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An Anonymous Treatise on Algebra from 395 H/1004–1005 CE: Translation and Commentary

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Abstract

We give an English translation and commentary of a treatise belonging to the tradition of algebra as a tool for geometrical problem solving. The explicit of the only extant witness, MS 5325 in the Astān-e Quds library in Meshed, Iran, states that it was composed in 395 H/1004–1005 CE, but its author is unknown because the first few folios are lacking. The treatise first covers different aspects of algebra related to the setting up of equations, and then gives both numerical and geometrical solutions and proofs for simplified first and second degree equations. Near the end, the author mentions the classification of irreducible cubic equations and that their solutions can only be found by conic sections. Other aspects of this treatise are of interest, among them the author's allusion to board calculations, his free use of geometric magnitudes of dimension greater than three, and his strict observance of the premodern ontology of numbers and algebraic monomials.

I Introduction

Medieval Arabic algebra, like the algebra in Diophantus, Brahmagupta, and the algebra later practiced in medieval Latin and Italian, was a technique of numerical problem solving. Whether the problem being solved originated in arithmetic, geometry, metrology, or even horology, algebra could be used to find the solution as long as the parameters were given numerical measure, and the answer was always found as a number. But some Arabic authors working in geometrical problem solving in the Greek tradition, where the lines, surfaces, and bodies are not intrinsically measured by numbers, saw algebra as a potential tool for constructing solutions. Given a problem in classical geometry, one can assign an arbitrarily chosen numerical length to one line and then set up and simplify an equation whose solution is the length of another, unknown line. The simplified equation can then be reinterpreted as a new geometry problem. If the equation is of first or second degree, the solution can be constructed with straightedge and compass. If the equation is of third degree, its solution can be found as a line extending to the intersection of two conic sections. Although the numerical assignment made at the beginning of the solution is arbitrary, the constructed solution does not depend on it. The most well known geometer to advocate for this approach is ʿUmar al-Khayyām (ca. 439–ca. 520 H/ca. 1048–ca. 1125 CE), who classified and gave geometrical constructions for the solutions to the twenty-five simplified equations of degree three and less in his *Risāla fī l-barāhīn ʿalā masāʾil al-jabr wa-l-muqābala* (*Treatise on the Proofs of Algebra Problems*, henceforth *Algebra*), composed ca. 467/1075.

The text we translate here, MS 5325 in the Astān-e Quds library in Meshed, Iran, belongs to this tradition of algebra as a tool for geometrical problem solving. It was composed in 395 H/1004–1005 CE, about seven decades before al-Khayyām, and its author is anonymous because the beginning of the only extant manuscript is lost. There is only one known, incomplete copy of the text. The treatise first covers different aspects of algebra related to the setting up of equations, and then gives both numerical and geometrical solutions and proofs for simplified first and second degree equations. Near the end of the treatise the author mentions the classification of irreducible cubic equations and that their solutions can only be found by conic sections. Other aspects of this treatise are of interest, too, among them the author's allusion to board calculations, his free use of geometric magnitudes of dimension greater than three, and his strict observance of the premodern ontology of numbers and algebraic monomials.

This article consists of two parts: an introduction with commentary followed by an English translation of the treatise. We begin the introduction with some general remarks about the treatise and a review the relevant aspects of Arabic algebra, with special attention to the terms used for the powers of the unknown. We continue with an overview of the contents of the treatise with occasional commentary, followed by a closer look into certain features of the treatise. We conclude the introduction with a description of the radically different nature of premodern numbers and monomials, which accounts for what seem to be oddities in the ways that the author explains the rules.

We make reference to several books on algebra by other authors in this study, and we list the more important ones here. The earliest complete extant Arabic book on algebra is the *Kitāb al-jabr wa-l-muqābala* (*Book of Algebra*) of Muḥammad ibn Mūsā al-Khwārazmī, written in Baghdad during the reign of al-Maʾmūn (198–218/813–833). Later in the 9th century Abū Kāmil wrote his *Kitāb fī l-jabr wa-l-muqābala* (*Book on Algebra*) in Egypt. Fakhr al-Dīn al-Karajī's guide to algebra is his *al-Fakhrī fī ṣināʿat al-jabr wa-l-muqābala* (*[Book of] al-Fakhrī on the Art of Algebra*), written in Baghdad ca. 402/1011–12. And in the far western part of the Islamicate world, Ibn al-Bannāʾ composed his *Kitāb al-uṣūl wa-l-muqaddimāt fī l-jabr wa-l-muqābala* (*Book on the Fundamentals and Preliminaries in Algebra*) in the late 7th/13th century in Marrakesh. Al-Khwārazmī's book is published in (al-Khwārazmī 2009), Abū Kāmil's in (Abū Kāmil 2012), and the books of al-Karajī and Ibn al-Bannāʾ are both published in (Saidan 1986). We abbreviate the titles of all of these books as *Algebra*. Another important book is the mid-6th/12th century *al-Bāhir fī ʿilm al-ḥisāb* (*The Dazzling [Book] on the Science of Calculation*) by al-Samawʾal ibn Yaḥyā al-Maghribī, which covers more advanced topics than are usually found in an introductory book on algebra. It is published in (Rashed 2021).

II Integrity of the Treatise

At the beginning of the first surviving page of the manuscript, the top margin has a note in Persian in another hand: "the first part of the treatise has been lost" (*awwal īn resālah uftādeh ast*). The loss must have occurred centuries ago, because the flyleaf of the manuscript was added after the loss had happened. The flyleaf contains old seals and barely legible owner's marks, and the following information in Arabic: "Treatise on algebra and arithmetical problems, composed by one of the ancient scholars, among the wise. The text was written in the year five hundred eighty-one. Parts of the beginning of this text have fallen away."[1] The "one" of the phrase "one of the ancient scholars" comes from *baʿḍu*. This word could also be read as "some," implying that the text is by more than one person. But the text is well organized and the style of writing is consistent throughout, so it was certainly written by a single author.

III Review of Arabic Algebra

III.1 Names of the Powers

Algebra stands out from among the several techniques of numerical problem solving that were practiced in Islamicate countries. In other methods, like single false position and double false position, operations were performed only on known numbers, but in algebra an unknown was named in terms of preassigned names of the powers, operations were performed on those names to establish an equation, and this equation was then simplified and solved.

The first degree unknown in Arabic algebra was called either a "root" (*jidhr*) or a "thing" (*shayʾ*), and its square was called a *māl*, literally "sum of money." We leave *māl* untranslated, and when plural we give it the English suffix: "*māl*s." The third degree unknown was called a "cube," which in most books is a *kaʿb*, but some early authors, ours included, called it by the related word *mukaʿʿab*. The fourth power was called a *māl māl*. Our author calls the fifth power "a *māl* cube or a cube *māl*," and still higher powers are written with combinations of the words *māl* and cube.

The terms for the first power, "root" and "thing," are interchangeable in Arabic algebra, but they are typically used in different settings. When listing the names of

[1] I thank Jan Hogendijk for deciphering the handwriting and helping with the translation.

the powers, discussing their multiplicative relationships, and classifying and solving simplified equations, authors usually preferred the term "root." But in naming unknowns and performing the operations on expressions necessary to set up equations in the solutions to problems, "thing" was the more commonly chosen term. Our author respects the different domains. He uses "root" when describing the proportional series one, root, *māl*, cube, etc., and when stating and solving simplified equations, while "thing" is the name he chooses for example calculations.

The words *jidhr* and *māl* are also common in Arabic arithmetic outside algebra, where they take different meanings. There, *jidhr* is the Arabic term for "square root," and our author naturally also uses it for this purpose. Two arithmetical meanings of *māl* are "an amount of money," and more commonly, a generic "quantity." *Māl* takes this latter meaning in a few places in our text.

The word for an arithmetical square, either a square number (like 9 or $\frac{25}{144}$) or the square *of* a number, was usually *murabbaʿ* ("square"), though *majdhūr* ("rooted," i.e., "has a root") was often used instead. Our author uses both words, and in two places he clearly uses the word *māl* to mean the "square" of a number. At [3.6.2, 454][2] he writes: "we divide a *māl* of the dividend by a *māl* of the divisor…," and at [3.6.7, 465]: "we find a *māl* of that root, by which we multiply that part by itself." This use of *māl* for "square" is very rare. The only other instance we know occurs in a proof in the late 13th century *Asās al-qawāʿid fī uṣūl al-Fawāʾid* (*Foundation of the Rules on Elements of Benefits*) of al-Fārisī (al-Fārisī 1994, 121.11).

The Arabic word for a geometrical or arithmetical cube is *mukaʿʿab*. The related word *kaʿb* usually meant "cube root," though some authors used it to mean "cube." Our author uses *mukaʿʿab* for the "cube" of a number and a geometric cube, and *kaʿb* for "cube root." In this book and in many others the term for the fourth power of a number, *māl māl*, was borrowed from algebra. Our text shows it with this meaning at [3.1.9, 257]: "… the multiplication of a *māl māl* of one of them by a *māl māl* of the other." These multiple meanings of *jidhr*, *māl*, *mukaʿʿab*, *māl māl*, as well as other words in this and other books, pose no problem of interpretation because the meanings are clear from the context.[3]

III.2 Stating Equations with *ʿadala*

Different words could be chosen to equate mathematical objects in medieval Arabic, whether numbers, geometric magnitudes, angles, or ratios. Both *sawiya* and *mithla* are very common across all texts for numbers, magnitudes, and angles, though sometimes the implied verb "to be" was also used. The prefix *ka-* typically equated ratios, but it could be used for other kinds of objects as well. The verb *ʿadala* was rarely used for equating outside algebra, and appears in the specific cases in which

[2] For an explanation of the references, see Section IV below.

[3] See (Oaks 2022b) for a study on the words used for the powers in Arabic arithmetic and algebra.

at least one of the equated sides consists of two or more objects, like in the following indeterminate problem from a book on finger reckoning of Ibn al-Hāʾim that equates two numbers to two numbers: "Two quantities (*māls*): a third of one of them and its fourth equal (*yaʿdil*) a fifth of the other and its sixth" (Ibn al-Hāʾim 1988, 333.16). There is no algebra in this book. Our book does not show *ʿadala* in this context.

In algebra the situation was different. The one verb used to equate the two sides of an equation was *ʿadala* in every text from al-Khwārazmī and Ibn Turk in the 9th century to the end of the Arabic algebraic tradition a thousand years later. This technical verb was used even when both sides consisted of a single term. Related words in algebra share the same root, such as the word for "equation" (*muʿādala*) and the word for the "two equated" expressions (*mutaʿādilān*).[4]

To distinguish between ordinary arithmetical and geometric equating on the one hand (with *sawiya*, *mithla*, etc.) and that of algebraic equations (with *ʿadala*) on the other, we translate "equal" for the former and "Equal" for the latter, and we capitalize the "E" for the related algebraic words "Equation" and "Equated" parts. Differentiating the verbs and their related forms is important to make sense of certain passages, such as when our author explains the meanings of equations. The first example is given at [4.1.1, 560]: "As for the single Equation, it is when one of those proportional species that we mentioned Equals (*yaʿdil*) another of those species, that is, (which) equals (*yusāwī*) it." The first "Equals" is conjugated from *ʿadala*, the technical verb for stating algebraic equations. To explain the use of the otherwise unusual verb, he writes that the other species "equals it," with a common word conjugated from *sawiya*.

III.3 Solving Problems by Algebra

The numerical solution to a problem via algebra passes through three basic stages. First, an unknown number is named in terms of the names of the powers and the conditions of the enunciation are applied to set up (ideally) a polynomial equation. In the second stage this equation is simplified to one of the six types of degree 1 or 2 first described by al-Khwārazmī. In the third stage the simplified equation is calculated by the prescribed rule, and the answer is found as a number.[5]

Later, in his *Algebra*, al-Khayyām gave a geometrical construction for each of the twenty-five equation types, so that for any geometry problem that can be reformulated as a simplified polynomial equation of degree 1, 2, or 3, his *Algebra* provides a prepackaged solution. He also gave, when possible, an arithmetical solution for each equation type following the standard rules in other books. (It was not until the work of Italian algebraists in the 16th century that exact numerical solutions were found

[4] See (Oaks 2010) for an account of the words meaning "equal" in Arabic mathematics.

[5] See (Abdeljaouad and Oaks 2021, 245–251) for translations of several short problems that illustrate these stages, as well as solutions by other methods.

for irreducible cubic equations.) Because he considered numbers properly speaking to be the collections of indivisible units of Euclid and Aristotle, his rules for arithmetical solutions to the equations presume that all calculations remain in the realm of whole numbers.

Our anonymous author covers only the six equations of degree 1 and 2, three of them "single" (*mufrad*), with a single term on each side, and three of them "combined" (*maqrūn*), with two terms combined on one side and one term on the other. In modern notation, the single equations are $bx = c$, $ax^2 = c$, and $ax^2 = bx$, and the three combined equations are $ax^2 + bx = c$, $ax^2 + c = bx$, and $bx + c = ax^2$. He gives arithmetical solutions to them all, where numbers include fractions and irrational roots, and he gives geometrical constructions for each of the three combined equations.

III.4 Proofs for the Rules to Solve the Simplified Equations

Proofs for the rules to solve the combined equations are given in the earliest books by al-Khwārazmī and Ibn Turk. These proofs are intuitive in that they directly compare the lengths and areas of lines and rectangles without recourse to ratio and proportion and without any implicit or explicit reference to Euclid. Jens Høyrup has plausibly linked these proofs with the kinds of arguments made by mensuration professionals (Høyrup 1986, 473–475). In the latter part of the 3rd/9th century, both Abū Kāmil and Thābit ibn Qurra wrote proofs that ground the numerical procedures in Propositions II.5 and II.6 of Euclid's *Elements*, and later authors found other ways of proving the rules—some citing the two propositions from the *Elements*, and others justifying the rules arithmetically—with varying ideas of what constitutes a proper proof (Oaks 2018b).

IV The Contents of the Treatise

The anonymous treatise is divided into four "categories" (sing. *nauʿ*), and each category is divided into "chapters" (sing. *bāb*). In the third category the chapters are further divided into sections that are given titles, but these sections are not designated by a term such as *faṣl* ("section"). Perhaps the author had set these titles apart either in red ink or on a separate line, but in our manuscript they run right along into the text that follows. We insert "(Section.)" in the translation to indicate them.

We have added reference numbers to the translation based in these divisions, so that, for example, [1.4.2] refers to the second paragraph in category 1, chapter 4. These numbers are followed by a reference to the line numbers of the Arabic text also found in this issue of the journal; so that, for example, [1.4.2, 80] refers to the material that is found in the paragraph marked [1.4.2] in our translation, and which begins on line 80 of the critical edition of the Arabic text. In our translation, the division of the chapters into paragraphs conforms to Jan Hogendijk's independent edition of the Arabic text. A notation such as [4a] or [7b] indicates the beginning of folio 4 recto and folio 7 verso in the manuscript. Parentheses () contain our explanatory remarks that are not part of the Arabic text.

IV.1 The First Category

The extant part of the treatise begins in the first chapter of the first category. In this category the names of the powers of the unknown are introduced, beginning with one (*wāḥid*, i.e., the unit), root (*jidhr*), and *māl*, with a focus on their proportionality. The author uses the word *ṭabaqa*, "level," to mean the position of a power in the proportional sequence one, root, *māl*, cube, etc. At [1.4.2, 78], for example, he writes of "the level of the root" and just after that "the level of the *māl māl*." We have not seen the term *ṭabaqa* used this way in other books.

The traditional operations of addition, subtraction, duplication, partition, multiplication, and division are introduced in the second chapter. Duplication and partition are generalizations of the operations of doubling and halving taught in the earliest books on Indian reckoning. Duplication sometimes covers just the doubling, tripling, quadrupling, etc. of an amount, though in some books, ours included, the multiples can be rational. Partitioning covers taking a half, a third, a fourth, etc. of an amount, and in our text this apparently includes common fractions as well. These operations were conceived of differently from multiplication and division and remained distinct in Arabic books on arithmetic and algebra. Our author uses them only on roots of numbers in chapters 3.1 and 3.2, such as to duplicate $\sqrt{8}$ two and a half times at [3.1.5, 234], or to take a third of $\sqrt{36}$ at [3.2.2, 280].

The first four operations are performed just as they are for ordinary numbers, since the species (power) of the terms do not change. For example, adding "five *māl*s" to "three *māl*s" gives "eight *māl*s," just as adding "five units" to "three units" gives "eight units." The rest of the category is devoted to multiplication and division, where the powers can change. The third and fourth chapters cover the rules for multiplication and division of the powers defined so far. Through multiplication the proportional levels extend to include the cube (*mukaʿʿab*), the *māl māl*, and higher powers, and through division one arrives at the reciprocals of the powers: "a part of a root," "a part of a *māl*," and so on. The fifth and sixth chapters cover the rules for multiplying and dividing those parts.

IV.2 The Second Category

Where the goal of the first category is to explain how operations affect the powers without regard to their multitudes (coefficients), the second category covers the addition, subtraction, multiplication, and division of polynomials. For the first three operations all of the example expressions are of first degree and consist of two terms, like “add ten and two things to ten less a thing” [2.1.2, 134] and “subtract ten and five things from fifteen and a thing” [2.2.4, 157]. The two examples of division are each two terms divided by one term.

Many of the rules presented in the first three categories are given proofs in one form or another. Eight of these proofs are based in a geometric diagram, one in this category and the other seven in the third category.[6] Euclid’s *Elements* is never cited in these proofs, but a few times the author makes use of the fact that the complement of two squares is the mean proportional between them. Euclid defines the complement in Proposition I.43 and proves that it is the mean between the squares in the Lemma to Proposition X.54. Other proofs are argued from arithmetical identities, the most involved being the one at [3.1.9, 250], which we break down below in Section V.2

IV.3 The Third Category

The third category leaves the realm of algebra to cover square, cube, and fourth roots of numbers. A number or expression is *muṭlaq* (“absolute”) if it is stated directly, like “three,” and not indirectly, like “a root of three” ($\sqrt{3}$). “A root of three” is not a three, but is a root, since “of three” is a prepositional phrase that modifies the noun (in Arabic the phrase is written in construct state, where “three” is genitive). The author also speaks of an “absolute binomial” [3.6.22, 539] and an “absolute apotome” [3.6.22, 543]. An example of an absolute binomial is “ten and a root of fifteen” [3.6.18, 507] (our $10+\sqrt{15}$), since its root is not taken. The antonym of *muṭlaq* is *mansūb*, meaning “ascribed.” A root is ascribed to a number, so in “a root of three” ($\sqrt{3}$), the root is ascribed to the three. Similarly, “eight and a root of sixty, and we take a root of that” [3.3.3, 296] ($\sqrt{8+\sqrt{60}}$) is a root ascribed to a binomial. We have not seen these terms “absolute” and “ascribed” in other books.

The chapters in this category address the duplication of roots, the partition of roots, then their addition, subtraction, multiplication, and division. Such rules are “useful in algebraic calculation,” as al-Karajī notes in the title of his chapter on the same topic (Saidan 1986, 121.14). An example of the first operation is to duplicate a root of eight two and a half times, whose meaning is “to make a root of eight two and a half times as much” [3.1.5, 231]. This operation was important in premodern arithmetic, where multiples and fractions of a root were properly converted to a single root (see Section VI.1 below). For the example just cited, the $2\frac{1}{2}$ is squared and then multiplied by the 8, then the square root of the product is taken to get “a root of fifty.” To partition a root is to take a fraction of it, and its rule is the same as the rule for duplication.

