ومعناه ان نجعله مثليه ولا فرق بين قولنا جذرا اربعة جذر اىّ مال هو فانّا نضرب عدد الامثال وهو هاهنا اثنان فى مثله فيكون اربعة ثمّ فى العدد المنسوب اليه وهو ايضًا اربعة فيكون ستّة عشر فجذر ذلك وهو اربعة هو ضعف جذر اربعة.

وعلى هذا القياس اذا اردنا ان نجعل جذر اربعة ثلثة امثاله وهو ايضًا كقولنا ثلثة اجذار اربعة جذر اىّ مال هو فانّا نضرب عدد الامثال وهو ثلثة فى مثلها ثمّ ما بلغ وهو تسعة فى اربعة فيكون ستّة وثلثين فجذر ستّة وثلثين وهو ستّة هو ثلثة امثال جذر اربعة.

(212) عشرة ... فيكون *marg.* | (229) فيكون : يكون.

وكذلك اذا اردنا ان نجعل جذر ثمنية مثلين ونصفًا فانّا نضرب عدد الامثال وهو اثنان ونصف فى مثله فيكون ستّة ورُبعًا ثمّ فى ثمنية فيكون خمسين فجذر ذلك وهو جذر خمسين هو مثل جذر ثمنية مرّتين ونصفًا.

[وايضًا فانّا اردنا جذرى تسعة اى نجعله مثلين فانّا نعرف اوّلًا جذرا تسعة جذر اىّ مال هو وهو على قياس ما قدّمنا وهو انّا نضرب اثنين فى مثله لاجل الجذرين فيكون اربعة ثمّ فى تسعة فيكون ستّة وثلثين فجذر ستّة وثلثين هو مثل جذرى تسعة فيصير قولنا كانّا نريد ان نضعّف جذر ستّة وثلثين اى نجعله مثليه].
وذلك قياسه.



برهان ذلك. نجعل لعلّة ما ذكرنا العدد الذى نريد ان نضعّف جذره مربّع اب [المعتدل] وجذره (7v) خطّ جب وليكن عدد الامثال خطّ بد وليكن خطّ بد قائمًا على بج [على زوايا قائمة] ونجعل على بد مربّع به ونتمّم مربّع ازهح.

فلان نسبة هد الى دز كنسبة مربّع به الى سطح بز لان ارتفاعهما واحد لكن زج مثل هد و اج مثل زد ونسبة زج الى اج كنسبة سطح بز الى مربّع اب فنسبة مربّع به الى سطح بز كنسبة سطح بز الى مربّع اب فسطح بز وهو المطلوب موسّط فى النسبة بين مربّعى اب به ويسمّى سطح بز احد المتمّمين لمربّعى اب به وسطح بح المتمّم الآخر وهما متساويان. فلهذه العلّة نضرب عدد الامثال وهو بد فى مثله ثمّ نضرب المبلغ وهو مربّع به فى العدد المجذور وهو اب ونأخذ جذر ذلك وهو سطح بز فيكون المطلوب لانّه مجتمع من ضرب جذر اب وهو خطّ بج فى عدد الامثال وهو خطّ بد. وذلك ما اردنا ان نبيّن.

(232) مثله : مثلها | ورُبعًا : وربع | (233) ونصفًا : ونصف | (234) نجعله : نجعلها | جذرا : جذرى | (235) هو وهو : هو هو.

تضعيف الكعاب المنسوبة. وهنالك استبان ان كلّ عددين نضرب احدهما فى الآخر ثمّ ما اجتمع فى مثله مثل ضرب مربّع احدهما فى مربّع الآخر. وعلى هذا القياس اذا اردنا ان نضعّف كعب عدد فانّا نضرب عدد الامثال فى مثله ثمّ ما بلغ فى عدد الامثال ايضًا ليصير المبلغ مكعّبًا ثمّ ما اجتمع فى العدد المنسوب اليه ونأخذ كعب المبلغ فيكون المطلوب.

والاصل فى ذلك ان كلّ عدد فهو مساوٍ لجذر مربّعه وكعب مكعّبه وجذر جذر مال ماله. وكلّ عددين فجذر ضرب مربّع احدهما فى مربّع الآخر مساوٍ لكعب (8^{r}) ضرب مكعّب احدهما فى مكعّب الآخر وهو ايضًا مساوٍ لجذر جذر ضرب مال مال احدهما فى مال مال الآخر ثمّ على هذا القياس وذلك للعلّة التى قدّمنا من ان كلّ عددين فضرب احدهما فى الآخر ثمّ ما اجتمع فى مثله مثل ضرب مربّع احدهما فى مربّع الآخر. فلان المطلوب فى تضعيف [كعب] الكعب هو ضرب [كعب] الكعب فى عدد الامثال ثمّ اذا كعّبنا ذلك المبلغ كان ذلك كضرب العدد المنسوب اليه فى مكعّب الاضعاف فلذلك نأخذ كعبه فيكون المطلوب.

تضعيف جذور الجذور المنسوبة وهى اضلاع اموال الاموال. وعلى هذا القياس اذا اردنا ان نضعّف جذر جذر عدد وهو تضعيف ضلع مال المال فانّا نضرب عدد الامثال فى مثله ثمّ ما اجتمع فى مثله ليصير مال مال ثمّ ما اجتمع فى العدد المنسوب اليه ونأخذ جذر جذر المبلغ فما كان فهو المطلوب.

والعلّة فى ذلك ما قدّمنا فى المثالين المتقدّمين من ان المطلوب هو ضرب ضلع مال المال فى عدد الاضعاف ثمّ اذا جعلنا ذلك المبلغ مال مال كان ذلك كضربنا العدد المنسوب اليه فى مال مال الاضعاف فلذلك نضرب كذلك ونأخذ جذر جذر المبلغ فيكون المطلوب.

الباب الثانى

فى تجزئتها

تجزئة الاجذار المنسوبة. وامّا التجزئة فعلى قياس التضعيف سواء وهو انّا اذا اردنا ان نجزّىء جذر عدد ومعناه ان نضرب جذر ذلك العدد فى نصف او ثُلث او رُبع

(269) كذلك : ذلك.

او ما كان من اجزاء الواحد فانّا نضرب ذلك الجزء فى مثله ثمّ ما بلغ فى العدد المنسوب اليه ونأخذ جذر المبلغ فما كان فهو المطلوب. (8^v)

مثال ذلك. اذا اردنا ان ننصّف جذر اربعة وهو كقولنا نصف جذر اربعة جذر ایّ مال هو فانّا نضرب الجزء وهو النصف فى مثله فيكون رُبعًا ثمّ فى العدد المنسوب اليه وهو اربعة فيكون واحدًا ونأخذ جذر ذلك وهو واحد وهو المطلوب.

وكذلك اذا اردنا ان نثلّث جذر ستّة وثلثين ومعناه ثُلث جذر ستّة وثلثين جذر ایّ مال هو فانّا نضرب ثُلثًا فى ثُلث فيكون تُسعًا ثمّ فى ستّة وثلثين فيكون اربعة فجذر ذلك وهو اثنان هو ثُلث جذر ستّة وثلثين.

وكذلك ايضًا تجزئة ضلع المكعّب وضلع مال المال على هذا القياس. والعلّة فى ذلك ما قدّمنا فى باب التضعيف بعينه.

الباب الثالث

فى جمع بعضها الى بعض

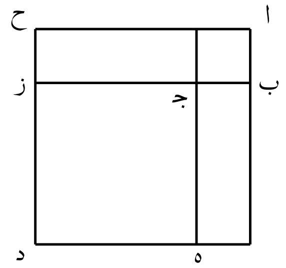
جمع الجذور المنسوبة بعضها الى بعض. اذا اردنا ان نجمع جذر عدد الى جذر عدد فانّا نجمع بين العددين المجذورين ونزيد على المبلغ ضعف جذر مبلغ ضرب احدهما فى الآخر ونأخذ جذر المبلغ فما كان فهو المطلوب.

مثال ذلك فى الجذور المنطقة. وهو ⟨انّا⟩ اذا اردنا ان نجمع جذر اربعة الى جذر تسعة فانّا نضرب اربعة فى تسعة فيكون ستّة وثلثين ونأخذ جذر ذلك وهو ستّة فنضعّفه فيكون اثنى عشر ونزيده على مجموع اربعة وتسعة فيكون المبلغ خمسة وعشرين فجذر ذلك وهو خمسة هو مجموع جذر اربعة وجذر تسعة.

وكذلك اذا اردنا ان نجمع جذر ثلثة الى جذر خمسة فانّا نضرب ثلثة فى خمسة فنأخذ جذر المبلغ وهو جذر خمسة عشر فنضعّفه على قياس ما تقدّم فى باب تضعيف الجذور فيكون جذر ستّين فنزيده على مجموع الثلثة والخمسة فيكون (9^r) المبلغ ثمنية وجذر ستّين ونأخذ جذر ذلك فيكون المطلوب.

(283) المكعّب : الكعب | (291) جذر *add. supra*.

[**وعلّة ذلك** هى ان كلّ عددين مربّعين اذا زدنا عليهما متمّميهما صار المبلغ مربّعًا واذا نقصناهما منهما كان الباقى مربّعًا.]



ونخطّ لبرهان ذلك مربّعين عليهما اج جد وضلع مربّع اج هو بج وضلع مربّع جد هو جز ونتمّم سطحى به جح المتمّمين.

وقد بيّنّا فى باب التضعيف ان كلّ واحد منهما موسّط فى النسبة بين مربّعى اج جد فلذلك نضرب مربّعى اج جد احدهما فى الآخر فيحصل عندنا سطح به مضروبًا فى مثله ونأخذ جذر ذلك فيكون سطح به فنضعّفه فيكون مجموع سطحى به جح ونزيد على ذلك مربّعى اج جد فيتمّ لنا مربّع اد فنأخذ جذره فيكون اح وهو مجموع ضلعى بج جز. وذلك ما اردنا بيانه.

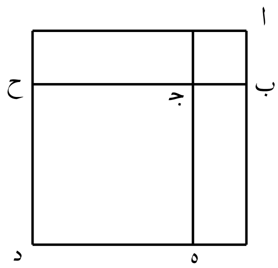
وعلى هذا القياس اذا اردنا ان نجمع جذر عدد الى عدد فانّا نضرب العدد المطلق فى مثله فيصير مجذورًا اعنى من جنس الآخر ثمّ نعمل بهما ما قدّمنا من العمل.

جمع الكعاب المنسوبة بعضها الى بعض. اذا اردنا ان نجمع كعب عدد الى كعب عدد فانّا نضرب مربّع احد العددين فى العدد الآخر ثمّ ما بلغ فى سبعة وعشرين ونأخذ كعب ذلك ونحفظه. ثمّ نضرب مربّع العدد الآخر فى العدد الاوّل ثمّ ما بلغ فى سبعة وعشرين ونأخذ كعبه ونزيده على ما حفظناه. ثمّ نزيد المبلغ على مجموع العددين المكعّبين الموضوعين ونأخذ كعب المبلغ فما كان فهو المطلوب.