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[6] These are the proofs beginning at [2.3.5, 191; 3.1.7, 239; 3.3.4, 300; 3.3.8, 325; 3.4.4, 358; 3.4.8, 380; 3.5.3, 403; 3.5.8, 424]. The proof at [3.4.8] relies on the diagram for the proof at [3.3.8], and the very brief proof at [3.5.8] relies on the diagram for the proof at [3.5.3].

The rule to add square roots is given in many Arabic algebra books, the earliest extant example being Abū Kāmil's *Algebra.* In modern notation, the result of adding $\sqrt{a}$ and $\sqrt{b}$ is $\sqrt{a+b+2\sqrt{ab}}$. If $\sqrt{a}$ and $\sqrt{b}$ are commensurable, then the sum is a root of a single number, like in al-Karajī's example "add a root of eight and a root of eighteen" from his *Algebra*, which gives $\sqrt{50}$ (Saidan 1986, 125.22). Our author gives the rational example of adding $\sqrt{4}$ to $\sqrt{9}$, but also the irrational example of adding $\sqrt{3}$ to $\sqrt{5}$, which gives the $\sqrt{8+\sqrt{60}}$ mentioned above [3.3.3, 294].

The rule to add cube roots is a little more complicated. In modern notation, adding $\sqrt[3]{a}$ to $\sqrt[3]{b}$ gives $\sqrt[3]{a+b+\sqrt[3]{27a^2b}+\sqrt[3]{27ab^2}}$. The rule is also found in a few other books, including al-Karajī's *Algebra* (Saidan 1986, 127.2) and al-Baghdādī's (d. 429/1038) *Takmila fī l-ḥisāb* (*Completion of arithmetic*) (al-Baghdādī 1985, 215.16), but our author gives a proof that other books do not show. He did not include a rule to add fourth roots, which can be done by two applications of the rule for adding square roots, as Ibn al-Bannā' illustrates in his *Algebra* (Saidan 1986, 530.15).

The multiplication and division of square roots, cube roots, and fourth roots are much simpler, but the rules to multiply and divide square roots with cube roots are less trivial. In the chapter on division the author explains division by binomials and apotomes. One of these calculations is put in context in Section V.4 below.

IV.4 The Fourth Category

The fourth and final category of the treatise covers the solutions to simplified equations, which are the same six single and combined equations of degree 1 and 2 first introduced by al-Khwārazmī. The single equations are stated as "roots Equal a number," "*māl*s Equal a number," and "*māl*s Equal roots," which in modern notation can be written as $bx = c$, $ax^2 = c$, and $ax^2 = bx$. The author first explains how to make the multitude of the highest power equal to 1. The operations the author follows are common in Arabic books, but different from how we do it today. In the example "three *māl*s and a third equal thirty" ($3\frac{1}{3}x^2 = 30$) [4.1.9, 605], we would either divide both sides by $3\frac{1}{3}$, or equivalently, multiply both sides by $\frac{3}{10}$. Instead, the author first converts the $3\frac{1}{3}$ to "ten thirds," and then subtracts from each term "seven of its tenths." Subtracting $\frac{7}{10}$ of $3\frac{1}{3}$ *māl*s from itself gives the required one *māl*, and subtracting $\frac{7}{10}$ of 30 from 30 gives 9, so the equation becomes "a *māl* Equals nine." When the multitude of the highest power is less than one, the author adds to each term a multiple of itself. For the way that these calculations were written down, see Section V.1 below. The author also notes that higher degree simple equations can be reduced in degree if neither term is of the level of a number [4.1.13, 637].

The combined equations are "*māl*s and roots Equal a number," "*māl*s and a number Equal roots," and "roots and a number Equal *māl*s," which transformed into modern notation are $ax^2 + bx = c$, $ax^2 + c = bx$, and $bx + c = ax^2$ [4.2.1, 645].

Again, equations can be reduced in degree if the lowest power is not already the level of a number [4.2.1, 648], and one should set the multitude of the highest power to 1 by the same method given for the single equations [4.2.2, 651].

For each of the combined equations, the author gives two different rules for the solution: first, a numerical recipe if one wanxts a number for the answer [4.2.3, 656; 4.2.13, 721; 4.2.26, 800], and then a geometric construction if one wishes to solve it geometrically [4.2.9, 709; 4.2.21, 775; 4.2.30, 822]. The arithmetical rules are the same that we find in other books, and each rule is given a proof based in a geometric diagram, but without reference to Euclid's *Elements.* These proofs are in the intuitive style of those in al-Khwārazmī, and in fact the proof for the rule for the first type, at [4.2.8, 699], is essentially the same as the second proof in al-Khwārazmī's *Algebra* (al-Khwārazmī 2009, 111.8). The rules for constructing geometric solutions, on the other hand, rely on propositions in Euclid's *Elements*, Proposition VI.29 for the first and third types and Proposition VI.28 for the second type. These propositions "treat the application of a surface of parallel sides to a known line that exceeds its completion or falls short of it by a square" [4.2.33, 836]. The constructions are then proven to solve their respective equations in Euclidean style with single line diagrams, and which rely on *Elements* Proposition II.6 (for the first and third types) and II.5 (for the second type). The arithmetical rules are presented in a way accessible to practitioners, while the constructions appear to have been composed for geometers working in the Greek tradition.

We know of one earlier text that gives geometric constructions for the solutions to equations. Nuʿaim ibn Muḥammad ibn Mūsā, who was probably a son of one of the Banū Mūsā, authored the treatise *Masāʾil handasiyya* (*Geometrical Problems*) probably in the late 3rd/9th century.[7] The fourth of his forty-two problems gives constructions for the solutions to the first two combined equations. For the equation type "a *māl* and a multitude of roots equal a known number," he constructs the solution by appealing to Euclid's *Elements* Proposition VI.29, but without citing it explicitly. He then proves that the construction works via *Elements* Proposition II.6. After that he gives another construction corresponding to the argument of the second proof in al-Khwārazmī, but proceeding synthetically.[8]

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[7] The text is edited with English translation by Hogendijk (2003), and with French translation by Rashed and Houzel (2004).

[8] Curiously, he introduces the proof with "There is another proof of it," and the second construction is prefaced with "There is a third proof of it." The other oddity is that he begins the treatment of the problem with a diagram based in an analysis, and which is then made irrelevant with the application of Euclid's Proposition VI.29. Both of these features are also present in the treatment of the second equation type.

For the equation type "a *māl* and a known number equal a multitude of roots[9] of the *māl*," Nuʿaim gives the same three part treatment he gave for the first type, with a construction, a proof, and then another construction. The first construction is based in *Elements* VI.28, the proof is based in *Elements* II.5. The second construction is difficult to analyze because the diagrams are corrupt, but it appears to be intuitive like the second construction for the first type. Nuʿaim does not cover the third combined type.

For his part, al-Khayyām constructed the solution to the first and second combined types by *Elements* VI.29 and VI.28 respectively, but he proved the validity of the constructions by citing Euclid's *Data*, which would be Propositions 59 and 58 respectively. For the third combined equation he simply gave a construction that follows the numerical rule for calculating the thing, and he did not give a proof.

Our anonymous author understood that the key to applying *Elements* VI.29 to the third combined equation is to make the unknown thing the entire line AG, and not a side of the appended square as it was for the first combined equation. Perhaps Nuʿaim and al-Khayyām were expecting the *māl* to be a square in the diagram, so they could not make it work. Nuʿaim then omitted the third type and al-Khayyām simply gave a construction based in the arithmetical solution.

IV.5 Concluding Remarks in the Fourth Category: Cubic Equations and Worked-Out Problems

In his *Quadrant* al-Khayyām recounted the work of earlier mathematicians in solving cubic equations (Rashed and Vahabzadeh 1999, 255.4–257.7). Proposition 4 of Book II of Archimedes' *On the Sphere and the Cylinder* calls for cutting a sphere by a plane in such a way that the two parts of the sphere are in a given ratio. At one point Archimedes constructed a point on a line, but without explaining how he derived it. Al-Māhānī attempted to determine the point via algebra, setting up an equation with "numbers, *māl*s, and cubes," but could not solve it. Given the problem as stated by al-Khayyām, the equation must have been of the form "a cube Equals *māl*s and numbers."[10] Abū Jaʿfar al-Khāzin (d. ca. 360/971) then solved it by conic sections and wrote a treatise on it.

Later, Abū Naṣr ibn ʿIrāq (d. before 427/1036) encountered another supposition of Archimedes while finding the side of a regular heptagon inscribed in a circle, from which he set up an equation of the form "a cube and *māl*s Equal numbers," which he also solved by conic sections.

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[9] We write "a multitude of roots of the *māl*" for what is evidently a corrupt passage. A different restoration is given by Rashed and Houzel (2004, 75.9).

[10] There are four choices of which line to make "a thing." Three of them yield four term equations, and the remaining one is of three terms, with the cube on one side.

In another direction, several prominent mathematicians, including Abū Sahl al-Qūhī, Abū l-Wafā', and Abū Ḥāmid al-Ṣāghānī, attempted to solve the arithmetic problem of dividing ten "into two parts such that the sum of their squares together with the result of dividing the greater by the smaller is seventy-two in number." The problem leads to an equation of the form "*māl*s Equal a cube and roots and numbers" (Rashed and Vahabzadeh 1999, 257.3). In fact, by naming the smaller part "a thing," the equation simplifies to what we would write as $10x^2 = x^3+13\frac{1}{2}x+5$. Abū l-Jūd ibn al-Layth finally solved it, again by a construction involving conic sections. Later, in a postscript to his *Algebra*, al-Khayyām related that he found out five years after completing his book that Abū l-Jūd had solved most of the irreducible cubic equations by conic sections, but that this treatment was not exhaustive.

Al-Khayyām made no claim that he was the first to classify the cubic equations, and we now know that they had been classified before. The author of our text writes of the irreducible cubic equations that cannot be solved "by the arithmetical rules we gave above," giving the numbers of equations in each category. These are "the two domains of triples, of which one of them is cubes, *māl*s, and number, and the other is cubes, roots, and number, consisting of six combined (Equations),[11] or in the single domain of quadruples, which are cubes, *māl*s, roots, and number, consisting of seven combined (Equations)"[12] [4.2.34, 842], which "can only be constructed by using conic sections." The reducible cubic equations are not included in this categorization, just as in al-Khayyām.

Another author who listed the cubic equations is ʿAlī al-Sulamī, though we are not certain whether he wrote before or after al-Khayyām. The one manuscript of his *al-Muqaddima al-kāfiyya fī ḥisāb al-jabr wa-l-muqābala wa mā yuʿrafu bihi qiyāsuhū min al-amthila* (*Sufficient Introduction on Calculation by Algebra and What One Can Learn from its Examples*) is dated to the 10th century (Rosenfeld and Ihsanoğlu 2003, #267, p. 99). This date comes from Paul Sbath's 1928 catalogue of the manuscript collection that he later donated to the Vatican Library (Sbath 1928 I, 9–10). All we know about al-Sulamī comes from this one manuscript, which was copied in 708H/1211. Although there is nothing in the manuscript indicating that it was written later than the 10th century—the author seems to not have known al-Karajī's works, for example—all we can definitively say is that he wrote before this particular manuscript was copied.

The "examples" in the title of al-Sulamī's book refer to the sixty-eight worked out problems of the third chapter, arranged into six sections, one for each of al-

[11] In modern notation these equations are $ax^3 + bx^2 = d$; $ax^3 + d = bx^2$; $bx^2 + d = ax^3$; $ax^3 + cx = d$; $ax^3 + d = cx$; and $cx + d = ax^3$.

[12] In modern notation these equations are $ax^3 = bx^2 + cx + d$; $bx^2 = ax^3 + cx + d$; $cx = ax^3 + bx^2 + d$; $d = ax^3 + bx^2 + cx$; $ax^3 + bx^2 = cx + d$; $ax^3 + cx = bx^2 + d$; and $ax^3 + d = bx^2 + cx$.

Khwārazmī's six equation types.[13] In the fourth chapter the author treats cubic and quartic equations. In the middle of his discussion he classifies the cubic equations, first the two term cubic equations for which he gives solutions, then the three term cubic equations, followed by the four term cubic equations, none of which he solves (Vat. Sbath 5, ff. 87a–88b).[14] Al-Khayyām's categorization differs from al-Sulamī's only by placing the reducible cases in a separate category. There is no mention of conic sections in al-Sulamī's book.

Before al-Khayyām, then, the cubic equations had already been classified according to the number of terms and the solutions to many of them had been found by conic sections. His contribution was to write a comprehensive guide for the geometer that incudes solutions to every equation type, and his introduction validates the numbers of the algebraists, which include fractions and irrationals, by grounding them in the domain of continuous magnitudes (Oaks 2011).

Last, our author explains that he does not give worked out problems. Such problems are of the kind that introductions to algebra often present after explaining the rules, like we find in the books of al-Khwārazmī, Abū Kāmil, al-Karajī, Ibn al-Bannā', and others. Equations are not problems in this sense. Equations are set up as part of the process of solving a problem. One example should suffice to clarify this. The enunciation of a simple problem from Abū Kāmil reads, "Ten: you divide it into two parts. You multiply each part by itself and you cast away the smaller from the greater, leaving eighty" (Abū Kāmil 2012, 335.11). We can translate this into a system of two equations and two unknowns, such as $x + y = 10$; $x^2 - y^2 = 80$, or a single equation in one unknown as $x^2 - (10 - x)^2 = 80$, or in other ways as well. By doing so, we have taken the steps of naming the unknowns as x and y, and of setting up algebraic equations. But as it is stated, the enunciation is simply a question in arithmetic, without any algebraic vocabulary. Abū Kāmil solves the problem by algebra three different ways with three different namings. In the first solution he names the smaller part "a thing" and sets up and solves the equation "a hundred dirhams less twenty things Equal eighty dirhams" ($100 - 20x = 80$). In the second solution he names the greater part "a thing" and sets up and solves "twenty things less a hundred dirhams Equal eighty dirhams" ($20x - 100 = 80$). And in the third solution he names the parts "five and a thing" and "five less a thing," and the

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[13] The equations set up in each of the problems in the first section simplify to the type $ax = b$, those in the second section simplify to the type $ax^2 = bx$, etc. Thirty-nine of the problems, more than half of the total, are in the first section (Oaks 2015).

[14] In modern notation, these are listed in order as: "equating cubes with one species": $ax^3 = d$, $ax^3 = cx$, $ax^3 = bx^2$; "two species equal a species": $ax^3 + bx^2 = d$, $ax^3 + bx^2 = cx$, $ax^3 + cx = d$, $ax^3 + cx = bx^2$, $ax^3 + d = cx$, $ax^3 + d = bx^2$, $d + cx = ax^3$, $d + bx^2 = ax^3$, $cx + bx^2 = ax^3$; "equating three species to a species": $ax^3 + bx^2 + cx = d$, $ax^3 + d + bx^2 = cx$, $ax^3 + cx + d = bx^2$, $d + cx + bx^2 = ax^3$; "equating two species to two species": $ax^3 + bx^2 = cx + d$, $ax^3 + cx = d + bx^2$, $ax^3 + d = cx + bx^2$.

equation becomes "twenty things Equal eighty dirhams" ($20x = 80$). The equation, which is always stated in terms of the names of the powers, arises in the course of solving of a problem.

The author explains in the last passage of the treatise, at [4.2.37, 860], that he has covered "the principles of algebra and the methods of the single and combined equations," that is, he has given the general rules for solving problems by algebra, including the names of the powers, the rules for operating on algebraic expressions and irrational roots necessary to set up equations, and the solutions to simplified equations. But "for the (worked out) problems leading to them (i.e., leading to the six equations), we do not pay attention to them … since they fall outside what we intended and proposed, and because they are types of practical applications that stem from the (theoretical) elements that we have indicated." The author has grounded the rules of algebra in proofs, and this theoretical foundation is all he intended to provide.

V Remarks on Various Aspects of the Treatise

V.1 Board calculations

In three places the author gives instructions related to performing operations on a dustboard or other surface. Such instructions are usually given in only elementary, practical books. At [1.6.2, 111] the author alludes to the rule for dividing a fraction by a fraction: "we multiply each of the two denominators by the parts of the other denominator multiplied crosswise, then of the two outcomes, we divide the dividend by the divisor." Fractions in Indian (i.e., "Arabic") numerals were written with the numerator over the denominator across the Islamicate world, the only difference being that in the West a horizontal bar was put between them like we do today. With the numerator over the denominator, the instruction to multiply "crosswise" presumes that the fractions are written on the board side-by-side.

Then, at [2.3.1, 168] he states the rule for multiplying two terms by two terms and how to work it if one or both of the amounts has a deleted term. His first example is to multiply "ten and a thing by ten and a thing," and his second is "ten less a thing by ten less a thing." The instruction specifies how to write the quantities: "we put down the multiplicand on a line and the multiplier on another line under it and parallel to it. Then we need four multiplications, two diagonal multiplications and two vertical multiplications." He also explains how to multiply three quantities by three quantities, where "one would need nine multiplications for it, six diagonal multiplications and three vertical multiplications."

Unfortunately, he does not explain how to represent algebraic terms on the dustboard. From the few hints that we have in our manuscripts, in the Islamic East around the time our text was written it seems that for simple expressions only the coefficients were written on the board, such as "1 2 = 99" for "a *māl* and two things Equal ninety-nine," where the Arabic letter *lam* (ل), the last letter in *taʿdil* ("equals") serves for the "=" (Oaks 2012, 64–65). The dustboard was for personal use, and because the user knew the powers for each number, it was not necessary to indicate them. For more complex calculations, including the higher degree polynomials that we find in al-Karajī and al-Samaw'al, it seems that the names of the powers, perhaps abbreviated, were indicated above the multitudes while working through the calculations. This practice, with single letter abbreviations, became standard in the notation specific to algebra that developed in the Maghreb and al-Andalus no later than the late 12th century. The practice of indicating the power as an abbreviation above the multitude is shown in some medieval Italian manuscripts as well.

Finally, at [4.1.3, 569], our author explains how to add to a number a fraction of that number, and how to subtract from a number a fraction of it, which is applied in setting the multitude of the highest term in an equation to 1.

> For the addition, if we want to add to a known number a known part of it, we put down the denominator of that part in two places and we add to one of them its part. What this comes to, we multiply it by the number and we divide the outcome [by] the denominator in the other place. What results from the division is the number with its part added to it.

One example where this rule is applied is the equation "a half and a third of a *māl* and two roots and a third Equal fourteen dirhams and a half" [4.2.6, 681], which we would write as $\frac{5}{6}x^2 + 2x = 14\frac{1}{2}$. This equation has only a fraction of a *māl*, and the goal is to convert it into an equation with a complete *māl*. To do this, he adds to the $\frac{5}{6}x^2$ "the same as its fifth." A "5," the denominator of the "fifth," is put down in two places. To one of them he adds its fifth to get 6. This 6 is then multiplied by each of the "two roots" and the "fourteen and a half dirhams" to get 12 roots and 87 dirhams respectively. These are then divided by the other 5 to get $2\frac{2}{5}$ roots and $16\frac{4}{5}$ dirhams, so the equation becomes: "a *māl* and two roots and four fifths of a root Equal seventeen and two fifths in number" (our $x^2 + 2\frac{4}{5}x = 17\frac{2}{5}$). Two 5s are needed because the first 5 is erased and written over with a 6 on the dustboard, so it becomes lost.