مثال ذلك (9^v) فى الكعاب المنطقة. اذا اردنا ان نجمع كعب ثمنية الى كعب مائة وخمسة وعشرين فانّا نضرب الثمنية فى مثلها ثمّ ما بلغ فى مائة وخمسة وعشرين فيكون المبلغ ثمنية آلاف ثمّ فى سبعة وعشرين فيكون المبلغ مأتى الف وستّة عشر الفًا ونأخذ كعب ذلك وهو ستّون فنحفظه. ثمّ نضرب مائة وخمسة وعشرين فى مثلها

(301) جح : حه | (308) بهما : بها.

فيبلغ خمسة عشر الفًا وستّمائة وخمسة وعشرين ثمّ فى ثمنية فيكون مائة الف وخمسة وعشرين الفًا ثمّ فى سبعة وعشرين فيكون المبلغ ثلثة آلاف الف وثلثمائة الف وخمسة وسبعين الفًا ونأخذ كعب ذلك وهو مائة وخمسون فنزيده على ما حفظنا وهو ستّون فيبلغ مائتان وعشرة ⟨ثمّ⟩ نزيد ذلك على مجموع العددين اللذين هما ثمنية ومائة وخمسة وعشرون فيكون المبلغ ثلثمائة وثلثة واربعين فكعب ذلك وهو سبعة هو مجموع كعب ثمنية وكعب مائة وخمسة وعشرين. وهذا قياسه.



ولبرهان ذلك نتوهّم مكعّبين مختلفين على قاعدتى اج جد المربّعين وليكن قطر قاعدة اج على استقامة قطر قاعدة جد وليكن اصغرهما اج ونتوهّم مربّع اد قاعدة المكعّب الذى يحوز المكعّبين الموضوعين اى يحيط بهما. ومعلوم ان هذا المكعّب الاعظم يزيد على المكعّبين الموضوعين بمجسّمين متساويين قاعدتاهما مثل سطح به وارتفاعهما مثل خطّ بج وبمجسّمين آخرين مختلفين ايضًا قاعدة احدهما مربّع اج وارتفاعه هج وقاعدة الآخر مربّع جد وارتفاعه بج.

لكن ضرب خطّ بج فى مثله ثمّ ما بلغ فى جه مجموعًا الى ضرب جه فى نفسه ثمّ ما بلغ فى بج هو مساوٍ لضرب بج فى جه (10r) ثمّ ما بلغ فى مجموع بج جح اعنى المجسّم الذى قاعدته به وارتفاعه بح. فاذن مجموع المجسّمين اللذين قاعدة احدهما اج وارتفاعه جه وقاعدة الآخر جد وارتفاعه بج ثُلث زيادة المكعّب الاعظم الذى قاعدته اد على مجموع المكعّبين اللذين قاعدتاهما اج جد. فاذا ضربنا كلّ واحد من هذين المجسّمين فى ثلثة صار المبلغ مساويًا لتلك الزيادة كلّها ومعلوم ان المجسّم الذى قاعدته اج وارتفاعه جه هو من ضرب بج فى نفسه ثمّ ما اجتمع فى جه.

(320) آلاف : الف | (323) واربعين : واربعون | (324) كعب *add. supra* | (336) مساويًا : مساو.

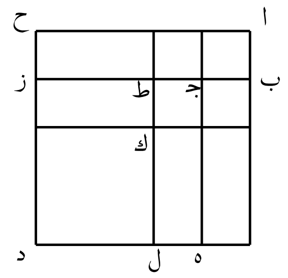
ومعلوم انّا اذا ضربنا مربّع بج فى جه ثمّ فى ثلثة ثمّ صيّرنا المبلغ مكعّبًا كان ذلك مثل ضرب المكعّب الذى يكون على ضلع بج فى مثله ثمّ ما بلغ فى المكعّب الذى ضلعه جه مضروبًا فى سبعة وعشرين. فلاجل هذا نضرب المكعّب الذى يكون على قاعدة اج فى مثله ثمّ فى المكعّب الذى يكون على قاعدة جد مضروبًا فى سبعة وعشرين ويؤخذ كعب ذلك فيكون مثل المجسّم الذى قاعدته مربّع اج وارتفاعه جه. ثمّ نضرب ايضًا المكعّب الذى قاعدته جد فى مثله ثمّ فى المكعّب الذى قاعدته اج مضروبًا فى سبعة وعشرين ونأخذ كعبه فيكون مثل المجسّم الذى قاعدته مربّع جد وارتفاعه بج. وقد بيّنّا ان هذين المجسّمين هما زيادة المكعّب الاعظم على المكعّبين الاصغرين الموضوعين. فلذلك نجمع ذينك المكعّبين الى مجموع العددين المكعّبين ليتمّ لنا المكعّب الاعظم ونأخذ كعب ذلك فيكون (10v) مبلغه مثل مجموع الكعبين المطلوب. وذلك ما اردنا ان نبيّن.

الباب الرابع

فى نقصان بعضها عن بعض

نقصان الجذور المنسوبة بعضها عن بعض. وامّا النقصان فاذا اردنا ان ننقّص جذر عدد من جذر عدد فانّا نجمع بين العددين المجذورين وننقّص من المبلغ ضعف جذر ضرب احدهما فى الآخر ونأخذ جذر المبلغ فيكون المطلوب.

مثال ذلك. اذا اردنا ان ننقّص جذر اربعة من جذر تسعة فانّا نضرب اربعة فى تسعة فيكون ستّة وثلثين ونأخذ جذر ذلك وهو ستّة فنضعّفه فيكون اثنى عشر وننقّصه من مجموع الاربعة والتسعة وهو ثلثة عشر فيبقى واحد ونأخذ جذره وهو واحد وهو الباقى من نقصان جذر اربعة من جذر تسعة.

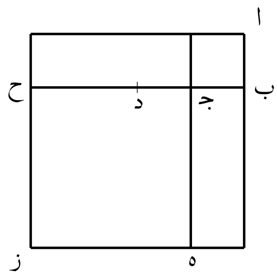


(346) نجمع : جمع | (348) المطلوب : المطلوبين | (355) فيكون ستّة *iter.*

ولبرهان ذلك نتوهم مربع اجـ اصغر من مربع جد وننقص من ضلعه جز مثل ضلع بجـ وليكن جط ونخرج خطّ طكل موازيًا لجه ⟨ونخرج خطّ طز موازيًا للد⟩. فلان سطح به مثل سطح هط يبقى سطح كز مثل سطح هط الّا مربّع جك لكن مربّع جك مثل مربّع اجـ فسطحا هط كز مع مربّع اجـ مثل متمّمى به جح. فلذلك ننقّص ذلك من مربّعى اجـ جد فيبقى مربّع كد ونأخذ جذره وهو لد اعنى طز فما كان فهو فضل جز على بجـ. وذلك ما اردنا ان نبيّن.

نقصان الكعاب المنسوبة بعضها عن بعض. اذا اردنا ان ننقّص كعب عدد من كعب عدد فانّا نضرب مربّع العدد الاصغر فى العدد الاعظم ثمّ ما يبلغ فى سبعة وعشرين ونأخذ كعبه ونزيده على العدد الاعظم ونحفظ ذلك. ثمّ نضرب (11r) مربّع العدد الاعظم فى العدد الاصغر ثمّ فى سبعة وعشرين ونأخذ كعب ذلك فنزيده على العدد الاصغر. وننقّص المبلغ من المحفوظ ونأخذ كعب ما يبقى فما كان فهو المطلوب.

مثال ذلك. اذا اردنا ان ننقّص كعب ثمنية من كعب مائة ⟨وخمسة⟩ وعشرين فانّا نضرب مربّع الثمنية وهو اربعة وستّون فى مائة وخمسة وعشرين فيكون ثمنية آلاف ثمّ فى سبعة وعشرين فيكون المبلغ مأتى الف وستّة عشر الفًا ونأخذ كعب ذلك وهو ستّون فنزيده على مائة وخمسة وعشرين ونحفظ المبلغ وهو مائة وخمسة وثمنون. ثمّ نضرب مربّع مائة وخمسة وعشرين وهو خمسة عشر الفًا وستّمائة وخمسة وعشرون فى ثمنية فيكون مائة وخمسة وعشرين الفًا ثمّ فى سبعة وعشرين فيكون المبلغ ثلثة آلاف الف وثلثمائة الف وخمسة وسبعين الفًا فنأخذ كعب ذلك وهو مائة وخمسون فنزيده على ثمنية فيكون المبلغ مائة وثمنية وخمسين. وننقّص ذلك ممّا حفظناه وهو مائة وخمسة وثمنون فيبقى سبعة وعشرون فكعب ذلك وهو ثلثة هو الباقى من نقصان كعب ثمانية من كعب مائة وخمسة وعشرين. وهذا قياسه.



(358) ضلعه جز : ضلع جد | (362) فما كان فهو : فيكون ما يبقى هو | (375) آلاف : الف.

برهان ذلك. والعلّة فى ذلك ما قدّمنا من البرهان فى باب الجمع وهو انّا اذا ضربنا المكعّب الذى قاعدته اج فى مثله ثمّ ما بلغ فى المكعّب الذى قاعدته جز مضروبًا فى سبعة وعشرين وأخذنا كعب المبلغ كان ذلك مثل ضرب قاعدة اج فى ضلع جح ثمّ ما بلغ فى ثلثة. فنفصل من ضلع جح مثل ضلع بج وليكن جد.

فعلى ما قدّمنا اذا زدنا مربّع بج اعنى مربّع جد المساوى له مضروبًا (11^v) فى جح ثمّ ما بلغ فى ثلثة اعنى المكعّب الكائن على ضلع جد ثلث مرّات مع ضرب مربّع جد فى دح ثلث مرّات على المكعّب الكائن على ضلع جح اعنى المكعّبين الكائنين على ضلعى جد دح وضرب مربّع جد فى دح ثلث مرّات مع ضرب مربّع دح فى جد ثلث مرّات وذلك جميعًا مثل مكعّب جد اربع مرّات ومكعّب دح مرّة وضرب مربّع جد فى دح ستّ مرّات وضرب مربّع دح فى جد ثلث مرّات.

و⟨اذا⟩ القينا ذلك من ضرب مربّع جح فى جد ثمّ فى ثلثة وذلك مثل ضرب كلّ واحد من مربّعى جد دح فى جد ثلث مرّات ومضروب مسطّح جد فى دح ثمّ فى جد مرّتين ثمّ فى ثلثة اعنى مثل ضرب مربّع جد فى دح ستّ مرّات مزيدًا ذلك على المكعّب الكائن من جد بقى مكعّب دح فلذلك نأخذ كعبه فيكون دح وهو ما بقى من نقصان كعب بج من كعب جح. وذلك ما اردنا ان نبيّن.

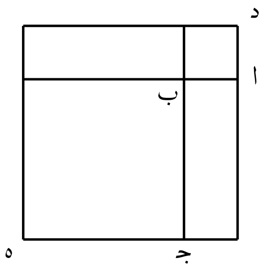
الباب الخامس

فى ضرب بعضها فى بعض

ضرب الجذور المنسوبة بعضها فى بعض. اذا اردنا ان نضرب جذر عدد فى جذر عدد فانّا نضرب احد العددين المجذورين فى الآخر ونأخذ جذر المبلغ فما كان فهو المطلوب.

مثال ذلك فى الجذور المفتوحة. اذا اردنا ان نضرب جذر اربعة فى جذر تسعة فانّا نضرب اربعة فى تسعة فيكون ستّة وثلثين ونأخذ جذر ذلك وهو ستّة وهو مضروب جذر اربعة فى جذر تسعة.