V.2 The Rule for Duplicating Cube Roots

The author gives a geometrical proof for the rule to duplicate a square root at [3.1.7, 239]. The argument is based in proportion and is straightforward, but it cannot be modified for the rule to duplicate a cube root. This is probably why the author chose to demonstrate the rule for the cube root through arithmetical identities instead. The proof, given at [3.1.9, 250], is not transparent and no example is given in the text, so we explain it in the context of an example of our own: to duplicate a cube root of five ten times. The rule states that we cube the 10 to get 1000, then multiply that by 5 to get 5000, and take a cube root of the result, giving $\sqrt[3]{5000}$. Here is the argument, annotated with our comments:

> And here it is clear that for any two numbers in which we multiply one of them by the other, then the product by itself, (the outcome) is equal to the multiplication of a square of one of them by a square of the other.

Let us call this "Identity 1," which in modern notation is $(ab)^2 = a^2b^2$. He does not state the versions for cubes and fourth powers, which would be $(ab)^3 = a^3b^3$ and $(ab)^4 = a^4b^4$.

> Based on this rule, if we want to duplicate a cube root of a number, we multiply the number of times by itself, then the outcome by the number of times again so the outcome becomes a cube, then the product by the number it is ascribed to, and we take a cube root of the outcome to get the desired [amount].

This is the general rule: to duplicate $\sqrt[3]{a}$ n times, one calculates $\sqrt[3]{n^3a}$.

> And the basis for this is that any number is equal to a root of its square and a cube root of its cube, and a root root of its *māl māl*.

We call this "Identity 2": $a = \sqrt{a^2} = \sqrt[3]{a^3} = \sqrt[4]{a^4}$.

> Therefore, for any two numbers, a root of the multiplication of a square of one of them by a square of the other is equal to a cube root of the multiplication of a cube of one of them by a cube of the other, which is also equal to a root root of the multiplication of a *māl māl* of one of them by a *māl māl* of the other.

We call this "Identity 3": $ab = \sqrt{a^2b^2} = \sqrt[3]{a^3b^3} = \sqrt[4]{a^4b^4}$.

> This is based on the reason that we have mentioned, that is, for any two numbers, multiplying one of them by the other, then the product by itself, is the same as multiplying a square of one of them by a square of the other.

Here he restates Identity 1, $(ab)^2 = a^2b^2$, and again omits $(ab)^3 = a^3b^3$ and $(ab)^4 = a^4b^4$. Identity 3 follows from Identities 2 and 1: $ab = \sqrt{(ab)^2} = \sqrt{a^2b^2}$, and similarly for the higher powers.

> Because the desired (amount) in duplicating the cube root is the multiplication of the cube root by the number of times, then if we cube that, its outcome is like multiplying the number it is ascribed to by the duplicated cube. Therefore, we take its cube root to get the desired (amount).

To duplicate $\sqrt[3]{a}$ n times, one multiplies $\sqrt[3]{a}$ by n. By Identity 1, $(\sqrt[3]{a} \cdot n)^3 = a \cdot n^3$. Therefore, by Identity 2, duplicating $\sqrt[3]{a}$ n times is found by working out $\sqrt[3]{a \cdot n^3}$.

V.3 Higher Dimensional Geometric Magnitudes

Despite the fact that geometric magnitudes of dimension greater than three were regarded as being impossible, several authors in Greek antiquity, the Islamicate world, and in medieval and early modern Europe worked with them anyway, including our author. All examples we have found stem from geometric proofs for rules of numerical calculation, whether the numbers arise in geometry or arithmetic. Because this has not been covered much in the secondary literature, we give an overview of it here.[15]

One setting that gives rise to such impossible magnitudes is Hero of Alexandria's rule for calculating the area of a triangle. If we call the sides of a triangle a, b, and c, and half the perimeter of the triangle s, then the rule for calculating its area, $\sqrt{s(s-a)(s-b)(s-c)}$, calls for the multiplication of four lengths. Hero states and proves the rule by geometry using a lettered diagram in Proposition I.8 of his *Metrica* and in *Dioptra* 30. The proof in *Metrica* concludes with: "But the [rectangle contained] by ΓΘ, EH is a side of a [square] on ΓΘ by that on EH; it thus results in: the area of triangle ABΓ multiplied by itself is equal to a [square] on ΓΘ by that on EH."[16] The product of two squares would be four dimensional, but that does not seem to have bothered Hero, since his goal was numerical calculation. In fact, he even sometimes calls his magnitudes "numbers," like in Proposition I.7: "If one has two numbers (*arithmoi*) AB, BΓ…" (Hero 2014, 163.1). Hero works with four dimensional magnitudes in other propositions, too, such as I.7, I.9, in general remarks after I.14, in I.17, and probably elsewhere.

Several later authors also give geometric proofs for Hero's rule. Some show no hesitation to work with four dimensional magnitudes, while others seem to have been uncomfortable with the idea. In *Derivation of Chords in the Circle* (*Istikhrāj*

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[15] The only historian we know to have addressed the issue is Liesbeth Cornelia de Wreede, in Chapter 7 of her PhD dissertation (2007). She reviews the various proofs of Hero's rule with the goal of assessing the solution to the problem of four dimensional magnitudes in Willebrord Snellius (1580–1626).

[16] Translated from the French in (Hero 2014, 165): "mais le [rectangle contenu] par ΓΘ, EH est un côté du [carré] sur ΓΘ par celui sur EH; il en résultera donc que l'aire du triangle ABΓ multipliée par elle-même est égal au [carré] sur ΓΘ par celui sur EH."

al-awtār fī l-dāʾira), al-Bīrūnī (973–1048) attributes Hero's rule to Archimedes, and gives a proof of Abū ʿAbdallāh al-Shannī (10th century). Near the end of the proof he writes: "And when we multiply surface EH by [the sum of] EA, AZ, [which is] one of the extremes, by surface GK by KA, [which is] the other extreme, it gives a square of the mean, that is, the area of the triangle."[17] The argument proceeds entirely in the context of the magnitudes, without reference to numbers. Al-Bīrūnī clearly saw no problem with the multiplication of a surface by a surface.

The Banū Mūsā (9th century) also stated the rule, giving essentially Hero's proof in their *Book on Knowledge of the Measurement of Planar and Spherical Figures* (*Kitāb maʿrifat misāḥat al-ashkāl al-basīṭa wa al-kuriyya*). The original version of this treatise is lost in Arabic, but it is extant in a 12th century Latin translation by Gerard of Cremona and in a 7th/13th century redaction of Naṣīr al-Dīn al-Ṭūsī. The Latin version runs through the proof referencing lines in the diagram until near the end, where the magnitudes would have become four dimensional. At that point the references to lines via the letters of the diagram are omitted, which speaks to a concern over the dimensions of the magnitudes:

> Therefore, the multiplication of the excesses of half of all sides of the triangle over each of the sides of the triangle, one of them by the other, and that which is aggregated by the third, then the multiplication of that which is aggregated by half of all sides of the triangle, is equal to the multiplication of the base of the triangle in itself. And that is what we wanted to show.[18]

Al-Ṭūsī's revision, however, completes the proof in the context of the diagram:

> … the square on ED by the square on AH is equal to the multiplication of BZ by ZG by AD by AH. And since ED by AH is equal to the area of the triangle, the square on ED by the square on AH is a square of the area of the triangle. Thus, the square of the area of the triangle is equal to the multiplication of BZ by ZG by AD by AH, which are the three excesses by half the sum of the sides, and that is what we wanted. (Rashed 1996, 89.5)

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[17] Translated from the Arabic edition based in the Bankipore manuscript (al-Bīrūnī 1948, 63.13). The German translation of Suter, from the Leiden manuscript, reads differently, but there, too, four dimensional magnitudes are present (Suter 1910–1911, 40.21).

[18] "Ergo multiplicatio superfluitatum medietatis omnium laterum trianguli super unumquodque laterum trianguli unius earum in alteram et eius quod agregatur in tertiam deinde multiplicatio eius quod agregatur in medietatem omnium laterum trianguli est equalis multiplicationi embadi trianguli in se. Et illud est quod declarare voluimus." (Clagett 1964, 288.102).

The Latin is a translation of the original version of the text, so either the Banū Mūsā or their translator Gerard made an effort to avoid stating four dimensional magnitudes, while al-Ṭūsī saw no problem with them.

It is Gerard's translation that became the source for later proofs in Latin Europe. One version of the proof that came to be attributed to Jordanus de Nemore (early 13th c.) expressed concern over the dimension: "The rule is said to have been written in Arabic. Since in the rule the third multiplication is evidently that of a line by a solid, which cannot exist among continuous magnitudes, one ought to assume in it a relation of numbers."[19]

Around the same time Fibonacci repeated the proof stating the four dimensional magnitudes in the context of the lines in their diagrams in his *De practica geometrie* (1220). The proof ends with:

> And the product of the square on ET by the square on AL equals the product of AE by the product of EB by BL, and the the result [by] AL. But the product of the square on ET by the square on AL equals the square of the area of triangle ABG.[20]

Although Fibonacci introduced the rule with a numerical example, his proof is situated entirely in the realm of geometry. Later, Luca Pacioli (1494) repeated Fibonacci's proof, and still later Nicolo Tartaglia (1560) copied the proof of Pacioli, and both of them stated the four dimensional magnitudes without reference to numbers.[21]

Petrus Ramus (1569), too, ran through the proof, citing both Jordanus and Tartaglia. But like Gerard's translation of the Banū Mūsā's text, he avoided stating the magnitudes in terms of the letters of the diagram: "Therefore to multiply these two squares of the sides is to square the given triangle, and its square will be the side of the given triangle."[22]

We give one last example, from the 1604 *Geometria practica* by Christoph Clavius, who added the phrase "the number produced" by the multiplications where the magnitudes would have entered the fourth dimension:

> That which is then made from the square of DE by AH, will be equal to that which is made from AE, by the product of EB by BH. Therefore, the number produced by

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[19] Translated by Clagett (1964, 643) and de Wreede (2007, 261).

[20] "Et ductus tetragoni *.et.* ad tetragonum *.al.* est sicut ductus *.ae.* ad ductum *.eb.* ad *.bl.*, et producti *.al.* Sed ductus tetragoni *.et.* ad tetragonum *.al.* est sicut tetragonum superficiei trigoni *.abg.*" (Fibonacci 1862, 41, 15th line from bottom). Translation adjusted from (Fibonacci 2008, 82).

[21] (Pacioli 1494 II, f. 11b.4; Tartaglia 1560, f. 7b, 6th line from bottom).

[22] "Itaque multiplicare haec duo laterum quadrata est quadrare datum triangulum, eiusque quadrati latus erit area dati trianguli." (Ramus 1569, 320, 5th line from bottom).

> the product of the square of DE by AH, multiplied by AH, will be equal to the number produced by the product of AE multiplied by the product of EB by BH, multiplied by the same AH.[23]

In our author higher dimensional magnitudes arise in three geometric proofs for rules of numerical calculation, one for adding square roots of numbers [3.3.4, 300], another for adding cube roots of numbers [3.3.8, 325], and the third for multiplying square roots of numbers [3.5.3, 403]. In the first proof, he observes that the rectangle BE is the mean in ratio between the two squares AG and GD, thus "we multiply the two squares AG, GD, one of them by the other, so we get surface BE multiplied by its same, and we take a root of that to get surface BE."

He works with nine dimensional magnitudes in the proof for adding cube roots: "if we multiply the square on BG by GE, then by three, and we form a cube from the outcome, then that is the same as multiplying the cube on side BG by its same, then the outcome by the cube whose side is GE multiplied by twenty-seven." Writing this with Cartesian exponents, the latter magnitude is $(\mathrm{BG}^3)^2 \cdot (\mathrm{GE}^3 \cdot 27)$. As in the first proof, this magnitude is the geometric version of part of the numerical calculation in the rule. The goal is not to do geometry, but to justify an arithmetical procedure.

In the third proof, to show that multiplying $\sqrt{a}$ by $\sqrt{b}$ gives $\sqrt{ab}$, he writes: "we therefore multiply … square DB by square BE, and we take a root of the outcome…" [3.5.3, 407]. Abū Kāmil also proved this rule with essentially the same line of reasoning, but unlike our anonymous author, he was careful to distinguish between the magnitudes and the numbers that measure them, much like we saw in Clavius. For this part of the proof Abū Kāmil wrote, "So the multiplication of the number that is in surface ZM by the number that is in surface EA is equal to the number that is in surface ZK by itself" (Abū Kāmil 2012, 305.14). He did not insert the phrase "the number that is in" in any of his other proofs.

The proofs in the authors just surveyed ground rules of numerical calculation in geometric proofs, where the constructions proceed without regard to the numerical sizes of the magnitudes. Ptolemy did nearly the same for the numerical calculations of tables in the *Almagest.* There he stated and proved not arithmetical rules, but propositions in Euclidean geometry, and those propositions were then applied to numerical calculations. Neither he nor any of his successors crossed over to four or more dimensions in their proofs until we get to François Viète (1540–1603). Viète worked with four dimensional magnitudes based in diagrams in Propositions XIIII and XV in *Effectionum geometricarum* (1593), and his theorem on angular sec-

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[23] "Qui ergo fit ex quadrato ipsius DE, in AH, aequalis erit ei, qui fit AE, in productum ex EB, in BH. Igitur ex numerus, qui ex producto ex quadrato ipsius DE, in AH, multiplicato in AH, gignitur, aequalis erit numero, qui ex producto ex AE, in productum ex EB, in BH, multiplicato in eundem AH, procreatur." (Clavius 1604, 177, 8th line from bottom).

tions in *Variorum de rebus mathematicis responsorum, liber VIII* (1593) deals with magnitudes of unlimited dimension. In fact, his unlimited scale of magnitudes had already been introduced in his 1591 *In artem analyticem isagoge.*[24] Viète's overall goal was the same as Ptolemy's, to ground astronomical calculations in propositions in nonarithmetized Euclidean geometry. Like Hero, al-Bīrūnī, al-Ṭūsī, Fibonacci, Pacioli, Tartaglia, and our author, Viète saw no problem working with higher dimensional magnitudes. What was important was that the numbers he calculated from them were correct (Oaks 2018a). Like his ancient and medieval predecessors, Viète did not address the nature of their existence.

V.4 Dividing a Number by a Binomial or an Apotome

In the third category, at [3.6.20, 518], the author explains how to find the result of dividing a number by a binomial or an apotome, first with the example of dividing 50 by $10+\sqrt{10}$. He multiplies the binomial by its apotome $10-\sqrt{10}$ to get 90, but instead of also multiplying the 50 by the apotome, he sets up a proportion. Dividing 90 by $10+\sqrt{10}$ gives $10-\sqrt{10}$, so "if we divide the fifty, which is the divisor, by ten and a root of ten, it results in a number such that the ratio of the fifty to that number resulting from the division is as the ratio of the ninety to ten less a root of ten." In modern notation, he wants x such that $50 : x :: 90 : 10-\sqrt{10}$. Then, following the procedure for the rule of three, he calculates the answer.

The problem of dividing a number by a binomial or apotome is solved differently in other books. Abū Kāmil solves the problem "divide ten dirhams by two and a root of three" by algebra. He names the result of the division "a thing" and multiplies it by the binomial to set up the equation "two things and a root of three *māl*s equal ten dirhams" ($2x+\sqrt{3x^2}=10$ in modern notation). He subtracts the two things from both sides to get "ten except two things Equal a root of three *māl*s," then squares both sides and solves the resulting quadratic equation (Abū Kāmil 2012, 413.14).

Another approach, corresponding to the one we practice today, is given by Ibn al-Bannāʾ in his *Talkhīṣ aʿmāl al-ḥisāb*: "For division by binomials and apotomes, you multiply the dividend and the divisor by an apotome of the divisor if it is a binomial, or by its binomial if it is an apotome. Then you divide the result of the dividend by the result of the divisor" (Abdeljaouad and Oaks 2021, 101). His student al-Hawārī illustrates the rule in his *Essential Commentary* with two examples, dividing 12 by $5+\sqrt{3}$ and dividing 10 by $3-\sqrt{7}$. Ibn al-Bannāʾ also gives the same rule, with several examples, in his *Algebra* (Saidan 1986, 523.11).

[24]For an overview of this theorem, see (Oaks 2018a, §3.5).

V.5 Borrowings from al-Khwārazmī

In his section on the third single equation, the author gives the example "two thirds of a fifth of a *māl* Equals a seventh of its root" [4.1.12, 622], which, contorted into modern notation, would become $\frac{2}{3}\frac{1}{5}x^2 = \frac{1}{7}x$. This example originates in al-Khwārazmī's problem (11), where our author has converted the arithmetical enunciation of the problem into an equation. Al-Khwārazmī's treatment of this problem is unusual in that he first gives the enunciation and a very brief solution, and then he explains in detail how he got the solution. Here is the enunciation and brief solution:

> If (someone) says, a *māl*: two thirds of its fifth equals (*mithla*) a seventh of its root. So the whole *māl* Equals (*yaʿdil*) a root and half a seventh of a root. So the root is fourteen parts of fifteen of the *māl*. (al-Khwārazmī 2009, 175.1)

The enunciation is not an algebraic equation. Like the enunciations in every collection of problems in books and chapters on Arabic algebra, it is a question in arithmetic, where the words *māl* and *jidhr* take the meanings of "quantity" and "square root" respectively. Not only is this enunciation situated with a group of other clearly arithmetical questions, but there are two linguistic indications that it is not an equation. The technical term for "equal" in stating algebraic equations was the verb *ʿadala*, and here we have instead the arithmetical *mithla*. The other indication is that al-Khwārazmī writes "of its root" rather than simply "of a root." The solution shifts to the realm of algebra with the equation "the whole *māl* Equals (*yaʿdil*, conjugated from *ʿadala*) a root and half a seventh of a root," where the meanings of *māl* and *jidhr* now conveniently take on their algebraic roles as the names of the second and first powers of the unknown. Because *jidhr* no longer means "square root," there is no need for the possessive suffix that we saw in the enunciation. Al-Khwārazmī next explains how the solution was found, beginning with: "Its rule is that you multiply two thirds of a fifth of a *māl* by seven and a half to complete the *māl*...," where the *māl* is the algebraic second power. Our author converted al-Khwārazmī's arithmetical enunciation into an algebraic equation by simply changing *mithla* into *yaʿdil*, but neglecting to change "a seventh of its root" into "a seventh of a root."

The passage that begins at [4.2.15, 733], where our author introduces his example of the second combined equation, also seems to have been copied from al-Khwārazmī: "a *māl* and twenty-one in number Equal (*taʿdil*) ten roots. And its meaning is, what *māl*, if you add to it twenty-one in number gives an outcome equal (*mithla*) to ten of its roots?" (al-Khwārazmī 2009, 105.3–5). Here the meaning of the algebraic equation is explained by transforming it into the enunciation of an arithmetic problem. Several other algebraists explain all three combined equations this way. Our author does it only for this equation type.

VI Premodern Numbers

There are many passages in our text which translated literally sound awkward to the modern ear. For example, the author regularly writes not of *the* (square) root of a number, but of *a* root of a number, like "if we want to add a root of three to a root of five" [3.3.3, 294]. Or, in several places he works not with the number at hand, but with one *the same as* that number, such as "we want to subtract from one and a third the same as its fourth" [4.1.7, 589]. Another example is that he calls fractions "parts" as was common in Arabic, but then he also calls the numerators of those fractions "parts" in the rule for multiplying fractions: "we multiply the parts by the parts and we divide the outcome by the denominators, one of them multiplied by the other" [1.5.2, 95]. And then there is an odd (to us) treatment of the coefficient and the power in an algebraic term. Writing about the squaring of half the multitude (coefficient) of "ten roots" (like our $10x$), he writes that the 25 thus obtained "is a number, since we multiply a number the same as the number of half the roots, and we do not multiply roots" [4.2.4, 665].

VI.1 A Number as a Multitude-Species Pair

Each of these curiosities is a consequence of a very different way of conceiving of numbers in medieval Arabic, and our author is careful to respect this conception. Today, whether one understands numbers to be elements of a set satisfying the ordered field axioms, or points on some imagined number line, or even the positive integers that one counts off without regard to anything counted, numbers are objects in themselves: they exist independently of any units they may count or measure. But in premodern arithmetic, including Euclid's *Elements* Books VII–IX and practical Arabic *ḥisāb*, numbers necessarily count or measure some material or intelligible unit. In Euclid a number is composed of intelligible, indivisible units, while in Arabic arithmetic numbers are again numbers *of something*, whether of horses, parsangs, mithqals, dirhams, hours, degrees, minutes, or abstract divisible units (*āḥād*).

Premodern numbers thus possess two aspects, what we can call the "multitude" and the "species."[25] For the number "three horses," the multitude is three and the species is horses. The number *is* the horses, seen from the perspective of quantity, just as a number in Euclid *is* the collection of units of which it is composed. The species can also be nested, like al-Khwārazmī's "three fourths of a dirham" (al-Khwārazmī 2009, 185.11), where "three fourths" has multitude "three" and species "fourths," and the whole phrase has multitude "three fourths" and species "dirhams."

Because the material of numbers lies with the species, there will be many distinct numbers that share the same multitude. The three horses on one farm is a number

[25] This use of "species" is unrelated to the philosophical "species" or "genus" of Aristotle or Nicomachus. It is simply a categorization.

equal to the three horses on another farm, and these distinct but equal threes are likewise distinct from the three pounds of flour in my kitchen cupboard and the three dollars in my wallet. They are all examples of threes, some of them of different species, some of the same species. This multiplicity applies to Euclid's numbers, too. Ian Mueller wrote: "In Greek arithmetic there are indefinitely many units and indefinitely many ways of combining them into multitudes. Clearly, then, there is no unique 2 or 3; any pair of units is a 2, for example" (Mueller 1981, 59). So where modern numbers are unique—there is only one "five," for example—premodern numbers admit multiplicity, and this is important for our text.

Arabic arithmeticians usually performed calculations in the abstract, that is, without specifying any material units. But they did not work with multitudes detached from their species. Instead, they worked with multitudes of intelligible, divisible units that are often counted in dirhams, a denomination of silver coin. For example, Abū Kāmil solves the problem of finding a quantity (*māl*) such that when "you add to it its root and a root of its half, then you multiply the sum by itself, it gives twenty dirhams" (Abū Kāmil 2012, 443.11). Here the *māl* and the dirhams are not real money, because the solution is irrational: in this problem the *māl* turns out to be $\sqrt{\frac{3}{8}+\sqrt{20}+\sqrt{\frac{1}{8}}}-(\frac{1}{2}+\sqrt{\frac{1}{8}})$. These dirhams are the intelligible and divisible units of *ḥisāb*, which are also often counted or measured in "units" (*āḥād*) or "in number" (*min al-ʿadad*), and it is common for the species to be left off altogether. Our author prefers "units," though he writes "dirhams" at [4.2.6, 681] and "in number" at [4.2.6, 687] and [4.2.27, 804].

Translations of passages in al-Bīrūnī, al-Fārābī, Ibn al-Hāʾim, and Ibn al-Bannāʾ that exhibit this number concept are given in (Oaks 2022a, 184–186). More relevant to our text is what al-Khayyām wrote in the third part of his commentary on Euclid's *Elements*:

> And those who make calculations, I mean those who make measurements, often speak of half of the unit, its third, and of other parts, although the unit be indivisible. But they do not mean by this a true absolute unit whereof true numbers are composed. On the contrary, they mean by this an assumed unit that, in their opinion, is divisible. They then act in whatsoever way they please in the management of the magnitudes in accordance with that divisible unit and in accordance with the numbers composed thereof; and they often speak of a root of five, of a root of a root of ten, and of other things that they constantly do in the course of their discussions, and in their constructions and their measurements. But they only mean by this a "five" composed of divisible units, as we have mentioned.[26]

[26] (Rashed and Vahabzadeh 1999, 379.15). Translation adapted slightly from (Rashed and Vahabzadeh 2000, 253).

Because numbers are not unique, one can have as many "roots of three" as one wishes. So when the author writes "a root of three" at [3.3.3, 294], he intends just one example of that root. In other places he writes of more than one or less than one root, and the word "root" becomes dual or plural depending on the multitude. At [3.1.4, 227] we find: "three roots of four is a root of what *māl*?" These "three roots of four" is a collection of three numbers, all of them $\sqrt{4}$. Think of it like $\sqrt{4}$, $\sqrt{4}$, $\sqrt{4}$. Naturally, he wants to express this as one number, so he gives instructions to convert it to $\sqrt{36}$. Today we can regard our $3\sqrt{4}$ as being a single number because of the multiplication implied by the concatenation of the signs. But the Arabic "three roots of four" is like "three chickens," an amount with a multitude and a species. Our author and every other medieval and early modern author we know insisted that multiple roots and fractions of a root be reexpressed as a single root. Descartes was perhaps first author to write terms like "$27\sqrt{3}$" and "$\frac{4}{9}\sqrt{3}$." All prior authors we have read would convert these to single roots, as $\sqrt{2187}$ and $\sqrt{\frac{16}{27}}$ (Oaks 2017, 151–153).

Our author states multiples and fractions of roots elsewhere, too, prior to making them a root of a single number, like "two roots of nine," "fifty roots of ten," "half of a root of four," "a third of a root of thirty-six," etc., and which are phrased exactly like "two fifths," "half of a sixth," and "a third of a tenth," etc., in fractions, and "two degrees," "fifty minutes," "a third of a second," etc., in sexagesimal arithmetic.

VI.2 Working With a Number "the Same As" Another Number

There is another subtlety in these operations that is connected with the premodern concept of a number as a multitude-species pair. In many instances in our text and across the whole of Arabic arithmetic and algebra we often find an author working with a number *the same as* (*mithla*) another number. For example, our author writes at [4.1.5, 579]: "we want to add to one and two thirds the same as its fifth." This is a particular $1\frac{2}{3}$ units that he possesses, and he wants to add to it a fifth of it. But if he takes its fifth, then his number becomes $\frac{1}{3}$, so he has lost his original amount. Think of this in material units. If we want to add to a loaf of bread its fifth, we can cut off a fifth of the loaf, but then we no longer have the whole loaf to which the fifth should be added. Instead, we need to introduce a new piece of bread equal to a fifth of our current loaf so the two can be joined. This is why our author adds a number "the same as its fifth" and not just "its fifth."

This way of introducing into the calculation a new number based in one we already have is very common in Arabic arithmetic (which includes algebra). Here are other examples from our text:

- "we want to subtract from one and a third the same as its fourth" [4.1.7, 589]

- "we subtract from everything we have of the roots and the number the same as three of its fourths" [4.1.8, 598]
- "we need to add to everything we have the same as its half" [4.1.9, 610]
- "we add to the outcome the same as a third of the *māl*" [4.2.5, 679]
- "if we add to that the same as seven of its roots" [4.2.5, 679]
- "we add to it and to everything with it of roots and number the same as its fifth" [4.2.6, 682]

This is also why authors usually multiply a number by "its same" (*mithlahu*) instead of by "itself" (*nafsuhā*). For any multiplication you need two numbers, so if you have one number that you want to square, you need to bring in another number equal to it so you have something to multiply it by. Our author writes "by its same" in this context 52 times, and only twice writes "by itself."

VI.3 The Meaning of "Number" (*ʿadad*)

Because our author works with fractions and irrational roots while at the same time he appeals to propositions from Euclid's *Elements* in which numbers are restricted to positive integers, we should ask what constitutes a "number" (*ʿadad*) for him. Most Arabic books on arithmetic, including books on algebra, do not give a definition of number, and most of those that do give some version of Euclid's definition and then proceed to operate on whole numbers, fractions, and irrational roots anyway (Oaks 2022a, 183). If our author defined "number," it was on a lost leaf from the beginning of the manuscript. Nevertheless, as with other authors, we can determine which quantities he considered to be numbers by observing how he used the word *ʿadad*.

We begin with irrational quantities. In one passage the author calls $\sqrt{10} + \sqrt{\frac{1}{3}}$, $10 + \sqrt{10}$, and other binomials "numbers":

> … since any binomial number, if multiplied by its apotome, results in a rational number. As for the absolute binomial, it is any number composed of two numbers rational in power, or one of them rational in length and the other rational in power, like a root of ten and a root of a third, or like ten and a root of ten, and the like. And for the apotome: any binomial number, if the smaller part is detached from the greater part, what remains of that is called the absolute apotome. [3.6.22, 538]

And just before this, in the passage at [3.6.21, 525], the amount that will be calculated to be $5\frac{5}{9} - \sqrt{3\frac{7}{81}}$ is called a "number." An example of a rational number is the following [4.1.5, 581]: "Then we multiply six by the number, which is one and two thirds, to get ten."

Our author is not unusual in using the word *ʿadad* with different meanings in other parts of the treatise. He writes at [1.1.2, 7]: "As for the root, it is any number or fraction that you want to multiply by itself," implying that the "number" is a whole number. Then, at [3.1.2, 219] the word "number" is apparently presumed to be rational: "If we want to duplicate a root ascribed to a number…"

The word "number" takes multiple meanings in other authors, too, including al-Karajī, al-Samawʾal, and Ibn al-Bannāʾ. I will give just one example from al-Samawʾal. He notes in his chapter on quadratic irrationals that "the ratio of a root of twenty to ten is not the ratio of a number to a number" (Rashed 2021, 176.14), where clearly "number" means a whole number. Then, only five lines below that on the same page, he writes: "Suppose there are two numbers rational in power, commensurable in (power), and let them be a root of 12 and a root of 6."

There is no contradiction here. The reader understands the local meaning by the context. Phrases like "a root of a number" and "a number with a fraction" designate forms that numbers take, the first being the square root of a rational number, and the second being a whole number with a fraction, and "the ratio of a number to a number" is simply language borrowed from Euclid. It is clear by the overall treatment of quantities in these books, including the present text, that the authors regarded any (positive) amount, including any fraction and irrational root, as being a "number." Of course, some authors might in some instances restrict the notion of "number," such as when working in Euclidean number theory, but in Arabic arithmetic and algebra numbers were universally regarded as including nonintegral amounts.

VI.4 Algebraic Terms as Multitude-Species Pairs

It is in the domain of these intelligible, continuous numbers described by al-Khayyām that algebraic calculations were performed. Just as intelligible numbers in sexagesimal form can be counted in the species degrees, minutes, seconds, etc., whole numbers in base ten can be counted in the species units, tens, hundreds, thousands, etc., and fractions of the unit can be counted in the species thirds, fourths, fifths, etc., the numbers in algebra can be counted in the species units, roots, *māl*s, cubes, etc. Al-Khwārazmī introduces the latter like this: "I found that the numbers that are necessary for calculation in algebra are of three species (*ḍurūb*, sing. *ḍarb*): roots, *māl*s, and simple number unrelated to a root or a *māl*" (al-Khwārazmī 2009, 97.9). Later authors extended these species to include cubes, *māl māl*s, etc.

Our author usually writes *ʿidda* for "multitude," though, like others, he sometimes uses *ʿadad* ("number") instead. And he typically writes *jins* for "species," though sometimes he uses *nauʿ*. Words used by other authors for the algebraic "species' include *nauʿ*, *jins*, *ḍarb*, and *aṣl*, so there was no standard term. Thus, a polynomial like "a hundred and a *māl* less twenty things" at [2.3.3, 185] consists of a 100 whose species is units and a 1 whose species is *māl*s, that are lacking a 20 whose species is things. It is like "a hundred pounds and a shilling less the twenty pence (that I kept for myself)" in predecimal British currency.

An algebraic term like "three things," where "three" is the multitude and "things" is the species, is not identical to its modern counterpart $3x$. In our $3x$ the 3 and the x are both numbers understood to be multiplied together, while the Arabic "three" and "things" are different aspects of a single amount, like we understand in phrases like "three bottles" or "three ounces." We can, of course, think of $3x$ as three xs, but this way of looking at it breaks down with terms like $\sqrt{5}x$ and $3xy$, which have no premodern equivalents. It is this fundamentally different way of understanding algebraic monomials, and thus also polynomials, that makes premodern algebra, whether in Greek, Arabic, Latin, or medieval and early modern European vernaculars, radically different from modern elementary algebra.

This understanding of monomials allows us to make sense of some seemingly odd wording in the solutions to equations in Arabic algebra. To pick a well known example, recall al-Khwārazmī's rule for finding the root in his equation "a *māl* and ten roots equal thirty-nine dirhams."[27] The rule begins: "you halve the roots, which in this problem are five…" We would not word it that way. Following the same reasoning with our modern $x^2 + 10x = 39$, we would take half of the 10, not half of the xs (whatever that would mean). The medieval wording is due to the "ten roots" being literally an aggregation, much like "ten euros" in the form of a pile of ten one-euro coins. The ten and the euros are not two separate entities that are brought together to form the term, but are two aspects of a single pile of coins, and similarly for "ten roots." One cannot separate the "ten" from the "roots" any more than one can separate the "ten" from the "euros."

Al-Khwārazmī wants half of those ten roots, so it suffices to say "you halve the roots." Taking half of the ten roots gives five roots, not the five units necessary for the rule to solve the equation. He and some later algebraists are concise in their wording, and do not explicitly announce that the species of this quantity shifts from roots to intelligible units, but others are more careful. Al-Karajī states it this way in his *Algebra*: "you halve the things, and you multiply the number (*ʿadad*) of its half by itself…" (Saidan 1986, 149.11), where the species of this pure "number" is abstract units. Ibn al-Bannāʾ similarly writes: "one always multiplies half the number of roots by its same…" (Saidan 1986, 547.23). Ibn al-Hāʾim writes in his 789/1387 commentary on the poem of Ibn al-Yāsamīn, "you halve the quantity (*qadr*) of the things" in one place and "subtract half the multitude of roots" in another, where the quantity/multitude is taken to be a number of intelligible units.[28]

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[27] (al-Khwārazmī 2009, 101.5). As per the critical apparatus, the original text did not read "ten of its roots," but simply "ten roots."

[28] *Sharḥ al-Urjūza al-Yāsmīnīyya* (*Commentary on the Poem of al-Yāsamīn*) (Ibn al-Hāʾim 2003, 76.24, 77.16).

Our author takes particular care to be precise about the species of numbers of which he writes. For the standard equation "a *māl* and ten roots equal thirty-nine," he explains:

> We halve the multitude of roots, which is ten, to get its half is five, and we multiply it by its same to get twenty-five, which is a number, since we multiply a number the same as the number of half the roots, and we do not multiply roots. [4.2.4, 664]

This clarifies that the 25 will be a number of units, and not the *māl*s that one would get if one multiplied "five roots" by its same. This next passage, which discusses the solution to the same equation type, is equally clear about the shift in species:

> … we multiply half the multitude of roots by its same, that is, a number the same as the multitude of half the roots [4.2.3, 660]

VI.5 Duplications and Fractions as Multitude-Species Pairs

The idea that a number is composed of its species, like "ten roots" being composed of roots, also surfaces at [3.1.9, 260] in the section on duplicating cube roots. In the problem, say, of duplicating $\sqrt[3]{10}$ five times, there are five duplications. Here "five" is the multitude and "duplications" is the species. The text calls for "a cube of the duplications," which in our example would be the number 125 units. We translate it as "a cube of the (number of) duplications" to make the reading clearer. The same wording occurs again in the following section on duplicating fourth roots. At [3.1.11, 269] we translate "a *māl māl* of the (number of) duplications."

This kind of wording appears again in the context of fractions. Fractions in Arabic algebra are expressed when possible using the species "half," "third," up to "tenth," like "eight ninths" or "two sevenths" or "four fifths and a fifth of a fifth." But where the denominator is not the product of numbers from among 2, 3, up to 10, the language of parts is used. For example, al-Khwārazmī writes of "thirteen parts of seventeen parts of a dirham" ($\frac{13}{17}$) (al-Khwārazmī 2009, 279.4). Here the dirham, or unit, is regarded as being partitioned into seventeen equal parts, and the fraction is thirteen of those parts. Often the phrase was shortened, so this fraction could be expressed as "thirteen parts of seventeen parts," or even just "thirteen parts of seventeen." In the same vein, unit fractions like $\frac{1}{11}$ could be expressed as "a part of eleven." This is why the reciprocals of the powers in algebra are called "a part of a thing," "a part of a *māl*," etc.

Suppose we want to multiply three parts of eleven parts of a unit by six parts of thirteen parts of a unit ($\frac{3}{11}$ by $\frac{6}{13}$). The first fraction is "three parts," where a "part" is one of the parts of eleven equal parts of a unit. As a quantity-species pair, the quantity is "three" and the species is "parts." When one wishes to work with the numerator 3, one can simply say “the parts,” much like saying “the roots” singles out the multitude 10 in the term “ten roots.” In this example one first multiplies “the parts,” which are 3 and 6, to get 18, and this is divided by “the product of the two denominators,” 11 and 13, to get eighteen parts of one hundred forty-three parts of a unit. Now the meaning of the rule our author gives for multiplying fractions will make sense: “we multiply the parts by the parts and we divide the outcome by the denominators, one of them multiplied by the other” [1.5.2, 95].

VII Summary

In both his *Quadrant* and *Algebra* al-Khayyām gave brief accounts of earlier progress that had been made in solving cubic equations via conic sections. It appears that he did not know our anonymous text, since his treatment of the third combined type is not based in Euclid, as is the one given by our author. Also, the particular classification of equations that al-Khayyām presents in both books had already been circulating before his time, as we see in our text and in al-Sulamī, if that author wrote before al-Khayyām.