(381) قاعدته (*post.*) : قاعدة | (393) فلذلك : فذلك.



ولبرهان ذلك نضع مربّعى دب به بمكان عددين مجذورين. نريد ان نضرب جذر احدهما فى جذر الآخر. وليكن ضلع مربّع دب هو اب (12r) وضلع مربّع به هو بج ونتمّم مربّع ده.

فلان سطح اج هو الذى يحيط به الجذران وهو كما بيّنّا فى باب التضعيف موسّط فى النسبة بين مربّعى دب به فلذلك نضرب احد المالين فى الآخر اعنى مربّع دب فى مربّع به ونأخذ جذر المبلغ فيكون المطلوب [هو المبلغ وهو سطح اج].

وعلى قياس ما ذكرنا اذا اردنا ان نضرب جذرى تسعة فى ثلثة اجذار اربعة فانّا نعرف اوّلًا جذرا تسعة جذر اىّ مال هو على قياس ما قدّمنا فيكون [جذر] ستّة وثلثين فنحفظه. ثمّ نعرف ايضًا ثلثة اجذار اربعة جذر اىّ مال هو فيكون ايضًا [جذر] ستّة وثلثين. فيصير كانّا نريد ان نضرب جذر ستّة وثلثين فى جذر ستّة وثلثين فنضرب ستّة وثلثين فى ستّة وثلثين ونأخذ جذر المبلغ فيكون ستّة وثلثين وهو مضروب جذرى تسعة فى ثلثة اجذار اربعة.

ضرب اجزاء الجذور المنسوبة بعضها فى بعض. اذا اردنا ان نضرب جزء جذر عدد فى جزء جذر عدد فانّا نضرب كلّ واحد من الجزئين فى مثله ثمّ فى العدد المنسوب اليه ثمّ نضرب المبلغين احدهما فى الآخر ونأخذ جذر ذلك. فما كان فهو المطلوب.

مثال ذلك. اذا اردنا ان نضرب ثُلثى جذر تسعة فى ثلثة اخماس جذر خمسة وعشرين فانّا نعرف اوّلًا ثُلثا جذر تسعة جذر اىّ مال هو على قياس ما قدّمنا فى باب التجزئة فيكون [جذر] اربعة. ثمّ نعرف ايضًا ثلثة اخماس جذر خمسة وعشرين (12v) جذر اىّ مال هو فيكون [جذر] تسعة. فيصير قولنا كانّا نريد ان نضرب جذر اربعة فى جذر تسعة. وذلك قياسه.

(410) جذرا : جذرى | (420) ثُلثا : ثلثى.

وعلّة ذلك ان المطلوب هو السطح الذى يحيط به الجذران وهو موسّط فى النسبة بين مربّعيهما. فلذلك نعرف مربّع كلّ واحد منهما ونضرب ذينك المالين احدهما فى الآخر ونأخذ جذر المبلغ فما كان فهو المطلوب.

وعلى هذا القياس اذا اردنا ان نضرب جذر عدد فى جزء جذر عدد فانّا نعرف ذلك الجزء جذر اىّ مال هو ثمّ نضرب ما يحصل من ماله فى العدد المجذور ونأخذ جذر المبلغ فيكون المطلوب. [وذلك للعلّة التى قدّمناها.]

ضرب الكعاب المنسوبة بعضها فى بعض. فان كان مكان جميع ما ذكرنا من الجذور فى هذا الباب كعاب فانّا نكعّب هاهنا حيث ربّعنا هناك ونأخذ الكعب هاهنا حيث أخذنا الجذر هناك سواء لا تختلف.

ضرب الجذور المنسوبة فى الكعاب المنسوبة. وكذلك اذا اردنا ان نضرب جذر عدد فى كعب عدد كانّا نريد ان نضرب جذر اربعة فى كعب ثمنية فانّا نجعل جذر اربعة مكعبًا وهو انّا نضربه فى مثله فيكون اربعة ثمّ فى جذر اربعة فيكون اربعة اجذار اربعة ثمّ نعلم اربعة اجذار اربعة [جذر] اىّ مال هو على قياس ما قدّمنا فيكون جذر اربعة وستّين وهو المكعّب الكائن من جذر اربعة وكعبه هو كعب جذر اربعة وستّين. فيصير قولنا كانّا نريد ان نضرب كعب ثمنية فى كعب جذر اربعة وستّين فعلى ما قدّمنا من القياس نضرب احد المكعّبين وليكن ثمنية فى المكعّب الآخر وهو جذر اربعة وستّين فيكون (13r) ثمنية اجذار اربعة وستّين. فينبغى ايضًا ان نعرف ثمنية اجذار اربعة وستّين جذر اىّ مال هو فيكون [جذر] اربعة آلاف وستّة وتسعين ونأخذ كعب جذره فيكون اربعة وهو مضروب كعب ثمنية فى جذر اربعة. وذلك قياسه.

ضرب جذور الجذور المنسوبة وهى اضلاع اموال الاموال. اذا اردنا ان نضرب جذر جذر عدد فى جذر جذر عدد فانّا نضرب احد العددين فى الآخر لانّهما متجانسان ونأخذ جذر جذر المبلغ فما كان فهو المطلوب.

وعلّة ذلك ما قدّمنا من ان كلّ عددين فجذر ضرب مربّع احدهما فى مربّع الآخر مثل جذر جذر ضرب مال مال احدهما فى مال مال الآخر.

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(424) الجذران : الجزان | (425) ذينك : ذلك | (431) كعاب : كعابا.

وعلى هذا القياس اذا اردنا ان نضرب جذر جذر عدد فى جذر عدد فانّا نضرب العدد المجذور مرّة واحدة فى مثله ليصير من جنس الآخر ثمّ نضرب احد المالين فى الآخر ونأخذ ⟨جذر⟩ جذر المبلغ فما كان فهو المطلوب.

الباب السادس

فى قسمة بعضها على بعض

قسمة الجذور المنسوبة بعضها على بعض. اذا اردنا ان نقسم جذر عدد على جذر عدد فانّا نقسم مال المقسوم على مال المقسوم عليه ونأخذ جذر الخارج بالقسم فما كان فهو الجواب.

مثال ذلك فى الجذور المفتوحة. ⟨اذا⟩ اردنا ان نقسم جذر ستّة وثلثين على جذر اربعة فانّا نقسم ستّة وثلثين على اربعة فيخرج تسعة فجذر تسعة وهو ثلثة هو الخارج بالقسم من جذر ستّة وثلثين على جذر اربعة.

وعلّة ذلك ما بيّنّا فى غير موضع من ان القسمة هى عكس الضرب. **(13ᵛ)**

قسمة اجزاء الجذور المنسوبة بعضها على بعض. اذا اردنا ان نقسم جزء جذر عدد على جزء جذر عدد فانّا نضرب احد الجزئين فى مثله ثمّ ما بلغ فى العدد المنسوب منه وكذلك نعمل بالجزء الآخر ثمّ نقسم مبلغ المقسوم على مبلغ المقسوم عليه ونأخذ جذر الخارج بالقسم فما كان فهو الجواب.

وعلى هذا القياس اذا اردنا ان نقسم جذر عدد على جزء جذر عدد فانّا نعرف مال ذلك الجذر وهو انّا نضرب ذلك الجزء فى مثله ثمّ ما بلغ فى العدد المنسوب منه ثمّ نقسم عدد المقسوم على ما بلغ المقسوم عليه ونأخذ جذر الخارج بالقسم فما كان فهو الجواب.

قسمة الكعاب المنسوبة بعضها على بعض. فان كان مكان ما ذكرنا من الجذور فى هذا الباب كعاب فان القياس فى ذلك واحد الّا انّا نكعّب هاهنا حيث ربّعنا هناك ونأخذ الكعب هاهنا حيث أخذنا الجذر هناك سواء لا يختلف.

قسمة الكعاب والجذور المنسوبتين بعضها على بعض. اذا اردنا ان نقسم كعب عدد على جذر عدد كانّا نريد ان نقسم كعب ثمنية على جذر اربعة فانّا نجعل جذر اربعة مكعّبًا ونعمل ما عملنا فى باب الضرب بعينه حتّى ننتهى فى العمل الى ان نريد ان نقسم كعب ثمنية على ⟨كعب⟩ جذر اربعة وستّين. فنحتاج هنالك ان نضرب

(450) المالين : المبلغين | (466) الجذر : الجزء | (467) عدد : العدد | (470) كعاب : كعابا | (473) كانّا : فانا | (474) ننتهى : ينتهى.

ثمنية فى مثلها ليصير من جنس الآخر ثمّ نقسم ما بلغ وهو اربعة وستّون على المقسوم عليه وهو ايضًا اربعة وستّون فيخرج واحد ونأخذ ⟨كعب⟩ جذره وهو ايضًا واحد وهو الجواب.

وكذلك اذا اردنا ان نقسم جذر عدد على كعب (**14^{r}**) عدد كانّا اردنا ان نقسم جذر اربعة وستّين على كعب ثمانية فنردّ جذر اربعة وستّين الى جنس المكعّب وهو انّا نضرب جذر اربعة وستّين فى مثله فيكون اربعة وستّين ثمّ فى جذر اربعة وستّين فيكون اربعة وستّين مرّة جذر اربعة وستّين فنعرف اربعة وستّون مرّة جذر اربعة وستّين جذر اىّ مال هو. وبابه ما قدّمنا ⟨من⟩ ان نضرب عدد الامثال وهو اربعة وستّون فى مثله ثمّ فى العدد المنسوب اليه وهو ايضًا اربعة وستّون فيكون مأتى الف واثنين وستّين الفًا ومائة واربعة واربعين فجذر كعب ذلك هو جذر اربعة وستّين مردود الى جنس المكعّب فيصير كانّا نريد ان نقسم جذر كعب مأتى الف واثنين وستّين الفًا ومائة واربعة واربعين على كعب ثمانية. فعلى ما قدّمنا من القياس نجعل كعب ثمنية مجذورًا ليصير من جنس المقسوم وهو انّا نضرب الثمنية فى مثلها فيكون اربعة وستّين ثمّ نقسم مأتى الف واثنين وستّين الفًا ومائة واربعة واربعين على اربعة وستّين فيكون الخارج بالقسم اربعة آلاف وستّة وتسعين. وذلك ما يخرج من قسمة جذر اربعة وستّين على كعب ثمنية ولان كعب اربعة آلاف وستّة وتسعين هو ستّة عشر وجذرها هو اربعة فهو الجواب.

قسمة جذور الجذور المنسوبة وهى اضلاع اموال الاموال بعضها على بعض. اذا اردنا ان نقسم جذر جذر عدد على جذر جذر عدد فانّا نقسم عدد المقسوم على عدد المقسوم عليه ونأخذ جذر جذر الخارج بالقسم فما كان فهو الجواب. (**14^{v}**)

وعلى هذا القياس اذا اردنا ان نقسم جذر جذر عدد على جذر عدد فانّا نضرب عدد المجذور مرّة فى مثله ليصير من جنس الآخر ثمّ نقسم المقسوم على ما بلغ المقسوم عليه ونأخذ جذر جذر الخارج بالقسم فما كان فهو الجواب.