It might seem that the geometric magnitudes of higher dimensions in some proofs are an accident of sloppy writing, and that what our author meant by a phrase like “we multiply the two squares AG, GD, one of them by the other,” was really “we multiply *the numbers of* the two squares AG, GD, one of them by the other.” But our author was not one to relegate such details to the background, as we see in the care with which he treats the multitudes and species of premodern arithmetic and algebra (Section VI.4, above). For example, he ensures that the reader knows that one should take “a number the same as the multitude of half the roots,” and not simply “half the roots” as some other algebraists wrote. Our author intended the operations to be performed on the geometric magnitudes themselves, just as others had done in their proofs of Hero’s rule and as Viète would do as a foundational principle in his own geometrical calculations.

These operations may contradict the accepted ontology of geometric magnitudes, but mathematicians committed similar violations in other settings. Arabic algebraists, al-Karajī more than others, sometimes admitted unresolved divisions into algebraic equations even if the two sides of an equation were understood to be aggregations of multitudes of the powers, and some 16th-century European authors violated their understanding of algebraic terms by admitting irrational multitudes. In arithmetic, Michael Stifel (1544) was certainly not unusual in denying that irrational numbers are “true numbers,” but yet he felt compelled to accept them because of “the results that follow from their use.” And it should have seemed outrageous that Cardano (1545), Bombelli (1572), and others would work with negative and complex numbers, which could be said to be even more impossible. Like these examples from arithmetic and algebra, Viète’s reason for working with higher dimensional magnitudes is that he had found a use for them (Oaks 2017; Oaks 2018a, §§ 3.6, 5.1). For our author, it appears that a proper proof by geometry should remain in the domain of geometry, even if the magnitudes might become fictitious. Looking over the various authors who committed these transgressions shows that they were typically the more brilliant, and such is the case with our anonymous author, too.

An important characteristic of the language in medieval Arabic books, one that is surely not restricted to books on mathematics, is that the meanings of words, even of technical terms, are determined in large part by context. *Māl* may be the technical name for the second power in algebra, but the word is also used with its nontechnical meanings in books and chapters on algebra. Sometimes the meanings of a single term are close, such as the term *māl māl*, which is the name of the fourth power of the unknown as well as the word for the fourth power of a number: the first is an algebraic name while the second is what one *does to* a number in arithmetic. These distinctions are important for understanding the mathematics, and can often be obliterated when converting the rhetorical rules into modern notation. Even the word "number" (*ʿadad*) takes different meanings, as the "multitude" of a multitude-species pair, as the first level of the proportional levels number, root, *māl*, cube, etc., and as a "number" in the ordinary premodern sense. Even within that latter sense a "number" can mean a whole number, a rational, number, or any number including irrational roots.

In the other direction, some technical concepts can be expressed with different words. In our text the "species" of a term is usually *jins*, but sometimes it is *nauʿ*, and other authors have used *ḍarb* or *aṣl*. Premodern authors rarely defined their terms, and we historians understand their meanings in the same manner that the first readers of the texts did, by considering the core meaning of the word in conjunction with the context in which it is used.

Last, we should address the intended readers of the treatise. On the one hand, the dustboard instructions and the arithmetical solutions to equations together with their intuitive proofs appear to have been intended for practitioners. On the other hand, the geometrical solutions to equations with their proofs based in Euclid's *Elements* and the deliberate lack of worked out problems point to a readership with a more theoretical interest. The book thus seems to have been written for two types of reader, practitioners who perform calculations in their jobs and who would not have been familiar with Euclid, and theoretical geometers. This difference in presentation may have also stemmed from what our author considered to be proper for each domain: the numerical algebra in the tradition of al-Khwārazmī belongs to the realm of practitioners, so he explained those propositions in practical terms, while the geometric solutions belongs to theoretical mathematics, so his presentation of those propositions are likewise theoretical.

There are other books in Arabic mathematics that cater to both a practical and a theoretical readership, too. The *Algebra* of Abū Kāmil also fits this description by giving ample proofs based in Euclid and also offering a large collection of worked out problems. The *Algebra* of al-Khayyām, by contrast, was intended specifically for people with a background in Euclid and Apollonius. His numerical solutions even presume the whole numbers of Euclid and Aristotle, making them distinctly impractical.

VIII Translation

In the translation, we adopt the following conventions: **Bold text** in the translation is written bold in the manuscript. (Parenthetical additions) are ours to make the reading easier. Words in <angle brackets> are restored to the text. We show the diagrams as they are in the manuscript, with the same relative scale, with transliterated letters in the same positions, and with the thick dots at the vertices.

VIII.1 Translation of Astan Quds 5325

[1.1.1] … according to the same ratio in which the first (quantity) is one. So if the first of the three quantities is one, then the second is a root and the third is a *māl*.

[1.1.2] As for the root, it is any number or fraction that you want to multiply by its same, and the *māl* is what is produced from multiplying the root by its same.[29]

[1.1.3] If the first of the three quantities is greater than the unit and we call it a number, then the second is (some) roots, its multitude[30] equal to the multitude of units of the first number, and the third is (some) *māl*s, again with the same multitude. By this rule, if the first of the three proportional quantities is a fraction less than the unit, then the second is a part or parts of the root in the ratio of the first fraction to the unit, and the third likewise is a part or parts of the *māl*, again in the same ratio.[31]

[1.1.4] For example, if we suppose the root is two, then the *māl* which comes from it is four, and the ratio of the unit to the two is as the ratio of the two to the four. Similarly, if we suppose the root is three, then the *māl* which comes from it is nine, and the ratio of the unit to the three is as the ratio of the three to the nine. And for fractions, if we suppose the root is a half, then the *māl* which comes from it is a fourth, and the ratio of the unit to the half is as the ratio of the half to the fourth.

[1.1.5] And by this rule, if the first of the three quantities is two units, then the second is two roots, no matter what they are equal to in number, and the third is

[29] Examples are given in the paragraph at [1.1.4].

[30] This is what we would call the coefficient.

[31] Examples are given in the paragraph at [1.1.5]. For "part" and "parts," see Euclid's *Elements* Book VII, definitions 3 and 4.

two *māl*s, each of them coming from the multiplication of one of the roots by its same. And similarly for the fractions: if we suppose the first of the three quantities is a half, then the second is half a root, an amount coming from the fraction of the unit, and the third is half a *māl*, from the whole *māl* which comes from multiplying that root by its same. Then on the basis of this rule the outcome can be attained for any numbers or fractions whatever.

[1.2] The second chapter, on what happens to the three proportional elements[32] with regard to the well-known cases.

[1.2.1] Taking into account what we mentioned of the three proportional elements, there are six cases that can occur with regard to their transformation by the kinds of operations prior to (setting up) the Equation. These are addition, subtraction, duplication, partition, multiplication, and division.

[1.2.2] As for what happens to them in the first four cases, which are addition, subtraction, duplication, and partition, the procedure in all of them is the same as the procedure for what happens in the case of absolute numbers, and is no different. As for the outcome in addition and duplication and for the remainder in subtraction and partition, the species does not change, even if it changes in quantity.[33]

[1.2.3] As for multiplication, it happens in many situations that one multiplies the root by the *māl*. They are unknowns, so the outcome is called a cube,[34] and it is third in the proportion stemming from the root and the *māl*. Because for any four proportional quantities, multiplying the first by the fourth is the same as multiplying the second by the third, and the first of these quantities, as we said, is one, and its multiplication by the fourth is the very same fourth, for this reason the outcome of multiplying the root by the *māl*, which is the second by the third, is the third stemming from them in the proportion, that is, the fourth from the first, which is the cube that we mentioned. These three names, which are the root, the *māl*, and the cube, are the single names[35] through which the three proportional levels are named.

[1.2.4] And from multiplying each of them by another there arise other levels in succession according to this ratio, and their names are composed of the three names we named, such as the *māl māl*, which follows the cube in this ratio. It comes from multiplying the root by the cube or from multiplying the *māl* by its same. And such as a *māl* cube or a cube *māl*, which follows the *māl māl* in this ratio. It

[32] The elements (*uṣūl*) here are the proportional levels: number, root, and *māl*.

[33] The word "quantity" is translated from *kammīya*, here meaning "value." This is the only instance of this word in the text.

[34] Because the root and the *māl* are unknowns, their product will also be an unknown, and thus must be given a name. That name is a "cube" (*mukaʿʿab*).

[35] I.e., not names combining two or more terms, like *māl māl* or *māl* cube.

originates from multiplying the root by the *māl māl* and from multiplying the *māl* by the cube. And such as the cube cube, which follows the *māl* cube in this ratio. It originates from multiplying the root by the *māl* cube or from multiplying the *māl* by the *māl māl* or from multiplying the cube by its same, and that is because of the continued proportion. We omit an explanation because we do not like to lengthen the discussion.

[1.3] The third chapter, on multiplying the proportional levels, one of them by another, and knowing the species of the outcome from what level it is.

[1.3.1] If we want to multiply a *māl* by a cube, we join the terms "*māl*" and "cube" and we say that the outcome of the multiplication is a *māl* cube or a cube *māl*. And if we want to multiply a root by a cube, we take the number of the place in which the root is (repeatedly) multiplied by its same until it gives a cube, which is three, and we add one to that because of the root, and we divide the outcome, which is four, into two parts, each of them greater than the unit, which are two (and) two, since it is not possible otherwise. We take a *māl* for each two, since the *māl* clearly comes from a root by a root of its same, and we say that the outcome of the multiplication is a *māl māl*.

[1.3.2] And according to this rule, if we want to multiply a root by a *māl* cube, we take five for the *māl* cube—two for the *māl* and three for the cube—and we add one to that because of the root, and we divide the outcome, which is six, into two parts in such a way that each of them is greater than one, and let them be three and three. We take a cube for each three, so the outcome is a cube cube. And if we were to divide the six into two other parts, one of them two and the other four, and we were to take a *māl* for the two and take a *māl māl* for the four and we were to join them, it would be a *māl māl māl* three times. That would be allowed, but the term cube cube is more specific and concise, since the repetition is twice, while in the *māl māl māl* it is three (times). And this is the rule.

[1.4] The fourth chapter, on dividing the proportional levels, one of them by another, and knowing the species of the result of the division from what level it is.

[1.4.1] If we want to divide one level of the proportional levels by another level and to know the species of the result of the division, then because division is the inverse of multiplication, we subtract the number of the one nearest to the level of the root from the number farther from it, so what remains, the result of the division is the species of that number. If the level of the dividend is farther from the level of the root, then the result of the division is a level. But if the level of the dividend is closer to the level of the root, then the result of the division is a part of that level.[36] And a part of each level is named for the number of its units, that is, if the root is two then its part is a half, and a part of the *māl* is a fourth, and a part of the cube

[36] A "part" of a number or an algebraic power is its reciprocal.

is an eighth, and a part of the *māl māl* is half an eighth, and so on according to this rule.

[1.4.2] For example, if we want to divide a *māl māl* by a root, we subtract the number of the root, which is one, from the number of the *māl māl*, which is four, leaving three, which is the number of the cube. So we say that the result of the division is a cube. And if the level of the dividend is the level of the root and the divisor is the level of the *māl māl*, then the result is a part of a cube. And similarly, if we want to divide a cube by a *māl*, we subtract the number of the *māl*, which is two, from the number of the cube, which is three, leaving one, which is the number of the root. So we say that the result of the division is the level of the root. And if the level of the dividend is the level of the *māl* and of the divisor the level of the cube, then the result of the division is a part of a thing. And if we want to divide a level by the same (level), then the result is one in number, since it is dividing the same by the same, and that is its rule.

[1.5] The fifth chapter, on multiplying parts of proportional levels, one of them by another, and knowing the produced part from what level it is.

[1.5.1] If we want to multiply a part of some level by a part of another level, and to know the level of the produced part from what species it is,[37] we multiply the two levels, one of them by the other. We know the species of the outcome from the preceding example, so a part of that level is the answer. For example, if we want to know what is produced from multiplying a part of a thing, that is, a part of a root, by a part of a *māl*, we multiply a thing by a *māl* to get a cube, and we take its part, which is a part of a cube. So we say that the outcome of the multiplication of a part of a thing by a part of a *māl* is a part of a cube.

[1.5.2] That is due to the rule for multiplying fractions by fractions, since there we multiply the parts by the parts and we divide the outcome by the denominators, one of them multiplied by the other. And here the parts in both the multiplicand and the multiplier are one part, so the outcome of multiplying one of them by the other is also one part, and dividing that[38] by the two multiplied denominators, that is, the two levels, one of them by the other, is a part of that outcome. So for this reason we multiply the two levels, one of them by the other, and we take a part of the outcome to get the desired amount.

[1.6] The sixth chapter, on dividing parts of proportional levels, one of them by another, and knowing the result of the division from what level it is.

[1.6.1] If we want to divide a part of a level by a part of (another) level and to know a part of the level of the result of the division, we divide the level whose part

[37] By writing “from what species it is” he means the species of the levels without regard to parts, which in the example is a cube. The produced part is then a part of a cube.

[38] Here, “that” is the product of the two 1s.

is the dividend by the level whose part is the divisor and we find the species of the result of the division. If the level whose part is the dividend is closer to the level of the root, then the desired amount is that same result of the division. And if the level whose part is the dividend is farther from the level of the root, then the desired amount is a part of the result of the division.

[1.6.2] For example, if we want to divide a part of a *māl* by a part of a cube, the result of the division is a thing, and if we want to divide a part of a cube by a part of a *māl*, then the result of the division is a part of a thing. And this is also due to the rule for dividing fractions by fractions. There, we multiply each of the two denominators by the parts of the other denominator multiplied crosswise, then of the two outcomes, we divide the dividend by the divisor. Since the parts in each of the two species is one part, we divide the two levels, the dividend by the divisor, and there is no need for crosswise multiplication. And that is its rule.

[2] **The second category**, concerning the proportional levels when they are absolute (and) combined, which is comprised of four chapters.

[2.1] The first chapter, on adding one of them to another.

[2.1.1] If it happens that there are two amounts[39] in the problem and in one of them are (some of) the same species as in the other, and it is required to increase what is in one of the two amounts by what is in the other, then one adds the multitude in each species of one of the amounts to the multitude of its counterpart in the other amount. If the two counterparts are appended then the sum is appended, and if they are both lacking, that is to say, excluded from another species, then the outcome is lacking, that is to say, excluded. And if one of them is appended and the other is lacking and the multitude of the appended is less than the multitude of the lacking, then one subtracts the smaller of the two multitudes from the greater. What remains is lacking, which is the outcome. And if the multitude of the appended is greater than it, then what remains is appended, which is the outcome.

[2.1.2] For example, if we want to add ten and a thing to ten and a thing, the outcome is twenty and two things. Or we add ten less a thing to ten less a thing, so the outcome is twenty less two things. Or we add ten less a thing to ten and a thing to get for the outcome a whole twenty.[40] Or we add ten and two things to ten less a thing to get the outcome twenty and one thing. Or we add ten less two things to ten and a thing, so the outcome is twenty less one thing. Or we add fifteen and a thing to a thing less ten to get the outcome two things and five. Or we add fifteen less two things to a thing less ten, so the outcome is five less one thing. Or we add

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[39] Lit., “two sides” or “two parts,” from *janba*.

[40] He writes “whole” because the twenty is not diminished by anything, like it is in the expression “twenty less two things” in the previous example.

ten less a thing to two things less fifteen, so the outcome is a thing less five. And that is its rule.

[2.2.0] The second chapter, on subtracting one of them from the other.

[2.2.1] As for subtraction, if it happens that there are two amounts[41] in the problem and in one of them are (some of) the same species as in the other, and it is required to subtract what is in one of the amounts from what is in the other amount, we subtract the multitude in each of the amounts: the one in the amount of the subtrahend from the multitude of its counterpart in the amount of the minuend.

[2.2.2] If the two counterparts are both appended and the (one in the) subtrahend is less, then the remainder is appended, and if it is greater, then the remainder, which is the difference between them, is lacking, which is to say excluded. And if the two counterparts are both lacking and the (one in the) subtrahend is less, then the remainder is lacking, and if it is greater, then the remainder, which is the difference between them, is appended, since that difference is an exclusion from the diminished.[42]

[2.2.3] And if only one of the counterparts is appended, suppose it to be the subtrahend, (whether) less or greater than the minuend, and the minuend is lacking, then the remainder, which is the amount of the two numbers, is lacking, that is, excluded from the element (i.e., the term) which we have mentioned before, and that is because an exclusion of an exclusion is appended in the diminished.[43] And if (the subtrahend is) lacking, (whether) less or greater than the minuend, and the minuend is appended, then the remainder, which is the sum of the two numbers, is appended.

[2.2.4] For example, if we want to subtract ten and a thing from fifteen and five things, the remainder is five and four things. Or (if) we (want) to subtract ten and five things from fifteen and a thing, the remainder is five less four things. Or (if) we (want) to subtract ten less a thing from twenty less ten things, the remainder is ten

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[41] Lit., "two sides" or "two parts," again from *janba.*

[42] This explanation seems to apply to the previous case, where the counterpart in the subtrahend is less. If the minuend is, say, "a *māl* less five units" and the subtrahend is "ten things less two units," then the remainder will be "a *māl* less three units and less ten things," where the difference "three units" is an exclusion from the diminished *māl.*

[43] Here, "an exclusion of an exclusion is appended in the diminished" applies to the next case, where the counterpart in the subtrahend is lacking. Take for example the minuend "a *māl*" and the subtrahend "a thing less five units." Algebraists routinely simplified problems like this by restoring the diminished thing in the subtrahend and adding five units to the minuend to balance the problem, to get a new subtraction problem where the minuend is "a *māl* and five units" and the subtrahend is "a thing," which makes the remainder "a *māl* and five less a thing." Here the five units, "an exclusion of an exclusion" becomes "appended in the diminished" *māl.*

less nine things. Or (if) we (want) to subtract ten less ten things from twenty less three things, the remainder is ten and seven things. Or (if) we (want) to subtract ten and a thing from fifteen less a thing, the remainder is five less two things. Or (if) we (want) to subtract ten less a thing from fifteen and a thing, (the remainder) is five and two things, and that is its rule.

[2.3] The third chapter, on multiplying one of them by another.

[2.3.1] As for multiplication, if there are two quantities and we want to multiply them by two other quantities, we put down the multiplicand on a line and the multiplier on another line under it and parallel to it. Then we need four multiplications: two diagonal multiplications and two vertical multiplications. And if there were three quantities by three quantities, then one would need nine multiplications for it: six diagonal multiplications and three vertical multiplications, and again following this rule as far as the (number of) quantities reaches. Also, when we multiply any two quantities, one of them by the other, and they are both appended or both lacking, then the outcome of the multiplication is appended; and if they are different,[44] then it is lacking.

[2.3.2] For example, if we want to multiply ten and a thing by ten and a thing, we put the ten under the ten and the thing under the thing. Then we multiply the ten by the thing which is diagonal to it to get ten things, then we multiply the other ten by the other thing which is diagonal to it to also get ten things for the outcome. Then we multiply the ten by the ten, which is vertical, to get one hundred for the outcome. Then we multiply the thing by the thing, which is also vertical, to get a *māl*, and we gather them to get a hundred and a *māl* and twenty things.