قسمة الطبقات المطلقة المنسوبة اذا كانت مفردة او مقرونة بعضها على بعض. **امّا القسمة فى هذا النوع** فالذى يتهيّأ اطّراده منها هو قسمة الاجناس المقترنة كم كانت

(482) وستّون : وستّين | (484) مثله : مثلها | (485) هو : وهو | (490) فيكون الخارج : فيخرج | (492) فهو : وهو | (494) عدد المقسوم : العدد المقسوم | (495) عدد المقسوم : العدد المقسوم | (500) فالذى : فالتى.

على جنس واحد كيف ما كان ذلك الجنس المقسوم عليه مطلقًا او منسوبًا. فامّا اذا كان المقسوم عليه اكثر من جنس واحد فلا يكاد يتهيّأ عليه القسمة الّا بحيل لا يلزمها وجه القياس. وذلك اذا لم يكن المقسوم عليه اكثر من جنسين وكان احدهما معلومًا والآخر ⟨جذرًا⟩ منسوبًا. فامّا اذا كان المقسوم عليه جنسين وكان احدهما مطلقًا اى شيئًا او مكعّبًا او ما اشبه ذلك او كان اكثر من جنسين كيف ما كانت الاجناس فلا سبيل الى معرفة الخارج بالقسم.

[**مثال ذلك فى المقسوم عليه اذا كان مفردًا.** وهو اذا اردنا ان نقسم عشرة وجذر خمسة عشر على شىء. قسمنا عشرة على شىء فيكون الخارج بالقسم عشرة اجزاء شىء. ثمّ قسمنا جذر خمسة عشر على شىء وهو انّا نضرب الشىء فى مثله فيكون مالًا ثمّ فى خمسة عشر فيكون خمسة عشر مالًا ونأخذ جذر ذلك وهو جذر خمسة عشر مالًا. ونجمع ذلك فيكون عشرة اجزاء شىء وجذر خمسة عشر مالًا. وذلك قياسه.]

وامّا مثاله فى المقسوم عليه اذا كان مفردًا منسوبًا. فهو اذا اردنا ان نقسم عشرة وجذر عشرين على جذر اربعة قسمنا عشرة على جذر اربعة وهو انّا نضرب عشرة فى مثلها فيكون مائة ثمّ نقسم مائة (15^r) على اربعة فيخرج خمسة وعشرون ونأخذ جذر ذلك وهو خمسة فنحتفظ به. ثمّ نقسم جذر عشرين على جذر اربعة على قياس ما قدّمنا فيكون الخارج بالقسم جذر خمسة. ونجمعه الى المحفوظ فيكون المبلغ خمسة وجذر خمسة. وذلك قياسه.

وامّا مثال اذا كان المقسوم مفردًا وكان المقسوم عليه مقرونًا. فهو اذا اردنا ان نقسم خمسين على عشرة وجذر عشرة فانّا نستعمل فى ذلك ضربًا من الحيلة وهو انّا ننقّص جذر عشرة من عشرة فيبقى عشرة الّا جذر عشرة ثمّ نضرب عشرة الّا جذر عشرة فى عشرة وجذر عشرة فيكون من ذلك تسعون.

فلان عشرة وجذر عشرة ضُرب فى عشرة الّا جذر عشرة فاجتمع من ذلك تسعون فاذا قسمنا التسعين على عشرة وجذر عشرة خرج بالقسم عشرة الّا جذر عشرة وذلك لان كلّ عددين ضُرب احدهما فى الآخر فان المبلغ اذا قُسم على احد العددين خرج الآخر. لكنّا اذا قسمنا الخمسين وهو المقسوم على عشرة وجذر عشرة فيخرج لنا عدد كانت نسبة الخمسين الى ذلك العدد الخارج بالقسم كنسبة التسعين الى عشرة الّا جذر عشرة وذلك لان كلّ واحد من التسعين والخمسين يكون مقسومًا على عدد واحد وهو عشرة وجذر عشرة. فيكون نسبة احد المقسومين الى الخارج له بالقسم

(505) مكعّبًا : كعبا | (521) تسعون : تسعين | (527) لان : ان.

كنسبة المقسوم الآخر الى الخارج له بالقسم. فهذه اربعة اعداد متناسبة ثلثة منها معلومة وواحد مجهول.

فنضرب الخمسين فى عشرة الّا جذر عشرة فيبلغ الضرب خمس مائة الّا خمسين جذر عشرة ونقسم ذلك على تسعين فيخرج من قسمة الخمس مائة على تسعين خمسة وخمسة اتساع ومن قسمة خمسين جذر عشرة الناقصة وهى جذر خمسة وعشرين الفًا ناقصًا على تسعين (15^{v}) جذر ثلثة وسبعة اجزاء من احد وثمنين ناقص ونجمع ذلك فيكون خمسة وخمسة اتساع الّا جذر ثلثة وسبعة اجزاء من احد وثمنين جزءًا من واحد. وذلك قياسه.

وعلّة ذلك انّما نقّصنا جذر عشرة من عشرة ثمّ ضربنا الباقى فى عشرة وجذر عشرة ليحصل لنا من ذلك عدد منطق لان كلّ عدد ذى اسمين اذا ضُرب فى منفصله فان المبلغ يكون عددًا منطقًا. فامّا ذو الاسمين المطلق فكلّ عدد مركّب من عددين منطقين فى القوّة او احدهما منطق فى الطول والآخر منطق فى القوّة وذلك مثل جذر عشرة وجذر ثلثة او مثل عشرة وجذر عشرة وما اشبه ذلك. وامّا المنفصل فكلّ عدد ذى اسمين اذا فُصل قسمه الاصغر من قسمه الاكبر فان الذى يبقى من ذلك يقال له المنفصل مطلقًا.

وامّا مثاله فى المقسوم والمقسوم عليه اذا كانا مقرونين. فهو اذا اردنا ان نقسم خمسين وجذر مأتين على عشرة وجذر عشرة فانّا نقسم اوّلًا خمسين على عشرة وجذر عشرة على ما عملنا فى المثال المتقدّم بعينه فيخرج لنا ما خرج هناك وهو خمسة وخمسة اتساع الّا جذر ثلثة وسبعة اجزاء من احد وثمنين جزءًا من واحد. ثمّ نقسم جذر مأتين على عشرة وجذر عشرة على مثال العمل المتقدّم بعينه وذلك انّا نضرب جذر مأتين فى عشرة الّا جذر عشرة فيكون عشرة اجذار مأتين اعنى جذر عشرين الفًا الّا جذر الفين ونقسم ذلك على تسعين فيخرج جذر اثنين وثمنية وثلثين جزءًا من احد وثمنين الّا جذر عشرين جزءًا من احد وثمنين ونجمع ذلك فيكون المبلغ خمسة وخمسة اتساع الّا جذر ثلثة وسبعة اجزاء من احد وثمنين جزءًا من واحد وجذر اثنين وثمنية وثلثين جزءًا من احد (16^{r}) وثمنين الّا جذر عشرين جزءًا من احد وثمنين جزءًا من واحد. وذلك قياسه.

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(538) عدد (*post.*) : عددين.

النوع الرابع
فيما يعرض للطبقات المتناسبة من المعادلة المفردة والمقترنة ويشتمل على بابين

الباب الاوّل
فى المعادلة المفردة

امّا المعادلة المفردة فهى ان يعدل نوع من هذه الانواع المتناسبة التى ذكرنا نوعًا آخر منها اى يساويه. فثلثة من هذه المعادلات المفردة وهى التى تقع بين الثلثة الانواع الاوَل المتناسبة التى هى العدد والجذر والمال وهى اصول لسائر المعادلات المفردة لان سائرها راجعة اليها ومنحطّة نحوها حتّى يصير من جنس ما ان لم يكن احدى المعادلتين طبقة عدد. وقد يعرض لجميعها فى بعض ما نعرف فيه من اعمال ان يكون عدّة ابعد النوعين المعادلين زائدة على الواحد او ناقصة عنه. فنحتاج عند ذلك ردّها الى الواحد الصحيح بجبر ذلك النقصان او حذف تلك الزيادة وكذلك العمل بما يعادله من النوع الاسفل [حتّى يصيران على النسبة الاولى].

فامّا العمل فى ردّ هذه الاجناس المعادلة الى واحدها اذا لم يكن معها كسر فسهل لا يحتاج فيه الى كثير شغل ومؤنة. فامّا اذا كانت كسور [او كان معهما كسور] فانّا نحتاج فى ذلك الى استعمال عملين احدهما ان نزيد على عدد معلوم جزءًا منه معلومًا والثانى ان ننقّص منه جزءًا منه معلومًا.

فامّا فى الزيادة فهو اذا اردنا ان نزيد على عدد معلوم جزءًا منه معلومًا فانّا نضع مخرج ذلك الجزء فى موضعين ونزيد على احدهما جزءه فما بلغ نضربه فى العدد (16^v) ونقسم المبلغ ⟨على⟩ المخرج فى الموضع الثانى فما خرج بالقسم فهو العدد مزيدًا عليه جزءه.

وفى النقصان فهو اذا اردنا ان ننقّص من عدد معلوم جزءًا منه معلومًا فانّا نعمل ما عملناه فى الزيادة سواء الّا ان ننقّص من احد الموضعين جزءه حيث زدناه هناك فقط فما كان الخارج بالقسم فهو المطلوب.

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(560) فهى : فهو | (566) ذلك : تلك | (569) كسور : كسورا.

مثال ذلك فى الزيادة. اذا اردنا ان نزيد على واحد وثُلثين مثل خُمسه فانّا نضع مخرج الخُمس وهو خمسة فى موضعين ونزيد على احد الموضعين خُمسه وهو واحد فيكون ستّة ثمّ نضرب ستّة فى العدد وهو واحد وثُلثان فيكون عشرة. ثمّ نقسم عشرة على المخرج فى الموضع الآخر وهو خمسة فهو الخارج بالقسم اثنان وهو واحد وثُلثان مزيدًا عليه خُمسه. وذلك قياسه.

وعلّة ذلك انّا اذا وضعنا الخمسة فى موضعين وزدنا على احدهما جزءه وهو الخُمس فبلغ ستّة حتّى صارت نسبة الخمسة الى الستّة كنسبة العدد الذى هو واحد وثُلثان الى المطلوب لان المطلوب انّما يحتاج ان يكون مثل واحد وثُلثين ومثل خُمسه فلذلك نضرب الستّة فى واحد وثُلثين وهو الثانى فى الثالث ونقسم ذلك على خمسة وهو الاوّل فيخرج المطلوب وهو الرابع.

وامّا مثاله فى النقصان فهو اذا اردنا ان ننقّص من واحد وثُلث مثل رُبعه. فنضع مخرج الرُبع وهو اربعة فى موضعين وننقّص من احد الموضعين رُبعه وهو واحد فيبقى ثلثه ثمّ نضرب ثلثة فى واحد وثُلث فيكون اربعة ونقسم ذلك على المخرج فى الموضع الثانى وهو ايضًا اربعة فيخرج واحد وهو الجواب.

فامّا المعادلات الثلثة التى تقع بين العدد والجذور والاموال

فالاولى (17^r) **منها جذور تعدل عددًا.** وهو كقولنا جذر يعدل ثلثة فالجذر ثلثة والمال الذى يكون منه تسعة.

وكقولنا اربعة اجذار تعدل اثنى عشر. فلان عدّة ابعد النوعين المعادلين وهى عدّة الاجذار زائدة على الواحد لانّها اربعة فالذى نحتاج اليه فى ردّها الى الواحد ان ننقّص من كلّ ما معنا من الجذور والعدد مثل ثلثة ارباعه. فيحصل عندنا بعد ذلك جذر يعدل ثلثة فالجذر ثلثة والمال الذى يكون منه تسعة.

وكقولنا نصف جذر يعدل واحدًا ونصفًا. فلان عدّة الجذور هاهنا اقلّ من واحد فالذى نحتاج اليه فى ردّها الى الواحد الصحيح ان نزيد على ما معنا مثله فيصير معنا جذر يعدل ثلثة فالجذر ثلثة والمال الذى يكون منه تسعة.

(596) اثنى : اثنا | (597) فالذى نحتاج اليه : ونحتاج | (601) فالذى : والذى | ردّها : رده.

وامّا المعادلة الثانية فهى اموال تعدل عددًا. وهو كقولنا مال يعدل تسعة. فالمال تسعة وجذره ثلثة.

وكقولنا ثلثة اموال وثُلث تعدل ثلثين. فلان عدّة الاموال اكثر من واحد ومعها كسر فنبسط ذلك من جنس الكسر اثلاثًا فيكون عشرة اثلاث فالذى يجب ان ننقّص من ذلك حتّى يرجع الى المال الواحد سبعة اثلاث وهى سبعة اعشاره وننقّص من الثلثين ايضًا سبعة اعشاره وهى احد وعشرون فيحصل معنا بعد ذلك مال يعدل تسعة.

وكقولنا ثُلثا مال يعدل ستّة فنحتاج هاهنا الى ان نزيد على كلّ ما معنا مثل نصفه فيحصل معنا بعد ذلك مال يعدل تسعة.

وامّا المعادلة الثالثة فهى اموال تعدل جذورًا. ⟨وهو⟩ كقولنا مال يعدل ثلثة اجذار. فلانّا بيّنّا ان نسبة المال الى الجذر كنسبة الجذر الى الواحد فنسبة المال الى الثلثة الاجذار كنسبة الجذر الى الثلثة الآحاد فجذر المال ثلثة والمال (**17ᵛ**) الذى يكون منه تسعة وهو مثل ثلثة اجذاره.

وكقولنا مالان وثُلث تعدل سبعة اجذار. فلان عدّة الاموال اكثر من واحد ومعها كسر فنبسط ذلك من جنس الكسر اثلاثًا فيكون سبعة اثلاث وهى اربعة اسباعه واذا نقّصنا من السبعة الاجذار اربعة اسباعها وهى اربعة بقى منها ثلثة اجذار وهى الجذور المعادلة للمال الواحد فنقول ان المال الواحد يعدل ثلثة اجذار فالجذر الواحد يعدل ثلثة والجذر ثلثة والمال تسعة ومثلا هذا المال وثُلثه هو احد وعشرون وذلك مثل سبعة اجذاره.

وكقولنا ثُلثا خُمس مال يعدل سُبع جذره. فلان عدّة الاموال اقلّ من واحد ومخرجه من ثلثة فى خمسة اعنى من خمسة عشر فثُلثا خُمسه جزءان من خمسة عشر فنحتاج فى ردّه الى المال الصحيح ان نزيد عليه ثلثة عشر ⟨مرّة نصفه⟩ اعنى ستّة امثاله ونصف مثله فامّا ان نضربه وما يعادله [فى سبعة ونصف] فنسلك فيه طريق القياس الذى قدّمنا وهو ان نضع واحدًا فى موضعين [لاجل المثل] ونزيد على ⟨احد⟩ الموضعين ستّة امثاله ونصف مثله فيبلغ سبعة ونصف ثمّ نضرب ذلك فى عدد الجذر وهو سُبع فيكون واحدًا ونصف سُبع ونقسم ذلك على الموضع الثانى وهو واحد

(612) كقولنا ... اجذار *rubro col.* | (616) وكقولنا ... اجذار *rubro col.* | (620) ومثلا : ومثلى | (622) وكقولنا ... جذره *crassa scriptura* | (625) فنسلك : ونسلك | (627) امثاله : مثاله (*ut vid.*).

[فيخرج من القسم واحد ونصف سُبع لان كلّ شىء ضُرب فى الواحد او قُسم عليه فانّه لا يتغيّر]. فنقول ان الجذور المعادلة للمال الواحد هى جذر ونصف سُبع جذر فعلى ما قدّمنا من التناسب يكون الجذر واحدًا ونصف سُبع واحد والمال الذى يكون منه واحد وسُبع وجزء من مائة وستّة وتسعين جزءًا من واحد. فاذا بسطنا المال من جنس الكسر اعنى اجزاء من مائة وستّة وتسعين كان مبلغه (18^r) مأتين وخمسة وعشرين وثُلثا خُمسه ثلثون فنحفظ ذلك. ثمّ نبسط الجذر من جنس هذا الكسر ⟨اعنى⟩ اجزاء من مائة وستّة وتسعين فيكون المبلغ مأتين وعشرة فسُبع ذلك ثلثون وهو مثل المحفوظ. [فقد صحّ التجزئة.] وذلك قياسه.

وقد يقع المعادلة بين كلّ طبقتين من سائر الطبقات المتناسبة التى سمّيناها غير ان حكم ذلك اذا لم يكن احديهما ⟨من⟩ طبقة العدد ان يحطّ كلّ واحدة منهما منزلةً او منازل حتّى يصير اقربهما [من طبقة العدد] من جنس العدد. مثل ان يكون مكعّبات تعدل جذور ⟨الجذور⟩ فنحطّ ذلك ثلث منازل فيصير المكعّبات عددًا واموال الاموال جذورًا. ومثل اموال اموال تعدل اموالًا فنحطّ ذلك منزلتين فيصير الاموال عددًا واموال الاموال اموالًا. وذلك قياسه.

الباب الثانى

فى المعادلة المقترنة

وامّا المعادلة المقترنة فالتى يتهيّأ اطّرادها فى صناعة الجبر والمقابلة هى ما يقع منها فى ثلثة اصول متناسبة من الاصول التى قدّمنا ذكرها. فثلثة من هذه المعادلاتُ المقترنة وهى التى تقع بين الاصول الاوَل الثلثة. **اوّلها اموال وجذور تعدل عددًا. والثانى اموال وعدد تعدل جذورًا. والثالث جذور وعدد تعدل اموالًا.** هى اسطقس لسائر المعادلات المقترنة لان سائرها راجعة اليها ومنحطّة نحوها حتّى يصير من جنسها كما بيّنّا ذلك فى المعادلات المفردة.

وقد يعرض لها فى بعض ما نعرّف فيه من الاعمال ان يكون عدّة اعلى المنازل المتعادلة زائدة على الواحد او ناقصة عنه فنحتاج عند ذلك الى ردّها الى الواحد

(638) ⟨من⟩ طبقة العدد : طبقة عدد *(corr. infra, l. 639, cod.)* | واحدة : واحد | (640) جذور : جذورا | (645) يتهيّأ : يتهيؤ | (646) المعادلاتُ : المعادلة | (652) او : و.

الصحيح بجبر ذلك النقصان (18^{v}) او حذف تلك الزيادة وكذلك العمل بما يعادلها من المنزلتين الاخريين [حتّى يصير الثلثة على نسبتها الاولى كما كانت]. والعمل فى ردّ هذه الانواع الى واحدها ما قد ارشدنا اليه فى المعادلات المفردة سواء لا يختلف.

وامّا معرفة ضلع المال فى المقترن الاوّل

فاعلم ان المجهول الذى نحتاج الى استخراجه بالحساب ومعرفته فى كلّ واحد من هذه المقترنات الثلثة هو ضلع المال المذكور فيها. والذى يجب استعماله فى معرفة ذلك فى المقترن الاوّل الذى هو اموال وجذور تعدل عددًا بعد ردّ عدّة الاموال الى واحدها الصحيح ان كانت اقّل او اكثر وما معها من الجذور والعدد هو ان نضرب نصف عدّة الاجذار فى مثله [اعنى عدد امثال عدّة نصف الاجذار] ويزاد المبلغ على العدد المعادل ويؤخذ جذر ما اجتمع فيُنقّص منه عدّة نصف الاجذار فما بقى فهو ضلع المال المجهول.

مثال ذلك مال وعشرة اجذار تعدل تسعة وثلثين. فننصّف عدّة الاجذار وهى عشرة فيكون نصفها خمسة ونضربها فى مثلها فيكون خمسة وعشرين [وهى عدد لانّا ضربنا عددًا مثل عدد نصف الاجذار ولم نضرب جذورًا]. ثمّ نزيد ذلك على العدد وهو تسعة وثلثون فيكون المبلغ اربعة وستّين ونأخذ جذره وهو ثمنية فننقّص منه نصف عدّة الاجذار وهو خمسة فيبقى ثلثة فهو جذر المال والمال تسعة وعشرة اجذاره ثلثون ومجموعهما تسعة وثلثون.

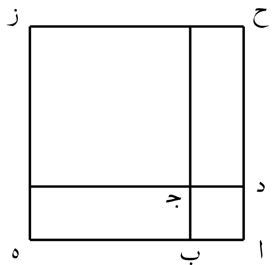
وكقولنا مالان وثُلث وسبعة اجذار تعدل اثنين واربعين. فلان عدّة الاموال اكثر من واحد فنردّها الى الواحد وهو انّا نبسطها من جنس الكسر اثلاثًا فيكون سبعة اثلاث ولانّا نحتاج ان ننقّص من ذلك ومن كلّ ما معنا من الجذور والعدد (19^{r}) اربعة اسباعه فعلى قياس ما قدّمنا من العمل نضرب كلّ واحد من الجذور والعدد فى ثلثة ونقسم المبلغ على سبعة فما خرج من كلّ واحد منهما فهو المال المعادل للواحد الصحيح واذا عملنا ذلك حصل معنا مال وثلثة اجذار تعدل ثمنية عشر. فنضرب نصف عدّة الاجذار وهو واحد ونصف فى مثله فيكون اثنين ورُبعًا ونزيد ذلك على العدد وهو ثمنية عشر فيبلغ عشرين ورُبعًا ونأخذ جذر ذلك وهو اربعة ونصف فننقّص من

(653) ذلك : تلك | وكذلك : وذلك | (658) الثلثة : الثلث | (660) كانت : كان | (668) وهو : وهى | (677) عشرين ورُبعًا : عشرين و ربع.

ذلك نصف عدّة الاجذار وهو واحد ونصف فيبقى ثلثة وهو جذر المال والمال تسعة واذا ضعّفناه وزدنا على المبلغ مثل ثُلث المال بلغ احدًا وعشرين واذا زدنا على ذلك مثل سبعة اجذاره وهى ايضًا احد وعشرون صار المبلغ اثنين واربعين.