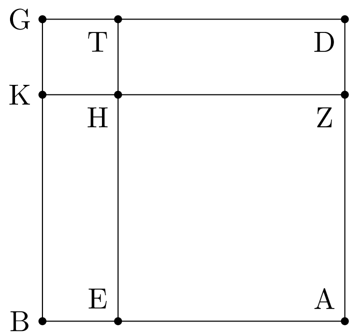
[2.3.3] Then we put for the two multipliers ten less a thing by ten less a thing the way they were put down before. Then we multiply ten by less a thing diagonal to it to get ten things lacking, that is, excluded. Then we also multiply the other ten by less a thing diagonal to it to also get ten things lacking. Then we multiply ten by ten to get one hundred appended, and we multiply less a thing by less a thing to get for the outcome a *māl* appended, and we gather them to get a hundred and a *māl* less twenty things.

[2.3.4] Again, we put for the two multipliers ten and a thing by ten less a thing the way they were first put down. We multiply ten by less a thing to get ten things lacking, then we multiply ten by a thing to get ten things appended. Then we multiply ten by ten to get one hundred appended, and we multiply a thing by less a thing to get a *māl* lacking, and we gather them to get a hundred less a *māl*, since the appended things vanished with the lacking things because they are equal in multitude.

[2.3.5] As for the reason that multiplying the lacking by the lacking is appended, we put down for this (purpose) line AB, and let it be ten in number, and we construct

[44] They are "different" when one is appended and the other is lacking.

on it square ABGD. We exclude from line AB a thing, and let it be BE, and from line AD (a line) equal to BE, which is DZ. And we draw line EHT perpendicular to AB and line ZHK perpendicular to AD. Then surface DH comes from multiplying DZ, which is a thing, by ZH, which is ten less a thing, and that is ten things less a *māl*. And surface DH is equal to surface HB. So both surfaces DH, HB are twenty things less two *māl*s. And surface HG is a *māl*, since it comes from multiplying a thing by its same. The three surfaces DH, HG, HB are thus twenty things less a *māl*, since the appended *māl* vanished with one of the two lacking *māl*s. And the whole surface ABGD comes from multiplying ten by ten, which is a hundred. And whenever we subtract twenty things less a *māl* from a hundred, the remainder is a hundred and a *māl* less twenty things, and that is the same as multiplying AE, which is ten less a thing, by its same, that is, surface AH. And that is what we wanted to show.



[2.4] The fourth chapter, on dividing one of them by another.

[2.4.1] As for division, in this absolute type one routinely encounters the division of combined species, however many there may be, by one species. If the divisor consists of more than one species, there is no way to know the result of the division unless it is prescribed in the problem. There, one uses multiplication based on a trick, which is, for any quantity divided by another quantity, the result of the division, if multiplied by the divisor, brings back the dividend.

[2.4.2] An example in which the method just mentioned will work: if we want to divide ten and a thing by five, we divide ten by five, resulting in two. Then we divide a thing by five, resulting in a fifth of a thing, and we gather them to get two and a fifth of a thing. And if we want to divide ten and five things by a thing, we divide ten by a thing to get ten parts of a thing, and we divide five things by a thing to get five in number, and we gather them to get the result of the division, (which is) five and ten parts of a thing, and that is its rule.

[3] **The third category**, concerning the proportional levels when they are single[45] (and) ascribed, which is comprised of six chapters.

[3.1] The first chapter, on duplicating them.

[3.1.1] (Section.) Duplicating ascribed (square) roots.

[3.1.2] If we want to duplicate a root ascribed to a number—and the meaning of duplication is that we make it twice as much or three times as much or however many times as much we wish—we multiply the number of times, and what fraction is with it if there is one, by its same, then by the number it is ascribed to, and we take a root of the outcome. What it gives is the desired amount.

[3.1.3] We first give a rational example, which is: if we want to duplicate a root of four one time—and its meaning is that we make it twice as much, and this is no different from saying: two roots of four is a root of what *māl*?—we multiply the number of times, which is here two, by its same to get four, then by the number it is ascribed to, which is also four, to get sixteen. So a root of that, which is four, is a duplicated root of four.

[3.1.4] By this rule, if we want to make a root of four three times as much—which is also like saying: three roots of four is a root of what *māl*?—we multiply the number of times, which is three, by its same, then the outcome, which is nine, by four, to get thirty-six. A root of thirty-six, which is six, is three times a root of four.

[3.1.5] And similarly, if we want to make a root of eight two and a half times as much, we multiply the number of times, which is two and a half, by its same to get six and a fourth, then by eight, to get fifty. A root of that, which is a root of fifty, is equal to a root of eight two and a half times.

[3.1.6] Also, if we want <to duplicate> two roots of nine, where we make it twice as much, we first find "two roots of nine is a root of what *māl*?" by the preceding rule, which is that we multiply two by its same, because of the two roots, to get four, then by nine, to get thirty-six. So a root of thirty-six is equal to two roots of nine. We then say: we want to duplicate a root of thirty-six, where we make it twice as much, and that is its rule.[46]

[3.1.7] Proof of this: to give the reason for what we mentioned, we make the number whose root we want to duplicate the medium-size square AB, and its root line GB, and let the number of times be line BD and let line BD be vertical to BG, at right angles. And we make on BD the square BE, and we complete square AZEH. Because the ratio of ED to DZ is as the ratio of square BE to surface BZ,

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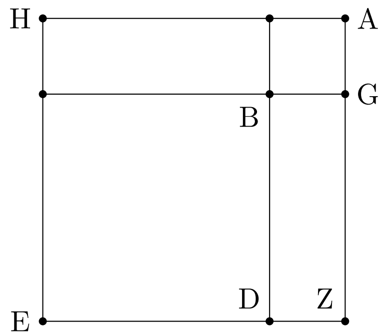
45 In Chapter 6, beginning at [3.6.16], he operates on combined numbers, so this category is not restricted to single numbers.

46 Since he begins with two roots, it would be easier to duplicate the two to get four, and then find out "four roots of nine is a root of what *māl*?"

since their altitudes are the same,[47] while ZG is equal to ED and AG is equal to ZD, and the ratio of ZG to AG is as the ratio of surface BZ to square AB,[48] therefore the ratio of square BE to surface BZ is as the ratio of surface BZ to square AB. So surface BZ, which is the desired amount, is the mean in the ratio between the squares AB, BE. And surface BZ is called one of two complements of the squares AB, BE, and surface BH is the other complement, and they are equal.[49] So for this reason, we multiply the number of times, which is BD, by its same, then we multiply the outcome, which is square BE, by the rooted number, which is AB, and we take a root of that, which is surface BZ, to get the desired amount, since it is what is produced from multiplying a root of AB, which is line BG, by the number of times, which is line BD. And that is what we wanted to show.



[3.1.8] (Section.) Duplicating ascribed cube roots.

[3.1.9] And here it is clear that when we multiply any two numbers, one of them by the other, then the product by its same, it is the same as multiplying a square of one of them by a square of the other. Based on this rule, if we want to duplicate a cube root of a number, we multiply the number of times by its same, then the outcome by the number of times again so the outcome becomes a cube, then the product by the number it is ascribed to, and we take a cube root of the outcome to get the desired amount. And the basis for this is that any number is equal to a root of its square and a cube root of its cube, and a root of a root of its *māl māl*. Therefore, for any two numbers, a root of the multiplication of a square of one of them by a square of the other is equal to a cube root of the multiplication of a cube of one of them by a cube of the other, which is also equal to a root of a root of the multiplication of a *māl māl* of one of them by a *māl māl* of the other, again by the rule. This is based on the reason that we have mentioned, that is, for any two numbers, multiplying one of them by the other, then the product by its same, is

[47] Euclid's *Elements* Proposition VI.1.

[48] Euclid's *Elements* Proposition VI.1.

[49] Euclid's *Elements* Proposition I.43.

the same as multiplying a square of one of them by a square of the other. Because the desired amount in duplicating the cube root[50] is the multiplication of the cube root[51] by the number of times, then if we cube that, its outcome is the same as multiplying the number it is ascribed to by a cube of the (number of) duplications. Therefore, we take its cube root to get the desired amount.

[3.1.10] (Section.) Duplicating ascribed roots of roots, which are the sides of *māl māl*s.

[3.1.11] By this rule, if we want to duplicate a root of a root of a number, which is duplicating the side of a *māl māl*, we multiply the number of times by its same, then the product by its same, so it becomes a *māl māl*, then the product by the number it is ascribed to, and we take a root of a root of the outcome. What this gives is the desired amount. And the reason for this is what we gave above in the two previous examples, that what is required is a multiplication of a side of a *māl māl* by the number of duplications. Then if we make that outcome a *māl māl*, it is the same as if we multiplied the number it is ascribed to by a *māl māl* of the (number of) duplications. Therefore, we multiply them and we take the root of a root of the outcome to get the desired amount.

[3.2] The second chapter, on partitioning them.

[3.2.1] (Section.) Partitioning ascribed (square) roots.

[3.2.2] As for partitioning, it is done by the same rule as for duplication, which is, if we want to partition a root of a number—and its meaning is that we multiply a root of that number by a half or a third or a fourth or whatever parts of the unit[52]—we multiply that part by its same, then the outcome by the number it is ascribed to,[53] and we take a root of the outcome. What this gives is the desired amount. For example, if we want to halve a root of four—which is like we say, "half of a root of four is a root of what *māl*?"—we multiply the part, which is the half, by its same to get a fourth, then by the number it is ascribed to, which is four, to get one. We take a root of that, which is one, which is the desired amount. Similarly, if we want to take a third of a root of thirty-six—and its meaning is: "a third of a root of thirty-six is a root of what *māl*?"—we multiply a third by a third to get a ninth, then by thirty-six to get four. A root of that, which is two, is a third of a root of thirty-six. And similarly, we also partition a side of the cube[54] or a side of the *māl māl* by this rule. And the reason for this is exactly the same as what we gave above in the chapter on duplication.

[50] Text reads "the cube root of the cube root."

[51] Text reads "the cube root of the cube root."

[52] He intends common fractions less than one, not just unit fractions.

[53] That is, the number that the root is ascribed to.

[54] The text has *kaʿb* (cube root) instead of *mukaʿʿab* (cube).

[3.3] The third chapter, on adding one of them to another.

[3.3.1] (Section.) Adding ascribed (square) roots, one of them to another.

[3.3.2] If we want to add a root of a number to a root of (another) number, we add the two rooted numbers, and we increase the outcome by double a root of the outcome from multiplying one of them by the other, and we take a root of the outcome. What this gives is the desired amount. For an example in rational roots, if we want to add a root of four to a root of nine, we multiply four by nine to get thirty-six, and we take a root of that, which is six. We double it to get twelve, and we increase it by the sum of four and nine to get twenty-five for the outcome. A root of that, which is five, is the sum of a root of four and a root of nine.

[3.3.3] Similarly, if we want to add a root of three to a root of five, we multiply three by five, then we take a root of the outcome, which is a root of fifteen. We double it according to the rule given above in the chapter on duplicating roots to get a root of sixty. We increase it by the sum of the three and the five to get the outcome eight and a root of sixty, and we take a root of that to get the desired amount.[55]

[3.3.4] The reason for this is that for any two square numbers, if we add their two complements to them the outcome becomes a square, and if we subtract them from them then the remainder is a square. For the proof of this we draw two squares, AG, GD, (where) a side of square AG is BG, and a side of square GD is GZ, and we complete the surfaces BE, GH,[56] the complements. And it was explained in the chapter on duplication that each of them is the mean in the ratio between the two squares AG, GD. Therefore we multiply the two squares AG, GD, one of them by the other, so we get surface BE multiplied by its same, and we take a root of that to get surface BE. We double it to get the sum of the surfaces BE, HG, and we add to that the two squares AG, GD, which completes for us the square AD. We take its root to get AH, which is the sum of the sides BG, GZ, and that is what we wanted to show. By this rule, if we want to add a root of a number to a number, we multiply the absolute number by its same so it becomes rooted, that is, of the other species, then we work it out as we explained above.

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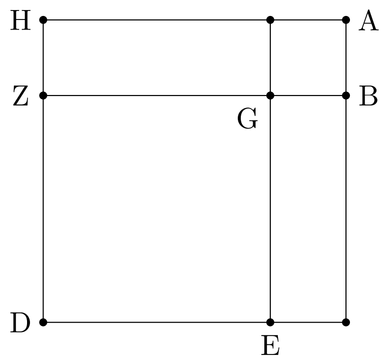
[55] This number, $\sqrt{8+\sqrt{60}}$ in modern notation, is not an absolute binomial. It is an ascribed root in which the number it is ascribed to is the binomial $8+\sqrt{60}$. See also note 75, below.

[56] The MS has GE in error.



[3.3.5] (Section.) Adding ascribed cube roots, one of them to another.

[3.3.6] If we want to add a cube root of a number to a cube root of (another) number, we multiply a square of one of the numbers by the other number, then the outcome by twenty-seven, and we take a cube root of that[57] and we keep it in mind. Then we multiply a square of the other number by the first number, then the outcome by twenty-seven, and we take its cube root, and we add it to what we remembered. Then we add the outcome to the sum of the two posited cubic numbers, and we take a cube root of the outcome. What comes from that is the desired amount.

[3.3.7] For an example in rational cube roots, if we want to add a cube root of eight to a cube root of one hundred twenty-five, we multiply the eight by its same, then the outcome by one hundred twenty-five to get eight thousand for the outcome, then by twenty-seven to get two hundred sixteen thousand for the outcome, and we take a cube root of that, which is sixty. We keep it in mind. Then we multiply one hundred twenty-five by its same to get fifteen thousand six hundred twenty-five, then by eight to get one hundred twenty-five thousand, then by twenty-seven to get three million three hundred seventy-five thousand for the outcome, and we take a cube root of that, which is one hundred fifty. We add it to what we remembered, which is sixty, to get two hundred ten. We add that to the sum of the two numbers, which are eight and one hundred twenty-five, to get three hundred forty-three for the outcome. So a cube root of that, which is seven, is the sum of a cube root of eight and a cube root of one hundred twenty-five, and that is its rule.

[3.3.8] And for a proof of this, we imagine two different cubes with square bases AG, GD, and let the diagonal of base AG be in a straight line with the diagonal of base GD, and let the smaller be AG. And we imagine square AD to be the base of a cube that bounds the two posited cubes, that is to say, which contains them. And it is known that this greater cube exceeds the two posited cubes by two equal solids,

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[57] This does not follow the pattern of the rule for adding square roots. If it did, then he would take the cube root of the product of a square of one of the numbers by the other, and then multiply it by three.

their bases equal to surface BE and their heights equal to line BH, and also by two other different solids, a base of one of them being square AG with height EG, and the other base square GD with height BG.

[3.3.9] But the multiplication of line BG by its same then the outcome by GE, gathered with the multiplication of GE by itself then the outcome by BG, is equal to the multiplication of BG by GE, then the outcome by the sum of BG, GE,[58] that is, the solid whose base is BE and whose height is BH.

[3.3.10] Consequently, the sum of the two solids for which the base of one of them is AG and whose height is GE, and the base of the other is GD and whose height is BG, is a third of the excess of the greater cube, whose base is AD, over the sum of the two cubes whose bases are AG, GD. So if we multiply each of these two solids by three, then the outcome becomes equal to the entire excess.

[3.3.11] And it is known that the solid whose base is AG and whose height is GE comes from multiplying BG by itself, then what is produced by GE. And it is known that if we multiply the square on BG by GE, then by three, and we form a cube from the outcome, then that is the same as multiplying the cube on side BG by its same, then the outcome by the cube whose side is GE multiplied by twenty-seven.

[3.3.12] Because of this, we multiply the cube which is on base AG by its same, then by the cube which is on base GD multiplied by twenty-seven, and taking a cube root of that to be three times[59] the solid whose base is square AG and whose height is GE. Then we also multiply the cube whose base is GD by its same, then by the cube whose base is AG multiplied by twenty-seven, and we take its cube root to get the same as three times[60] the solid whose base is the square GD and whose height is BG.

[3.3.13] We have explained that these two solids are the excess of the greater cube over the two posited smaller cubes. Therefore, we add those two cubes to the sum of the two cubic numbers in order to complete the greater cube, and we take a cube root of that. Its outcome is then equal to the desired sum of the cube roots,[61] and that is what we wanted to show.[62]

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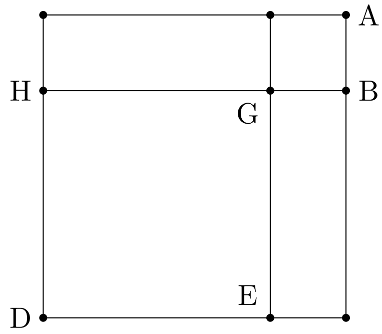
[58] The MS has GH in error.

[59] The text has *mithla* ("equal") instead of *thalāthat amthāl* ("three times").

[60] The text again has *mithla* ("equal") instead of *thalāthat amthāl* ("three times").

[61] The manuscript has "the sum of the desired two cubes," probably a scribal error.

[62] The figure also contains a number of characters in a later hand, which we have not been able to decipher.

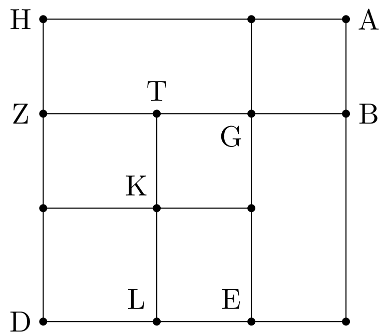


[3.4] The fourth chapter, on subtracting one of them from another.

[3.4.1] (Section.) The subtraction of ascribed (square) roots, one of them from another.

[3.4.3] For example, if we want to subtract a root of four from a root of nine, we multiply four by nine to get thirty-six, and we take a root of that, which is six. We double it to get twelve and we subtract it from the sum of the four and the nine, which is thirteen, leaving one, and we take its root, which is one, which is the remainder from subtracting a root of four from a root of nine.

[3.4.4] And for a proof of this, we imagine[63] square AG smaller than square GD, and we subtract from side GZ[64] the equal of side BG, and let it be GT. And we draw line TKL parallel to GE. Since surface BE is equal to surface ET, it leaves surface KZ equal to surface ET less the square GK. But square GK is equal to square AG, so the surfaces ET, KZ with square AG are equal to the complements BE, GH. Therefore, we subtract that from the two squares AG, GD, leaving square KD, and we take its root, which is LD, that is, TZ, so that what remains is the excess of GZ over BG, and that is what we wanted to show.



[63] It seems that he should write that we "draw" the squares, since they are only two-dimensional. Perhaps "imagine" was echoing in his mind from the last proof.

[64] The MS has GD in error.

[3.4.5] (Section.) Subtracting ascribed cube roots, one of them from the other.

[3.4.6] If we want to subtract a cube root of a number from a cube root of (another) number, we multiply a square of the smaller number by the greater number, then the outcome by twenty-seven, and we take its cube root, and we add it to the greater number and we keep it in mind. Then we multiply a square of the greater number by the smaller number, then by twenty-seven, and we take a cube root of that. We add it to the smaller number, and we subtract the outcome from the remembered amount, and we take a cube root of what remains. It is the desired amount.

[3.4.7] For example, if we want to subtract a cube root of eight from a cube root of one hundred twenty<-five>, we multiply a square of the eight, which is sixty-four, by one hundred twenty-five to get eight thousand, then by twenty-seven to get two hundred sixteen thousand for the outcome, and we take a cube root of that, which is sixty. We add (it) to one hundred twenty-five and we keep the outcome in mind, which is one hundred eighty-five. Then we multiply a square of one hundred twenty-five, which is fifteen thousand six hundred twenty-five, by eight, to get one hundred twenty-five thousand, then by twenty-seven to get three million three hundred seventy-five thousand for the outcome. We take a cube root of that, which is one hundred fifty. We add it to eight to get one hundred fifty-eight for the outcome, and we subtract that from what we remembered, which is one hundred eighty-five, leaving twenty-seven. A cube root of that, which is three, is the remainder from subtracting a cube root of eight from a cube root of one hundred twenty-five, and this is its rule.

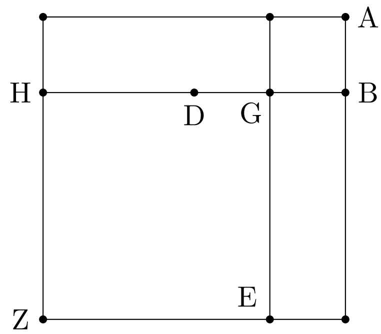
[3.4.8] The proof of this and the reason for this is the proof we gave above in the chapter on addition, which is, if we multiply the cube whose base is AG by its same, then the outcome by the cube whose base is GZ multiplied by twenty-seven, and we take a cube root of the outcome, it is the same as multiplying base AG by side HG, then the outcome by three.

[3.4.9] So we cut off from side GH the equal of side BG, and let it be GD. By what we gave above, if we add (1)[65] the square on BG, that is, the square on GD which is equal to it, multiplied by GH, then the outcome by three, that is, the cube which comes from side GD three times, with the multiplication of the square on GD by DH three times to (2) the cube which comes from side GH, that is, the two cubes which come from the sides GD, DH and the multiplication of the square on GD by DH three times with the multiplication of the square on DH by GD three times, then that total is equal to (3) the cube on GD four times and the cube on DH once

[65] This addition takes up ten lines of the translation. We have inserted "(1)," "(2)," and "(3)" so the addition takes the form "if we add (1) to (2), then that total is equal to (3)." In anachronistic notation, (1) is $3\mathrm{BG}^2 \cdot \mathrm{GH} = 3\mathrm{GD}^2 \cdot \mathrm{GH} = 3\mathrm{GD}^3 + 3\mathrm{GD}^2 \cdot \mathrm{DH}$, (2) is $\mathrm{GH}^3 = \mathrm{GD}^3 + \mathrm{DH}^3 + 3\mathrm{GD}^2 \cdot \mathrm{DH} + 3\mathrm{DH}^2 \cdot \mathrm{GD}$, and (3) is $4\mathrm{GD}^3 + \mathrm{DH}^3 + 6\mathrm{GD}^2 \cdot \mathrm{DH} + 3\mathrm{DH}^2 \cdot \mathrm{GD}$.

and the multiplication of the square on GD by DH six times and the multiplication of the square on DH by GD three times.

[3.4.10] From that[66] we cast away the multiplication of the square on GH by GD, then by three, and that is equal to the multiplication of each of the two squares on GD, DH by GD three times, and surface GD multiplied by DH, then by GD twice, then by three, that is, equal to the multiplication of the square on GD by DH six times, adding that to the cube that comes from GD, leaving the cube on DH.[67] We then take its cube root to get DH, which is what remains from subtracting a cube root of BG from a cube root of GH, and that is what we wanted to show.



[3.5] The fifth chapter, on multiplying one of them by another.

[3.5.1] (Section.) Multiplying ascribed (square) roots, one of them by another.

[3.5.2] If we want to multiply a root of a number by a root of (another) number, we multiply one of the rooted numbers by the other, and we take a root of the outcome, which gives the desired amount. For an example in rational roots, if we want to multiply a root of four by a root of nine, we multiply four by nine to get thirty-six, and we take a root of that, which is six, and it is a root of four multiplied by a root of nine.

[3.5.3] For a proof of this, we put down two squares DB, BE in place of the two rooted numbers for which we want to multiply a root of one of them by a root of the other. And let a side of square DB be AB, and a side of square BE be BG, and we complete square DE. Since surface AG, which is bounded by the two roots, is, as we have shown in the chapter on duplicating, a mean in ratio between two squares DB, BE, we therefore multiply one of the two *māl*s by the other, that is, square DB by square BE, and we take a root of the outcome to get the desired amount, which is surface AG.

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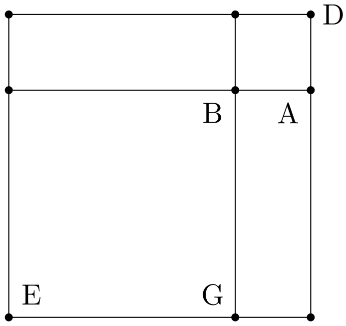
[66] The MS has in error "that from" instead of "from that."

[67] In modern notation, he expands $3\mathrm{GH}^2 \cdot \mathrm{GD}$ to get $3\mathrm{GD}^3 + 3\mathrm{DH}^2 \cdot \mathrm{GD} + 6\mathrm{GD}^2 \cdot \mathrm{DH}$. Adding GD^3 to this and casting the sum away from the amount calculated in [3.4.9] leaves GH^3.



[3.5.4] According to the rule we mentioned, if we want to multiply two roots of nine by three roots of four, we first find "two roots of nine is a root of what *māl*?" according to the rule we gave above, to get a root of thirty-six. We keep it in mind. Then we also find "three roots of four is a root of what *māl*?" to also get a root of thirty-six. So it is like we want to multiply a root of thirty-six by a root of thirty-six. We multiply thirty-six by thirty-six and we take a root of the outcome to get thirty-six, which is two roots of nine multiplied by three roots of four.

[3.5.5] (Section.) Multiplying parts of ascribed (square) roots, one of them by another.

[3.5.6] If we want to multiply a part of a root of a number by a part of a root of (another) number, we multiply each of the parts by its same, then by the number it is ascribed to. Then we multiply the two outcomes, one of them by the other, and we take a root of that. What it gives is the desired amount.

[3.5.7] For example, if we want to multiply two thirds of a root of nine by three fifths of a root of twenty-five, we first find "two thirds of a root of nine is a root of what *māl*?" according to the rule we gave above in the chapter on partitioning to get a root of four. Then we also find "three fifths of a root of twenty-five is a root of what *māl*?" to get a root of nine. So it is like we say, "if we want to multiply a root of four by a root of nine," and that is its rule.

[3.5.8] The reason for this is that the desired amount is the surface which is bounded by the two parts, which (surface) is the mean in the ratio between the two squares. Therefore, we find a square of each of them and we multiply those[68] two *māl*s, one of them by the other, and we take a root of the outcome. What it gives is the desired amount.

[3.5.9] By this rule, if we want to multiply a root of a number by a part of a root of (another) number, we find "that part is a root of what *māl*?" then we multiply what is obtained from its *māl* by the rooted number, and we take a root of the outcome to get the desired amount, and that is due to the reason that we gave above.

[3.5.10] (Section.) Multiplying ascribed cube roots, one of them by another.

[68] The MS has *dhālika* in error.

[3.5.11] If there are cube roots in place of roots in everything we have mentioned in this chapter, we cube here where we squared there, and we take the cube root here where we took the root there. Otherwise it is no different.

[3.5.12] (Section.) Multiplying an ascribed (square) root by an ascribed cube root.

[3.5.13] Similarly, if we want to multiply a root of a number by a cube root of (another) number, such as if we want to multiply a root of four by a cube root of eight, we make a root of four a cube, that is, we multiply it by its same to get four, then by a root of four to get four roots of four. Then we find "four roots of four is a root of what *māl*?" by the rule we gave above, to get a root of sixty-four, which is a cube that comes from a root of four, and its cube root is a cube root of a root of sixty-four, so it is like we want to multiply a cube root of eight by a cube root of a root of sixty-four. According to the rule we gave above, we multiply one of the cubes, and let it be eight, by the other cube, which is a root of sixty-four, to get eight roots of sixty-four. Therefore, we should also find "eight roots of sixty-four is a root of what *māl*?" to get a root of four thousand ninety-six. We take a cube root of its root to get four, which is a cube root of eight multiplied by a root of four, and that is its rule.

[3.5.14] (Section.) Multiplying ascribed roots of roots, which are sides of *māl māl*s.

[3.5.15] If we want to multiply a root of a root of a number by a root of a root of (another) number, we multiply one of the numbers by the other since they are of the same species,[69] and we take a root of a root of the outcome. What this gives is the desired amount.

[3.5.16] And the reason for this is what we gave above, that for any two numbers, a root of the multiplication of a square of one of them by a square of the other is the same as a root of a root of the multiplication of a *māl māl* of one of them by a *māl māl* of the other.

[3.5.17] And by this rule, if we want to multiply a root of a root of a number by a root[70] of (another) number, we multiply the rooted number one time by its same so it becomes the other species. Then we multiply one of the outcomes by the other and we take a root <of a root> of the outcome. What this gives is the desired amount.

[3.6] The sixth chapter, on dividing one of them by another.

[3.6.1] (Section.) Dividing ascribed (square) roots, one of them by another.

[3.6.2] If we want to divide a root of a number by a root of (another) number, we divide a *māl* of the dividend by a *māl* of the divisor, and we take a root of the result of the division. What this gives is the desired amount.

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[69] That is, they are both fourth powers of the multipliers.

[70] The MS has "root of a root."

[3.6.3] For an example of that in rational roots, we want to divide a root of thirty-six by a root of four. We divide thirty-six by four, resulting in nine. A root of nine, which is three, is the result of the division of a root of thirty-six by a root of four.

[3.6.4] The reason for this is what we have shown elsewhere, namely, that division is the inverse of multiplication.

[3.6.5] (Section.) Dividing parts of ascribed (square) roots, one of them by another.

[3.6.6] If we want to divide a part of a root of a number by a part of a root of (another) number, we multiply one of the two parts by its same, then the outcome by the number it is ascribed to. We work out the part of the other similarly, then we divide the outcome of the dividend by the outcome of the divisor, and we take a root of the result of the division. What this gives is the answer.

[3.6.7] By this rule, if we want to divide a part of a number by a part of a root of (another) number, we find a *māl* of that root, by which we multiply that part by its same, then the outcome by the number it is ascribed to, then we divide the number of the dividend by the outcome from the divisor, and we take a root of the result of the division. What this gives is the answer.

[3.6.8] (Section.) Dividing ascribed cube roots, one of them by another.

[3.6.9] If wherever we mentioned roots in this chapter there were cube roots, then the rule is the same except that we cube here where we squared there, and we take a cube root here where we took a root there. Otherwise it is no different.

[3.6.10] (Section.) Dividing ascribed cube roots and roots, one of them by another.

[3.6.11] If we want to divide a cube root of a number by a root of (another) number, (such as if) we want to divide a cube root of eight by a root of four, we make a root of four a cube and we work it out exactly like we did in the chapter on multiplication, until we get to the point in the calculation where we want to divide a cube root of eight by <a cube root of> a root of sixty-four. There we should multiply eight by its same so that it becomes the other species,[71] then we divide the outcome, which is sixty-four, by the divisor, which is also sixty-four, resulting in one, and we take its root,[72] which is also one, which is the answer.

[3.6.12] Similarly, if we want to divide a root of a number by a cube root of (another) number, such as if we want to divide a root of sixty-four by a cube root of eight, we return a root of sixty-four to the species of the cube, by which we multiply a root of sixty-four by its same to get sixty-four, then by a root of sixty-four to get sixty-four times[73] a root of sixty-four. So we find "sixty-four times a root of

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[71] The other species is that of the *māl* cube, or the fifth power.

[72] We should also take its cube root.

[73] The word "times" is not the quasi-preposition indicating the operation of multiplication that it often is in English. Here it takes a meaning like "I ran around the house sixty-four times," and thus is linked instead with the operation of duplication.

sixty-four is a root of what *māl*?" and we gave the way to do it above: we multiply the number of times, which is sixty-four, by its same, then by the number it is ascribed to, which is also sixty-four, to get two hundred sixty-two thousand one hundred forty-four. A root of a cube root of that is a root of sixty-four returned to the species of the cube. So it is as if we want to divide a root of a cube root of two hundred sixty-two thousand one hundred forty-four by a cube root of eight. By what we gave above for the rule, we make a cube root of eight rooted, so that it becomes the species of the dividend, which is that we multiply the eight by its same to get sixty-four, then we divide two hundred sixty-two thousand one hundred forty-four by sixty-four. The result of the division is four thousand ninety-six.[74] A cube root of four thousand ninety-six is sixteen, and its root is four, which is the answer.

[3.6.13] (Section.) Dividing ascribed roots of roots, which are sides of *māl māl*s, one of them by another.

[3.6.14] If we want to divide a root of a root of a number by a root of a root of (another) number, we divide the number of the dividend by the number of the divisor and we take a root of a root of the result of the division, which is the answer.

[3.6.15] By this rule, if we want to divide a root of a root of a number by a root of (another) number, we multiply the rooted number once by its same so it becomes the other species, then we divide the dividend by the outcome of the divisor and we take a root of a root of the result of the division. What this gives is the answer.

[3.6.16] (Section.) Dividing absolute (and) ascribed levels, whether they are single or combined, one of them by another.

[3.6.17] As for division in this category, what routinely presents itself is the division of combined species, however many there may be, by one species, whether the species of the divisor is absolute or ascribed.[75] When the divisor has more than one species, the division is hardly possible except by a particular technique that does not adhere to the rule, and that (works only) if the divisor is not more than two species, and one of them is known and the other is ascribed.[76] And if the divisor is of two species, and each of them is absolute, like a thing or a cube[77] or the like, or

[74] The MS has next "and that is what results from dividing a root of sixty-four by a cube root of eight, and since," which may be a marginal remark that crept into the text. If placed in the text, it should appear after the answer, four, has been found.

[75] That is, whether the species of the divisor is a level like things, *māl*s, cubes, etc., or if it is an ascribed root like $\sqrt{8}$.

[76] That is, if one of them is expressed without a root and the other with a root, like the $10+\sqrt{10}$ in the example below. But it will also work if the two are both ascribed, like $\sqrt{12}+\sqrt{10}$.

[77] Here he writes *kaʿb* where he would ordinarily have written *mukaʿʿab*.

more than two species, whatever species they may be, there is no way to know the result of the division.

[3.6.18] An example of this in which the divisor is single: if we want to divide ten and a root of fifteen by a thing, we divide ten by a thing to get ten parts of a thing for the result of the division. Then we divide a root of fifteen by a thing, by which we multiply the thing by its same to get a *māl*, then by fifteen to get fifteen *māl*s, and we take a root of that, which is a root of fifteen *māl*s, and we gather them to get ten parts of a thing and a root of fifteen *māl*s, and that is its rule.[78]

[3.6.19] And for an example of this in which the divisor is single (and) ascribed: if we want to divide ten and a root of twenty by a root of four, we divide ten by a root of four, by which we multiply ten by its same to get a hundred, then we divide a hundred by four, resulting in twenty-five, and we take a root of that, which is five, and we keep it in mind. Then we divide a root of twenty by a root of four by the rule we gave above to get for the result of the division a root of five, and we add it to the remembered amount to get the outcome five and a root of five, and that is its rule.

[3.6.20] As for an example in which the dividend is single and the divisor is combined, such as if we want to divide fifty by ten and a root of ten, we work out this type by a particular technique, which is that we subtract a root of ten from ten, leaving ten less a root of ten. Then we multiply ten less a root of ten by ten and a root of ten to get from that ninety. Since ten and a root of ten multiplied by ten less a root of ten amounts to ninety, if we divide the ninety by ten and a root of ten, the result of the division is ten less a root of ten, and that is because for any two numbers in which one of them is multiplied by the other, the outcome, if divided by one of the two numbers, results in the other.

[3.6.21] But if we divide the fifty, which is the divisor, by ten and a root of ten, it results in a number such that the ratio of the fifty to that number resulting from the division is as the ratio of the ninety to ten less a root of ten. And that is (because) both the ninety and the fifty are divided by the same number, which is ten and a root of ten, so the ratio of one of the dividends to the result of its division is as the ratio of the other dividend to the result of its division. These are four proportional numbers, three of them known and one unknown. So we multiply the fifty by ten less a root of ten. The outcome of the multiplication is five hundred less fifty roots of ten, and we divide that by ninety. The result of dividing the five hundred by ninety is five and five ninths, and of dividing the lacking fifty roots of ten, which is a lacking root of twenty-five thousand, by ninety is a root of three and seven parts of eighty-one lacking, and gathering them gives five and five ninths less a root of three and seven parts of eighty-one parts, and that is its rule.

[78] The author made an error here. He should have divided fifteen by a *māl* to get fifteen parts of a *māl*, and then take a root of the result.

[3.6.22] The reason for this is that we subtract a root of ten from ten, then we multiply the remainder by ten and a root of ten, so that the outcome will be a rational number, since any binomial number,[79] if multiplied by its apotome, results in a rational number. As for the absolute binomial, it is any number composed of two numbers rational in power, or one of them rational in length and the other rational in power, like a root of ten and a root of a third, or like ten and a root of ten, and the like. And for the apotome: any binomial number, if the smaller part is detached from the greater part, what remains of that is called the absolute apotome.[80]

[3.6.23] And for an example in which the divisor and the dividend are combined, such as if we want to divide fifty and a root of two hundred by ten and a root of ten, we first divide fifty by ten and a root of ten exactly as we worked it out in the previous example, which results in the result we got there, which is five and five ninths less a root of three and seven parts of eighty-one parts of a unit. Then we divide a root of two hundred by ten and a root of ten exactly like the work in the previous example, and for that we multiply a root of two hundred by ten less a root of ten to get ten roots of two hundred, that is, a root of twenty thousand, less a root of two thousand. We divide that by ninety, resulting in a root of eighty-two and thirty parts of eighty-one less a root of twenty parts of eighty-one. We gather them, so the outcome is five and five ninths less a root of three and seven parts of eighty-one parts of a unit and a root of eighty-two and thirty parts of eighty-one less a root of twenty parts of eighty-one parts of a unit, and that is its rule.

[4] **The fourth category**, concerning the proportional levels of the single and combined Equation, which is comprised of two chapters.

[4.1] The first chapter, on the single Equation.

[4.1.1] As for the single Equation, it is when one of those proportional species which we mentioned Equals another of those species, that is, (which) equals it. Three of these single Equations are those which arise between the first three proportional species, which are the number, the root, and the *māl*. These are the elementary cases (*uṣūl*) for the remaining single Equations, since the others revert to them and are lowered in their direction until they become (equations) in those species, unless one of the Equated amounts is the level of a number.

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[79] The MS has "two numbers" in error.

[80] These are *absolute* binomials and apotomes because one is not working with their roots. For example, $5+\sqrt{45}$ is an absolute binomial, but $\sqrt{5+\sqrt{45}}$ is a root ascribed to a binomial.

[4.1.2] It may happen in any of them[81] that in some of our manipulations of the operations, the multitude of the farther of the two Equated species[82] exceeds the unit or falls short of it. In this case we need to return it to the whole unit by restoring the shortfall or by deducting the excess. The same operation (should be performed) for the lower species that it is Equated to, in order to bring them into the original ratio. And for the operation of returning these Equated species to one of them,[83] if there is no fraction with it, it is easy; there is no need for much preoccupation and trouble for it.