وكقولنا نصف وثُلث مال وجذران وثُلث تعدل اربعة عشر درهمًا ونصفًا . فنحتاج هاهنا ان نكمّل المال وهو انّا نزيد عليه وعلى كلّ ما معه من الجذور والعدد مثل خُمسه. فعلى قياس ما قدّمنا من العمل نضع مخرج الخُمس فى موضعين ونزيد على احدهما خُمسه وهو واحد فيبلغ ستّة ثمّ نضرب كلّ واحد من الجذور والعدد فى ستّة ونقسم المبلغ على الموضع الثانى وهو خمسة فيكون ما يخرج من كلّ واحد منهما هو المعادل للمال الواحد الصحيح. واذا عملنا ذلك حصل معنا مال وجذران واربعة اخماس جذر تعدل سبعة عشر وخُمسين من العدد. فنضرب نصف عدّة الاجذار وهو واحد وخُمسان فى مثله ونزيد المبلغ وهو واحد واربعة اخماس واربعة اخماس خُمس على العدد وهو سبعة عشر وخُمسان فيكون المبلغ تسعة عشر وخُمسًا (**19ᵛ**) واربعة اخماس خُمس ونأخذ جذر ذلك وهو اربعة وخُمسان فننقّص منه نصف عدّة الاجذار وهو واحد وخُمسان فيبقى ثلثة وهو جذر المال والمال تسعة.

وذلك قياس جميع ما يكون ويؤتى فى هذا النوع ان شاء اللّه تعالى.

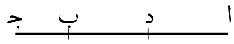


صورة هذا العمل. لمّا كان مجموع العدد ومربّع نصف عدّة الاجذار عددًا مربّعًا علمنا ان صورة العدد علم على مربّع نصف الاجذار. وكلّ علم فانّه مساوٍ لمربّع ومتمّمين. فاذن العدد مساوٍ لمربّع ومتمّمين. لكنّه مساوٍ لمال وعشرة اجذار. فاذن كلّ واحد من المتمّمين خمسة اجذار وذلك لان كلّ واحد منهما سطح يحيط به ضلعان وأحدهما جذر لانّه ضلع مربّع والآخر مساوٍ لنصف عدّة الاجذار اعنى خمسة من العدد.

(679) ضعّفناه : اضعفناه | (684) خُمسه : خمسة | (689) وخُمسًا : وخمس | (692) ويؤتى : وياتى.

رأينا ان نجعل صورة المال المجهول مربّعًا [متساوى الاضلاع والزوايا] مثل مربّع ابجد ونخرج ضلع اب منه على استقامته الى نقطة ه ونجعل به مساويًا لنصف عدّة الاجذار ونجعل على اه مربّع اهزح. ونخرج ضلعى بج دج على استقامتهما الى ضلعى حز هز.

فلان كلّ واحد من ضلعى بج جد جذر وكلّ واحد من ضلعى دح به خمسة فكلّ واحد من سطحى حج جه خمسة اجذار ومجموعهما مع مربّع ابجد اعنى العلم مال وعشرة اجذار وذلك يعدل تسعة وثلثين. فلان مربّع جز خمسة وعشرون فاذن جميع سطح از اربعة وستّون واه جذره فهو ثمنية. فاذا نقّصنا من ذلك به وهو نصف عدّة الاجذار (20ʳ) وهو خمسة بقى اب ثلثة وهو جذر المال المجهول. وذلك ما اردنا ان نبيّن.



عمل هذه المسئلة بالهندسة والبرهان عليه وعلى علّة تنصيف الاجذار فيه بالخطوط.

اذا اردنا معرفة ضلع المال المجهول وضعنا خطّ اب مساويًا لعدّة الاجذار واضفنا اليه سطحًا قائمَ الزوايا مساويًا للعدد المعلوم يزيد على تمامه مربّعًا كما تبيّن ذلك فى التاسع والعشرين من المقالة السادسة من كتاب الاصول. وليكن سطح اج فى جب وضلع المربّع الزائد بج. فاقول ان بج ضلع المال المطلوب.

برهانه انّا نقسم اب بنصفين على د فخطّ اب قُسم بنصفين على د وزيد فيه زيادة وهى بج. فضرب اج فى جب مع مربّع دب مثل مربّع دج. لكن ضرب اج فى بج معلوم لانّه مساوٍ للعدد المعلوم فلذلك يزيد مربّع نصف عدّة الاجذار اعنى مربّع دب على العدد المعلوم وهو سطح اج فى جب ليصير لنا مربّع دج معلومًا. فنأخذ جذره وهو دج فننقّص منه نصف الاجذار وهو دب فيبقى بج معلومًا وهو ضلع المال. وذلك ما اردنا ان نبيّن.

معرفة ضلع المال فى المقترن الثانى

وامّا العمل فى معرفة ضلع المال فى المقترن الثانى الذى هو اموال وعدد تعدل جذورًا بعد ردّ الاموال الى واحدها ان كانت اقلّ او اكثر فهو ان ننصّف عدّة

(700) استقامته : استقامة | مساويًا : مساو (*add. supra* يا) | (711) للعدد المعلوم : لعدة المعلوم | (715) زيادة : زيادتا (*ut vid.*).

الاجذار ونضرب نصف عدّتها فى مثله وننقّص من المبلغ العدد المعلوم ونأخذ جذر ما يبقى وننقّصه من عدّة نصف الاجذار او نزيد عليها فما بلغ فهو جذر المال المجهول.

وانّما قلنا ننقّص جذر ما يبقى او نزيده لان ⟨من الاعمال فى تلك⟩ مسائل الجبر والمقابلة ما يخرج ⟨كما⟩ فى (20$^{\mathrm{v}}$) هذا المقترن بالزيادة والنقصان جميعًا ومنها ما يخرج بالنقصان وحده او بالزيادة فقط. فينبغى ان نمتحن جميع ما يردّ من المسائل المردّية الى هذا المقترن بكلّ واحد من الوجوه التى ذكرنا حتّى يخرج الى حدّ الجواب. وليس يكون مربّع نصف عدّة الاجذار اقلّ من العدد الذى يكون مع المال فى هذا المقترن ابدًا واذا اتّفق ان يكون اقلّ فتلك المسئلة مستحيلة. وان كانا متساويين فجذر المال المجهول مثل نصف عدّة الاجذار.

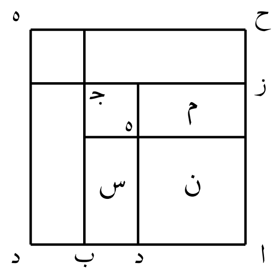
مثال ذلك مال وأحد وعشرون من العدد تعدل عشرة اجذار. ومعناه اى مال اذا زدت عليه احدًا وعشرين من العدد كان المبلغ مثل عشرة اجذاره. فننصّف عدّة الاجذار ونضرب نصف عدّتها فى مثله وهو خمسة فيكون خمسة وعشرين وننقّص من ذلك العددَ وهو احد وعشرون فيبقى اربعة ونأخذ جذرها وهو اثنان فننقّصه من نصف عدّة الاجذار وهو خمسة فيبقى ثلثة وهو جذر المال والمال الذى يكون منه تسعة او نزيده عليه فيكون سبعة فهو جذر المال والمال الذى يكون منه تسعة واربعون. فمتى زدنا على اىّ هذين المالين احدًا وعشرين من العدد صار المبلغ مثل عشرة اجذاره.

فامّا ان يكون فيه مربّع نصف ⟨عدّة⟩ الاجذار مساويًا للعدد الذى مع المال فمثل قولنا مال وخمسة وعشرون من العدد تعدل عشرة اجذار. فاذا ضربنا نصف عدّة الاجذار فى مثله كان خمسة وعشرين ⟨وهو⟩ مثل العدد سواء. فقلنا ان جذر المال هو مثل نصف عدّة الاجذار وهو خمسة والمال الذى يكون منه خمسة وعشرون. فاذا زدنا على هذا المال خمسة وعشرين من العدد بلغ خمسين وهو مثل عشرة اجذاره. وذلك قياس ما يؤتى فى هذا (21$^{\mathrm{r}}$) النوع ان شاء اللّه تعالى.

صورة هذا العمل. لـمّا كان فضل مربّع نصف عدّة الاجذار على العدد المعلوم عددًا مجذورًا علمنا ان صورة العدد علم على مربّع نصف عدّة الاجذار. لكن العدد

(723) مثله : مثلها | (735) وهو : وهى | (741) ان : ما | (743) مثله : مثلها | (744) وعشرون : وعشرن (*corr. supra*) | (746) يؤتى : ياتى | (748) على : فى.

المعلوم مع المال المجهول مساوٍ لعشرة اجذار فنصف العلم ونصف المال مساويان لخمسة اجذار فهو اذن سطح يحيط به خطّان احدهما جذر المال المجهول والثانى خطّ مساوٍ لنصف عدّة الاجذار.



رأينا ان نضع خطّ اب مساويًا لنصف عدّة الاجذار ونعمل عليه مربّع اج ونجعل ضلع المال المجهول اد امّا فى النقصان فاصغر من اب وامّا فى الزيادة فاعظم من اب. ونعمل على اد فى كلى الوضعين مربّع اه ونخرج خطوط الشكل. فلان خطّ اب خمسة وخطّ اد فى كلى الوضعين جذر ⟨المال المجهول⟩ فعلم منس مع مربّع اه فى كلى الوضعين مساوٍ لضرب اب فى اد مرّتين اعنى عشرة اجذار. فامّا فى النقصان فان ذلك فيه بيّن.

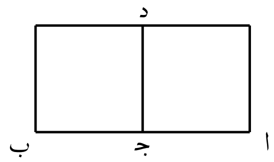
وامّا فى الزيادة فان علم منس مع مربّع اه الاعظم مساوٍ لسطحى حب زد وذلك لان علم منس مع مربّع اه الاعظم مثل علم منس مرّتين ومربّع هج مرّتين وسطح جح مرّتين وذلك جميعًا مثل سطح حب مرّتين اعنى سطحى حب زد المتساويين وكلّ واحد منهما خمسة اجذار. فاذن العدد المعلوم مع مربّع اه المجهول مثل عشرة اجذار.

فاذن منس مثل العدد المعلوم. فلذلك نسقطه من مربّع اج فيبقى مربّع هج معلومًا فنأخذ جذره وهو دب وننقّصه من اب او نزيده عليه فيكون الباقى (**21v**) او المبلغ اد وهو جذر المال المطلوب.

وامّا صورته فى المساواة وهو انّه لمّا كان العدد مساويًا لمربّع نصف عدّة الاجذار علمنا ان صورة العدد ⟨مربّع⟩ ولمّا كان ايضًا مجموع العدد والمال يعدل اجذارًا

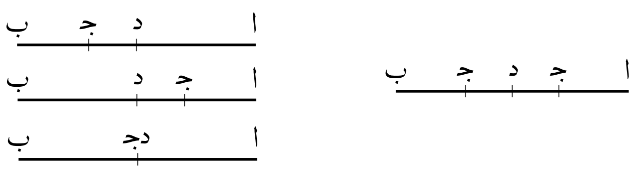
(749) مساويان : مساو | (758) مساوٍ : مساويان | (761) فاذن : لكن | (766) وهو : فهو.

معلومة علمنا ان صورة ذلك المجموع سطح يحيط به خطان احدهما جذر [لا محاله] ⟨المال⟩ والآخر عدد مساوٍ لعدّة الاجذار. لكن ذلك السطح ينقسم بنصفين مربّعين احدهما المال والآخر العدد ومجموع ضلعيهما مساوٍ لعدّة الاجذار. فلذلك يكون نصف عدّة الاجذار هو جذر المال المجهول.



رأينا ان نضع خطّ اب مساويًا لعدّة الاجذار المعلومة وننصّفه على جـ ونعمل على كلّ واحد من اجـ جب مربّعى اد بد فيكون احد خطّى اجـ جب جذر المال المجهول وهو مساوٍ لنصف عدّة الاجذار.

عمل هذه المسئلة بالهندسة والبرهان عليه وعلى علّة تنصيف الاجذار فيه بالخطوط. اذا اردنا معرفة ضلع المال المجهول وضعنا خطّ اب ⟨مساويًا لعدّة الاجذار ونضع عليه نقطًا⟩ فى ثلثة مواضع للزيادة والنقصان والمساواة [مساويًا لعدّة الاجذار] واضفنا اليه سطحًا قائم الزوايا مساويًا للعدد المعلوم ينقص عن تمامه مربّعًا كما يُبيّن ذلك من الشكل الثامن والعشرين من ⟨المقالة السادسة من⟩ كتاب الاصول وليكن السطح المضاف سطح اجـ فى جب وضلع المربّع الناقص جب. فاقول ان جب ضلع المال المجهول.



برهانه انّا ننصّف خطّ اب على نقطة د فيقع نقطة د فى احد الشكلين الذى هو للنقصان فيما بين نقطتى ا جـ على خطّ اجـ وفى الشكل الآخر الذى هو للزيادة فيما بين نقطتى جـ ب (22r) على خطّ جب وفى الشكل الثالث الذى هو للمساواة على نقطة جـ بعينها. فلان خطّ اب قُسم بنصفين على د وبقسمين مختلفين على جـ فضرب اجـ فى جب ⟨مع مربّع دجـ مثل مربّع دب. لكن ضرب اجـ فى جب⟩ معلوم لانّه

(770) المال : مال | العدد : عدد | (782) فى *postea insertum* | (783) الذى *add. supra*.

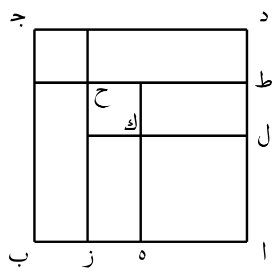
مساوٍ للعدد المعلوم ومربّع بد معلوم لان بد مثل نصف عدّة الاجذار. فلذلك ننقّص العدد المعلوم وهو سطح اجـ فى جب من مربّع نصف عدّة الاجذار اعنى مربّع بد ونأخذ جذر ما يبقى اعنى دجـ فننقّصه من نصف عدّة الاجذار او نزيده عليه فيكون الباقى او المبلغ ضلع المال المجهول الذى هو بجـ.

[**وفى الاستحالة** نضع خطّ اب مساويًا لعدّة الاجذار ونجعل بجـ منه ضلع المال المجهول فيكون ضرب اب فى بجـ مساويًا لعدّة الاجذار. لكن عدّة الاجذار مثل العدد المعلوم والمال المجهول فان سطح اب فى بجـ مساوٍ لضرب اجـ فى جب مع مربّع جب. وقد جعلنا جب ضلع المال المجهول فسطح اجـ في جب مساوٍ للعدد المعلوم. [فاذا نصّفنا خطّ اب وقع التنصيف امّا على قسم اجـ او على قسم جب من خطّ اب. فيكون فى كلى الأمرين ضرب اجـ فى جب مع مربّع جد مساويًا لمربّع دب.] لكن ضرب اجـ فى جب الذى هو العدد وضع اعظم من مربّع دب. وذلك غير ممكن.]

معرفة ضلع المال فى المقترن الثالث

وامّا العمل فى معرفة ضلع المال فى المقترن الثالث الذى هو جذور وعدد تعدل اموالًا بعد ردّ الاموال الى واحدها ان كانت اقلّ او اكثر فهو ان يُضرب نصف عدّة الاجذار فى مثله ويزاد ذلك على العدد ويؤخذ جذر المبلغ (22v) ويزاد على عدّة نصف الاجذار فما بلغ فهو جذر المال المجهول.

مثاله. ثلثة اجذار واربعة من العدد تعدل مالًا. فاذا اردنا معرفة ضلع المال ضربنا نصف عدّة الاجذار وهو واحد ونصف فى مثله فيكون اثنين ورُبعًا فزدنا ذلك على العدد الذى هو اربعة فيبلغ ستّة ورُبعًا ونأخذ جذره وهو اثنان ونصف فنزيده على عدّة نصف الاجذار وهى واحد ونصف فيبلغ اربعة وهو جذر المال المجهول.



(796) مساويًا *add. supra* يا | (805) اثنين ورُبعًا : اثنان و ربع | فزدنا : زدنا | (807) وهى : وهو.

صورة هذا العمل. لمّا كان مجموع العدد ومربّع نصف عدّة الاجذار مربّعًا علمنا ان صورة العدد علم على مربّع نصف عدّة الاجذار. ولمّا كان ضلع هذا المربّع المجتمع مزيدًا على نصف عدّة الاجذار هو جذر المال المجهول.

رأينا ان نضع صورة المال المجهول مربّعًا [متساوى الاضلاع والزوايا] عليه ابجد وليكن به من ضلع اب مساويًا لعدّة الاجذار. وننصّف هب على نقطة ز ونعمل على از مربّعًا عليه ازحط ونخرج خطّى زح حط على استقامتهما الى ضلعى دج بج ونعمل على اه مربّع اهكل ونخرج خطّى هك لك مستقيمين الى ضلعى طح زح.

فلان دج جذر ⟨المال⟩ ودط واحد ونصف لانّه مساوٍ لنصف عدّة الاجذار فسطح طج جذر ونصف جذر. وكذلك ايضًا نبيّن ان سطح جزً جذر ونصف جذر. فسطوح دح مرّة وجح مرّتين وحب مرّة مساوية لثلثة اجذار. لكن سطح لح مثل سطح حج. فيبقى سطوح طك كا كز اعنى العلم مثل العدد المعلوم. فلذلك نزيد العدد (23r) على مربّع نصف عدّة الاجذار ليحصل عندنا مربّع اح فنأخذ جذره وهو از فنزيده على نصف عدّة الاجذار وهو زب فيكون المبلغ اب وهو جذر المال المجهول. وذلك ما اردنا ان نبيّن.

عمل هذه المسئلة بالهندسة والبرهان عليه وعلى علّة تنصيف الاجذار فيه بالخطوط. اذا اردنا معرفة ضلع المال المجهول فى هذه المسئلة وضعنا خطّ اب مساويًا لعدّة الاجذار المعلومة واضفنا اليه سطحًا قائم الزوايا مساويًا للعدد يزيد على تمامه مربّعًا. وليكن السطح المضاف سطح اج فى جب وضلع المربّع الزائد بج. فاقول ان اج ضلع المال المجهول.

برهانه انّا نقسم خطّ اب المعلوم بنصفين على نقطة د. فخطّ اب قُسم بنصفين على نقطة د وزيد فيه زيادة وهى بج. فضرب اج فى جب مع مربّع دب مثل مربّع دج. لكن ضرب اج فى جب معلوم لانّه مثل العدد المعلوم ومربّع دب معلوم لان دب مثل ⟨نصف⟩ عدّة الاجذار. فيكون مربّع دج معلومًا. فلذلك يزاد مربّع نصف

(808) نصف *add. supra* | (809) ضلع *add. supra* | (816) جذر (3^{um}, 4^{um}) : جذرٍ | (824) يزيد : نزيد.

الاجذار اعنى مربّع دب على العدد المعلوم وهو سطح اج فى جب ونأخذ جذر المبلغ وهو جذر مربّع دج اعنى دج. فنزيد عليه عدّة نصف الاجذار اعنى اد فيكون المبلغ اج وهو جذر المال. وذلك ما اردنا ان نبيّن.

وقد تبيّن ممّا قدّمنا ان التدبير الذى خرجت به اضلاع الاموال المجهولة فى كلّ واحد من هذه المقترنات الثلثة هو التدبير الذى اورده اوقليدس فى اواخر المقالة السادسة من كتابه فى الاصول وهو اضافة سطح (23ᵛ) متوازى الاضلاع الى خطّ معلوم يزيد على تمامه او ينقص عنه مربّعًا. وذلك ان ضلع المربّع الزائد هو ضلع المال المجهول فى المقترن الاوّل وفى المقترن الثانى هو ضلع المربّع الناقص وفى المقترن الثالث هو مجموع الخطّ المضاف اليه السطح وضلع المربّع الزائد. وذلك ما اردنا بيانه.

فامّا ما يقع من الاقترانات المعادلة بين ثلثة اصول غير متناسبة ثمّ ما زاد عليها متناسبة كانت او غير متناسبة

مثل الذى يمكن ان يقع فى الحيّزين الثلاثيين اللذين احدهما مكعّبات واموال وعدد والثانى مكعّبات وجذور وعدد من المقترنات الستّة او فى الحيّز الواحد الرباعى الذى هو مكعّبات واموال وجذور وعدد من المقترنات السبعة او فى غيرها ممّا يشتمل على ما فوق هذه المنازل. فلا يكاد يطّرد ذلك بما قدّمنا من القياسات العددية الّا من جهة التقدير المسّاحية بتقديم القطوع المخروطية.

امّا الحيّزان الثلاثيان اللذان ذكرناهما فلخروج ما يشتمل عليه كلّ واحد منهما من الانواع الثلثة عن حال النسبة المتوالية وذلك ان نسبة المكعّب الى المال ليست كنسبة المال الى العدد لان بين المال والعدد منزلة واحدة وهى منزلة الجذر ولا نسبة المكعّب الى الجذر كنسبة الجذر الى ⟨العدد⟩ [المال] لان بين المكعّب والجذر منزلة واحدة وهى منزلة المال فيمتنع كلّ واحدة من قرائنها الستّ عن قول ما قدّمنا من القياسات العددية وذلك لان المجهول الذى نحتاج الى استخراجه ومعرفته فى كلّ واحد من هذه المقترنات هو ضلع المكعّب المذكور فيها ويؤدّى تحليله الى اضافة مجسّم (24ʳ) متوازى السطوح معلوم الى خطّ معلوم يزيد على تمامه او ينقص عنه مكعّبًا ولا يتركّب ذلك الّا باستعمال القطوع المخروطية.

(835) فى *add. supra* | (845) يشتمل : يستعمل.

وامّا الحيّز الرباعى فلزيادة ما يشتمل عليه من الانواع على الثلثة وان كان يتّفق فيها حال النسبة المتوالية فامتنعت قرائنه السبع عن لزوم القياسات المطّردة لان المجهول الذى نحتاج الى معرفته هو ضلع المكعّب المذكور ولا يكاد يُستخرج بما قدّمنا من القياسات العددية الّا بما ذكرنا من القطوع المخروطية.

فهذه هى اصول الجبر والمقابلة ووجوه المعادلات المفردة والمقترنة التى تُبنى عليها انواع المسائل العددية اللازمة للقياسات الصحيحة المطّردة. بيّنّاها باوضح البيان واصحّ البرهان ووفّينا حقوقها من التقسيم والترتيب والتهذيب والتقريب. فامّا المسائل المؤدّية اليها فلم تصرف عنايتنا الى ايراد شىء منها اذ كان خروجًا ممّا قصدناه ونواناه ولانّها من اجناس الفروع اللاحقة بما ذكرنا من الاصول. فلنختم الكلام بالحمد للّه ربّ العالمين والصلوة على محمّد وآله الطيّبين.

عمل فى سنة شصه للهجرة
فرغ من تحريره يوم الجمعة الثانى عشر
من ربيع الآخر سنة احدى وثمانين وخمس مائة
غفر اللّه لكاتبه ولمن نظر فيه
وحسبنا اللّه وحده وتوفيقه

---

(863) ونواناه : ونويناه.

IV Glossary

ا

اتى (I) : 692, 746.

اخذ (I) : 53, 56, 58, 61, 62, 93, [100], 221, 247, 253, … .

آخر: 39, 62 (2), 64, 168, 169, 177 (2), 206, 246, 329, 330, 445, … .

ادى (II) : 853.

مؤدّىً: 862.

اسطقس: 648.

اصل: 25, 26, [152], 255, 562, 646 (2), 647, 840, 860, 864; (كتاب الاصول) 712, 779, 836.

امر: [796].

اوقليدس: 835.

اوّل: 6 (2), 9, [10], 11, [12], 18, 20, 28, 34 (2), 36, 186, …; (اوّلًا) 223, [234], 410, 420, 545.

ب

برهان: 239, 300, 325, 358, 380 (2), 403, 709, 714, 775, 782, 822, 827, 862.

بسط (I) : 606, 617, 632, 634, 671.

ابعد: 70, 71, 107, 565, 596.

بغى (VII) : 440, 728.

بقى (I) : 70, 77, 81, 126, 127, 356, 360, 362, (363), 368, 378, 393 (2), 520, … .

باقٍ: 30, 147, 148, 149 (2), 151, 154, 156, 157, 159, … .

بلغ (I) : 228, 252, 275, 311, 312, 316, 319 (cf. 322), 322, 331, 332 (2), 339, …; (بالغة ما بلغت) 23, 172.

مبلغ: 30, 33, [35], 41, 50, 51, 55, 57, 59, 61, 90, 94, [96], [97], … .

بنى (I) : 860.

باب: 24, 48, 65, 86, 101, 119, …; 483.

بان (II): 56, 200, 249, 302, 345, 348, 363, 394, 406, 460, 613, 650, 708, … .

بان (V): 711, 834.

بان (X): 250.

بيّن: 757.

بيان: 306, 839, 861.

ت

ترك (I): 46.

تمّ (I): 305, 346.

تمّ (II): 241, 301, 405.

تمام: 711, 778, 824, 837, 854.

متمّم: 245, 246, [298], 301, 361, 695 (2), 696.

ث

ثلث (II): 280.

(*pro* ثمنية) ثمانية: 379, 480, 487.

ثنى (X): 192.

استثناء: 150, [152 (2)].

مستثنىً: 124, 125, 148, [150], 152, [153], 182.

ج

جبر: 566, 653; (جبر ومقابلة) [1], 645, 726-7, 860.

جذر (pl. اجذار, جذور): 7, [7], [8], 9, 12, [14], [15], [17], 19, [20], 21, [22], 32, 33, [36], 38, 42, …; (جذر جذر) 255, 257, 263, 264, 266, 269, 444, 445 (2), …; (جذر كعب) 485, 486.

مجذور: 247, 288, 308, 352, 398, 403, 428, 450, 488, 497, 748.

جزأ (II): 274.

جزء : 11, 12 (2), 13, 73, [73], [74 (4)], 83, 87, 88, 89 (3), 91, 92 (2), 93 (3), ...
تجزئة : 27, 29, 30, 272, 273 (2), 283, 421, [636].
مجسّم: 328, 329, 333 (2), 336 (2), 342, 344, 345, 853.
جعل (I) : 220, 223, 224, 227, 231, [234], [237], 239, 241, 268, 434, 473, ...
جمع (I) : 51, 54, 59, 63, 129, 131, 133, 134, 135, 137, 138, 140, . . . ; 352 (بين).
جمع (VIII) : [8], 92, 251, 253, 259, 265 (2), 337, 522, 662.
جمع : 27, 28, 30, 120, 286, 287, 310, 380.
جميع : [22], 29, 430, 564, 692, 728, . . . ; (جميًا) 388, 727, 760.
مجموع: 292, 293, 296, 304, 305, 313, 322, 324, 331, 332, 333, ...
مجتمع: 88, 90, 248, 810.
جنبة : 121, 122, 123 (2), 144, 145 (2), 146 (2).
جنس : 31, 50, 67, 69, 70, 90 (2), 104, [105], [113], 121, 123, [124], 145, 146, 203, 204, ...
متجانس : 446.
مجهول: 33, 530, 657, 663, 699, 707, 710, 725, 732, 749, ...
جواب : 91, 456, 464, 468, 478, 492, 495, 498, 592, 730.
جائز : 63, 208.

ح

حدّ : 729.
حذف : 566, 653.
حذاء : 169.
حساب : 657.
حسابي : [1].
حصل (I) : 303, 428, 538, 598, 608, 611, 675, 686, 819.
حطّ (I) : 638, 640, 641.
منحطّة: 563, 649.
حفظ (I) : 312, 313, 318, 321, 366, 373, 377, 411, 634.
حفظ (VIII) : 515.
محفوظ: 368, 516, 636.

حقّ : 862.
حكم : 638.
تحليل : 853.
حاج (VIII) : 122, 145, 169, 171, 475, 565, 569, 570, 586, ...
حاز (I) : 327.
حيّز : 842, 843, 847, 856.
حاط (IV) : 327, 406, 424, 696, 750, 768.
حال : 25, 27, 28, 848, 857.
حيلة : 206, 502, 519.
استحالة : [791].
مستحيل : 731.

خ

خرج (I) : 209 (2), 458, 477, 490, 514, 523, ...
خرج (IV) : (خطوط) 193, 359, 700, 701, 754, 813, 814.
خرج (X) : 858.
خروج : 847, 863.
خارج : *v.* قسم.
مخرج : [96], [98], [112 (2)], 573, 574, 580, 582, 590, 591, 623, 683.
استخراج : 657, 852.
اخصر : 64.
خطّ (I) : 300.
خطّ : 191, 192, 193 (3), 240 (3), 248, 249, 329, 331, ...
خلف (VIII) : 30, 174, 432, 471, 655.
مختلف : 325, 329, 785.

د

تدبير: 834, 835.

درهم: 681.

ذ

ذكر (I): 26, [37], 208, 239, 409, 430, 469, 560, 729, 847, 859, 864.

ذكر: 646.

مذكور: 658, 853, 858.

ذهب (I): 190, 197.

ر

رأى (I): 699, 752, 772, 811.

ربع (II): 431, 470.

مربّع: 192, 239, 241 (2), 242, 243 (2), 244, 245 (2), 247, ...

ترتيب: 862.

رجع (I): 607.

راجع: 563, 649.

ردّ (I): 480, 671, 728.

ردّ: 565, 568, 597, 601, 624, 652, 654, 659, 722, 801.

مردود: 486.

مردّى: 729.

رشد (IV): 655.

ارتفاع: 242, 329, 330 (2), 333, 334 (2), 337, 342, 345.

ركب (V): 854.

مركّب: 539.

متركّب: 40.

ز

زاوية: [241], [699], 711, 778, [811], 824.

زاد (I): 122, 288, 292, 296, [298], 305, 313 (2), 321, 322, ...; 328 (ب); 714, 828 (فى).

زيادة: 122, [153], 173, 334, 336, 345, 572, 577, 579, 715, 727, ...

مزيد: 392, 575, 583, 810.

زائد: 124 (2), 125 (2), 127 (2), 147 (2), 149, 150, 154 (2), ...; 652 (على).

س

مسئلة: [1], 121, 144, 205, 709, 726, 728, 731, 775, 822, 823, 861, 862.

سبيل: 204, 506.

سطح: 194, 195 (3), 196 (2), 197, 200, 242, 243, 244 (3), ...

مسطّح: 391.

سطر: 169 (2).

سقط (IV): 763.

سلك (I): 625.

سمى (II): 9, 33, 40, 245, 637.

سمى (V): 38.

اسم: 38 (2), 40 (2); ذو اسمين: 538, 539, 541-2.

مسمّىً: [73].

سهل: 568.

سوى (III) : 561.

مساواة : 766, 777, 784.

مساوٍ : 255, 256, 257, 332, 336, 384, 694, 695 (2), 697 … .

متساوٍ : [19], 190, 246, 328, [699], 732, 761, [811].

سائر : 562, 563, 637, 648, 649.

ش

شبه (IV) : 505, 541.

شرح : 46.

شغل : 569.

شكل : 754, 779, 782, 783, 784.

شمل (VIII) : 118, 216, 557, 845, 847, 856.

شاء (I) : 220.

شىء : 83 ($n.$ 45), 92, 93, 94, 109, 110, 129 (2), 130, …; (فى الّا شىء) 182, 183, 184, 187, 189.

ص

صحّ (I) : [636].

صحيح : 566, 601, 624, 653, 660, 675, 686, 861.

اصحّ : 861.

صرف (I) : 863.

اصغر : 326, 346, 358, 365, 367, 368, 542, 753.

صناعة : 645 (الجبر والمقابلة).

صورة : 693, 694, 699, 747, 748, 766, 767, … .

صار (I) : [237], 253, [298], 308, 336, 412, 422, … .

صار (II) : 338.

ض

ضرب (I): [7], 32, 51, 53 (2), 58, 89, 90, 93, [95], [111], 168, … .

ضرب: [8], [20], [22], 27, 32, [34 (2)], [35 (2)], 39, …; 519.

ضربة: 170 (3), 171 (3); (ضربات متقاطرة) 170, 171; (ضربات قائمة) 170, 171-2.

مضروب: [96], [97], [97 (فيه)], [98], 169 (2), 181 (factors), 186, 303, … .

ضعف (II): 219, 223 ($n.$ 96), [237], 239, 252, 264, 292, 295, 304, 355 (double), 679.

ضعف: 226, 288, 352; (اضعاف) 262, 268, 269.

تضعيف: 27, 28, 30, 218, 219, 220, 250, 260, 263, 264, … .

ضلع: 263, 264, 267, 283 (2), 300 (2), 306, 339 (2), … .

ضاف (IV): 710, 777, 824.

اضافة: 836, 853.

مضاف: 780, 825, 839.

ط

طبقة: 39 (2), 49, 50, 66, 67, 68 (3), 70, 71 (2), 72 (3), … .

طرد (VIII): 845.

اطّراد: 203, 500, 645.

مطّرد: 857, 861.

طريق: 625.

مطلوب: [100], 106, 108, 222, 244, 248, 254, 260, … .

مطلق: 29, 117, 203, 308, 499, 501, 504, 539, 543 (مطلقًا).

ع

عدّة: (quantity = coefficient) 9, 10 (2), 123 (2), 125, 126 (2), 127 (2) …; (quantity = exponent) 69, 70, 71, 76, 77 (2), 80 (2), 81. See $n.$ 41.

عدد: [7], 9, 10, [19], 23, 29, [73], 219, 221, ...; (من العدد) 192, 212, 687, 697-8, 734, ...; 84 (بالعدد).

عددى: 845, 852, 859, 861.

معدود: 25.

عدل (I): 560, 594 (2), 596, 599, 600, 602, 603 (2), ...

عدل (III): 567, 625, 653.

معادلة: 27, 556, 559, 560, 561, 562, 564 (term of an equation), 593, 603, ...

معادل: 565, 568, 596, 619, 630, 662, 674, 686, 840.

متعادل: 652.

معتدل: [240].

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