[4.1.3] If it is fractions or there are fractions with it,[84] we need to perform (one of) two operations. One of them is that we add to a known number a known part of it, and the second is that we subtract from it a known part of it. For the addition, if we want to add to a known number a known part of it, we put down the denominator of that part in two places[85] and we add to one of them its part. What this comes to, we multiply it by the number and we divide the outcome <by> the denominator in the other place. What results from the division is the number with its part added to it.

[4.1.4] And in subtraction, if we want to subtract from a known number a known part of it, we do what we did for addition, except that (here) we subtract its part from one of the places where we added it there, and that's all. Then the result of the division is the desired amount.

[4.1.5] An example of addition: if we want to add to one and two-thirds the same as its fifth, we put the denominator of the fifth, which is five, in two places, and we add to one of the places its fifth, which is one, to get six. Then we multiply six by the number, which is one and two thirds, to get ten. Then we divide ten by the denominator in the other place, which is five, <to get>[86] the result of the division two, which is one and two thirds with its fifth added <to it>, and that is its rule.

[4.1.6] The reason for this is <if we put down> the five in two places and we add to one of them <one, which is> the fifth, then it comes to six. Then the ratio of the five <to the six> is as the ratio of the number, which is one and two thirds, <to the desired amount>, since the desired amount needs to be <the same> as one and two thirds and its fifth. Therefore, we multiply the six by one and two thirds, which is

[81] I.e., in any of these types of equation.

[82] The "farther of the two Equated species" is the farther from the unit, which means the higher power.

[83] I.e., to a multitude of one.

[84] The word "fractions" is plural because they are often compound, like "a half and a third" or "four fifths and a fifth of a fifth."

[85] This duplication is necessary because the denominator will be subject to two operations.

[86] Here and in the next few lines we have restored several words to the text that are now illegible because they are covered by a piece of tape that was used to repair the manuscript.

the second,[87] by the third, and we divide that by five, which is the first, resulting in the desired amount, which is the fourth.

[4.1.7] And for an example of subtraction, it is if we want to subtract from one and a third the same as its fourth. We put down the denominator <of the fourth>,[88] which is four, in two places, and we subtract from one <of the places> its fourth, which is one, leaving three. Then we multiply <three by> one and a third to get four, and we divide that by <the denominator in the> other <place>, which is also four. The result <is one>, which is the answer.

[4.1.8] As for the three Equations which arise between the number, the roots, and the *māl*s, the first of them is roots Equal a number, such as when we say, a root Equals three. So the root is three and the *māl* which comes from it is nine. And such as when we say, four roots Equal twelve. Since the multitude of the farthest of the two Equated species, which is the multitude of the roots, exceeds the unit, since it is four, we need to return it to the unit: we subtract from everything we have of the roots and the number the same as three of its fourths. So the outcome we get after that is: a root Equals three. So the root is three and the *māl* which comes from it is nine. And such as when we say, half a root Equals one and a half. Because the multitude of roots here is less than one, what is needed to return it to the whole unit is that we add to what we have its same, so we get: a root Equals three. So the root is three and the *māl* which comes from it is nine.

[4.1.9] As for the second Equation, it is *māl*s Equal a number, such as when we say: a *māl* Equals nine. So the *māl* is nine and its root is three. And such as when we say, three *māl*s and a third Equal thirty. Since the multitude of the *māl*s is greater than one and there is a fraction with it, we numerate that for the species of fraction "thirds" to get ten thirds. What one needs to subtract from that so that it reverts to the one *māl* is seven thirds, which is seven of its tenths. And we also subtract from the thirty seven of its tenths, which is twenty-one, so the outcome we get after that is: a *māl* Equals nine. And such as when we say, two thirds of a *māl* Equal six. Here we need to add to everything we have the same as its half, so the outcome we get after that is: a *māl* Equals nine.

[4.1.10] And for the third Equation, it is *māl*s Equal roots, such as when we say, a *māl* Equals three roots. Since it is clear that the ratio of the *māl* <to> the root is as the ratio of the root to the unit, the ratio of the *māl* to the three roots is as

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[87] As was common, the author calls the four numbers in a proportion "the first," "the second," "the third," and "the fourth," where the first is to the second as the third is to the fourth.

[88] Again, in this passage we have restored words that were made illegible because they were covered by a piece of tape.

the ratio of the root to the three units. So a root of the *māl* is three, and the *māl* which comes from it is nine, which is the same as (*mithla*) three of its roots.[89]

[4.1.11] <And such as when we say, two *māl*s and a third Equal seven roots.>[90] Since the multitude of *māl*s is greater than one and there is a fraction with it, we numerate that for the species of fraction "thirds" to get seven thirds, (so the excess) is four of its sevenths, and if we subtract from the seven roots four of its sevenths, which is four, it leaves three roots, which are the roots Equated to the one *māl*. So we say that the one *māl* Equals three roots, so the one root Equals three, and the root is three and the *māl* is nine. And twice that *māl* and its third is twenty-one, and that is the same as seven of its roots.

[4.1.12] And such as when we say, two thirds of a fifth of a *māl* Equals a seventh of its root. Since the multitude of the *māl*s is less than one, and its denominator comes from three by five, that is, it comes from fifteen, two thirds of its fifth are two parts of fifteen. So we need to return it to the whole *māl*, by which we add to it thirteen (of its parts of fifteen), that is, six times it and a half of it.[91] For that, we either multiply it and what it is Equated to by six and a half, or we follow[92] the method of the rule we gave above, which is that we put down one in two places, because of the "times,"[93] and we add to <one> of the places six times it and half of it,[94] so the outcome is seven and a half. Then we multiply that by the multitude of roots, which is a seventh, to get one and half a seventh, and we divide that by the other (number) put down, which is one. The result of the division is one and half a seventh, since anything multiplied by the unit or divided by it does not change it. So we say, the roots that are Equated to the one *māl* are a root and half a seventh of a root. By what we stated above on proportions, the root is then one and half a seventh of a unit and the *māl* which comes from it is one and a seventh and a part of one hundred ninety-six parts of a unit. If we numerate the *māl* for the species of fraction, namely, parts of one hundred ninety-six, then its outcome is two hundred twenty-five, and two thirds of its fifth is thirty. We keep that in mind. Then, we numerate the root for the species of this fraction, parts of one hundred ninety-six. This gives the outcome two hundred ten. A seventh of that is thirty, which equals the remembered amount. Therefore the partition is correct, and that is its rule.

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[89] This is the only book we have seen that solves this equation type by proportion. This is another indication of the theoretical leaning of the author.

[90] In modern notation, $2\frac{1}{3}x^2 = 7x$.

[91] This could be translated a little more literally as "six copies of it and half a copy of it," which is $6\frac{1}{2}$ of its same.

[92] We emend "we follow" for mathematical sense.

[93] I.e., "copies of," not "times" in the sense of multiplication.

[94] We would say "six and a half times it".

[4.1.13] The Equation might (also) arise between any two levels of the other proportional levels that we have named. But the rule (*ḥukm*) for that (case) is that if neither of them is of the level of a number, we reduce each of them by one or more places[95] so that the smaller of them becomes the level of a number, consisting of the species "number," like what one gets from cubes Equal *māl māl*s.[96] We reduce that by three places, so the cubes become a number and *māl māl*s roots; and like *māl māl*s Equal *māl*s: we reduce that by two places, so the *māl*s become a number and *māl māl*s (become) *māl*s, and that is its rule.

[4.2] The second chapter, on the combined Equation.

[4.2.1] As for the combined Equation, what routinely presents itself in the art of algebra (*al-jabr wa-l-muqābala*) is what is encountered with three proportional elements from among the elements which we mentioned previously. Three of these combined Equations are those which occur between the first three elements. The first is *māl*s and roots Equal a number, the second is *māl*s and a number Equal roots, and the third is roots and a number Equal *māl*s. These are the elementary cases[97] for the other combined Equations, since the others revert to them and are lowered in their direction until they become of their species, like we explained for the single Equations.

[4.2.2] It may happen in some of the operations that we attend to that the multitude of the higher of the Equated places exceeds the unit or falls short of it. In that case we need to return it to the whole unit by restoring the shortfall or by deducting the excess. There is also the same operation for the other two places that it is Equated to, so that they can be brought into the original ratio, as it was before. For the process of returning these species to one of them we have guided (the reader) towards it in (the chapter) on the single Equations, where it is no different.

[4.2.3] As for finding a side of the *māl* in the first combined (Equation), know that the unknown which we need to solve for through calculation and (need) to find is, in each of these three combined (Equations), the value of the *māl* cited in it. The required procedure to find that for the first combined (Equation), which is *māl*s and roots Equal a number, after returning the multitude of the *māl*s to a whole one of them if it is smaller or greater, and (returning) what is with it of the roots and the number, is that we multiply half the multitude of roots by its same, that is, a number the same as the multitude of half the roots, and we add the outcome to

[95] Literally, "a place or places."

[96] The manuscript has "roots" instead of "*māl māl*s" in error.

[97] The word is *usṭuqis*, in the singular. This is an Arabized form of the Greek word *stoicheia* used in the title of Euclid's *Elements*. In the similar passage at [4.1.1, 562] the plural word *uṣūl* appears instead.

the Equated number. And taking a root of the sum, then subtracting from it the multitude of half the roots, what remains is a side of the unknown *māl*.

[4.2.4] For example, a *māl* and ten roots Equal thirty-nine. We halve the multitude of roots, which is ten, to get its half is five, and we multiply it by its same to get twenty-five, which is a number, since we multiply a number the same as the multitude of half the roots, and we do not multiply roots. Then, we add that to the number, which is thirty-nine, to get the outcome sixty-four, and we take its root, which is eight. We subtract from it half the multitude of roots, which is five, leaving three. It is a root of the *māl*, and the *māl* is nine, and ten of its roots are thirty, and they add up to thirty-nine.

[4.2.5] And such as when we say, two *māl*s and a third and seven roots Equal forty-two. Since the multitude of *māl*s is greater than one, we return it to the unit, by which we numerate it for the species of fraction "thirds," to get seven thirds. And since we need to subtract from that and from everything we have of the roots and the number four of their sevenths, by the rule we gave above for the procedure we multiply each of the roots and the number by three and we divide the outcome by seven. What results from each of them is Equated to the one whole *māl*. And if we work that out, we get a *māl* and three roots Equal eighteen. We multiply half the multitude of roots, which is one and a half, by its same to get two and a fourth, and we add that to the number, which is eighteen, giving twenty and a fourth. We take a root of that, which is four and a half, then from that we subtract half the multitude of roots, which is one and a half, leaving three, which is a root of the *māl*, and the *māl* is nine. And if we double it and we add to the outcome the same as a third of the *māl*, it gives twenty-one. And if we add to that the same as seven of its roots, which is also twenty-one, the outcome becomes forty-two.

[4.2.6] And such as when we say, a half and a third of a *māl* and two roots and a third Equal fourteen dirhams[98] and a half. Here we need to complete the *māl*, by which we add to it and to everything with it of roots and number the same as its fifth. By the rule we gave above of the procedure, we put down the denominator of the fifth in two places and we add to one of them its fifth,[99] which is one, giving six. Then we multiply both the roots and the number by six and we divide the outcome by the other (number) put down, which is five. What results from each of them (taken together) is Equated to the one whole *māl*. And if we work that out, we get: a *māl* and two roots and four fifths of a root equal seventeen and two fifths in number. We multiply half the multitude of roots, which is one <and> two fifths, by its same, and we add the outcome, which is one and four fifths and four fifths of a fifth, to the number, which is seventeen and two fifths, giving the outcome nineteen

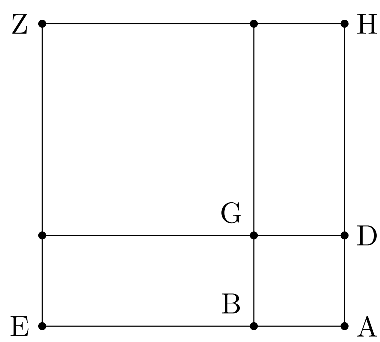
[98] In modern notation, $\frac{5}{6}x^2 + 2\frac{1}{3}x = 14$. This is the only instance of dirhams for units in the extant part of the treatise.

[99] We emended the text with "its fifth."

and a fifth and four fifths of a fifth. And we take a root of that, which is four and two fifths. We subtract from it half the multitude of roots, which is one and two fifths, leaving three, which is a root of the *māl*, and the *māl* is nine. And that is the rule for everything that ends up in this kind (of Equation),[100] if almighty God wishes.

[4.2.7] Figure for this procedure. Since the sum of the number and a square of half the multitude of roots is a square number,[101] we know that the figure for the number is a gnomon (added) to a square of half the roots,[102] and every gnomon is equal to a square and two complements. Thus, the number is equal to a square and two complements. But it is equal to a *māl* and ten roots. Thus each of the complements is five roots, and because each of them is a surface bounded by two sides, one of them is a root since it is a side of the square, and the other is equal to half the multitude of roots, namely, five in number.

[4.2.8] We decided to make the figure of the unknown *māl* a square of equal sides and angles as square ABGD, and we extend its side AB in a straight line to point E, and we make BE equal to half the multitude of roots, and we make on AE square AEZH and we draw the sides BG, DG in a straight line to the sides HZ, EZ. Because each of the sides BG, GD is a root and each of sides DH, BE is five, each of the surfaces HG, GE is five roots, and together with square ABGD, that is, the gnomon, they are a *māl* and ten roots, and that Equals thirty-nine. Because square GZ is twenty-five, therefore the whole surface AZ is sixty-four, and AE is its root, which is eight. So if from that we subtract BE, which is half the multitude of roots, which is five, it leaves AB is three, which is a root of the unknown *māl*, and that is what we wanted to show.



[100] I.e., for any problem solved by algebra in which the equation that is set up simplifies to the type "*māl*s and roots Equal a number."

[101] This only means that the sum will be shown as a square in the diagram. It does not mean that the sum is a perfect rational or whole square.

[102] Here the author or copyist resorts the the usual, short way by writing "half the roots" instead of "the multitude of half the roots."

[4.2.9] Construction of this problem by geometry and a proof of it and of the reason for halving the roots in it[103] by lines.

[4.2.10] If we want to find a side of the unknown *māl*, we put down line AB equal to the multitude of roots, and on it we put down a right-angled surface equal to the known number,[104] exceeding its completion by a square, as explained in the twenty-ninth (proposition) of the sixth book of the book of *Elements*, and let it be surface AG by GB, and a side of the exceeding square be BG. I say that BG is a side of the sought-after *māl*.

[4.2.11] Its proof is that we divide AB into two halves at D, so line AB is divided into two halves at D and is extended by an extension, which is BG. So the multiplication of AG by GB with the square on DB is equal to the square on DG.[105] But the multiplication of AG by BG is known, since it is equal to the known number. Therefore, we add the square of half the multitude of roots, that is, the square on DB, to the known number, which is surface AG by GB, so we get that the square on DG is known. We take its root, which is DG, we subtract from it half the roots, which is DB, leaving BG known, which is a side of the *māl*, and that is what we wanted to show.

[4.2.12] Finding a side of the *māl* in the second combined (Equation).

[4.2.13] As for the procedure to find a side of the *māl* in the second combined (Equation), which is *māl*s and a number Equal roots, after returning the *māl*s to one of them, whether it is smaller or greater, it is that we halve the multitude of roots and we multiply half of their multitude by its same and we subtract the known number from the outcome and we take a root of what remains, and we subtract it[106] from the multitude of half the roots or we add it[107] to it. The outcome is a root of the unknown *māl*.

[4.2.14] We say that we subtract a root of what remains or we add it, because some problems of algebra (*al-jabr wa-l-muqābala*) are solved for this combined (Equation) by both addition and subtraction, and some of them are solved by subtraction alone or by addition only. So one should examine all of the problems leading to this combined (Equation) for each of the procedures which we mentioned until it results in the answer. And the square of half the multitude of roots will never be less than

[103] I.e., in the equation.

[104] We emended "to … the number."

[105] Euclid's *Elements* Proposition II.6.

[106] We emended "we subtract it."

[107] We emended "we add it."

the number which is with the *māl* in this combined (Equation). If it happens to be less, then the problem is impossible. And if they are equal, then a root of the unknown *māl* is equal to half the multitude of roots.

[4.2.15] For example, a *māl* and twenty-one in number Equal ten roots. And its meaning is, what *māl*, if you add to it twenty-one in number gives an outcome equal (*mithla*) to ten of its roots? We halve the multitude of roots and we multiply half of its multitude by its same, which is five, to get twenty-five, and from that we subtract the number, which is twenty-one, leaving four, and we take its root, which is two. We subtract it from half the multitude of roots, which is five, leaving three, which is a root of the *māl*, and the *māl* which comes from it is nine. Or, we add it to it to get seven, which is a root of the *māl*, and the *māl* which comes from it is forty-nine. Whenever we add to either of these two *māl*s twenty-one in number, the outcome becomes the same as ten of its roots.

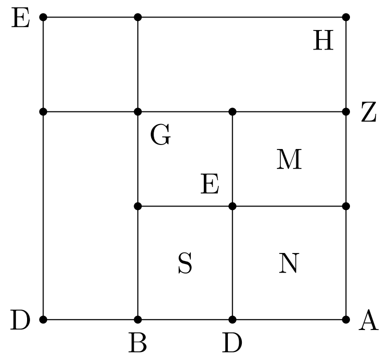
[4.2.16] As for when a square of half the roots is equal to the number which is with the *māl*, such as when we say, a *māl* and twenty-five in number Equal ten roots, if we multiply half the multitude of roots by its same it gives twenty-five, just the same as the number. So we say that a root of the *māl* is the same as half the multitude of roots, which is five, and the *māl* which comes from it is twenty-five. If we add to this *māl* twenty-five in number it gives fifty, which is equal to ten of its roots, and that is the rule for whatever ends up in this kind, if almighty God wishes.

[4.2.17] Figure for this procedure. Since the excess of a square of half the multitude of roots over the known number is a rooted number, we know that the figure for the number is a gnomon in a square of half the multitude of roots. But the known number with the unknown *māl* is equal[108] to ten roots. So half the gnomon and half the *māl* are equal to five roots. They are thus a surface contained by two lines, one of them a root of the unknown *māl* and the other a line equal to half the multitude of roots.

[4.2.18] We decided to put line AB equal to half the multitude of roots, and we construct on it square AG, and we make a side of the unknown *māl* AD: for subtraction it is smaller than AB and for addition it is greater than AB. And we construct on AD in both positions square AE, and we draw the lines of the figure. Because line AB is five and line AD in each of the two positions is a root, gnomon MNS with square AE in each of the positions is equal to the multiplication of AB by AD twice, that is, ten roots. For the case of subtraction, this is clear. For the case of addition, the gnomon MNS with greater square AE is equal to two surfaces HB, ZD, and that is because gnomon MNS with greater square AE is equal to gnomon MNS twice and square EG twice and surface HG twice, and that sum is equal to surface HB twice, that is, the two equal surfaces HB, ZD, and each of them is five

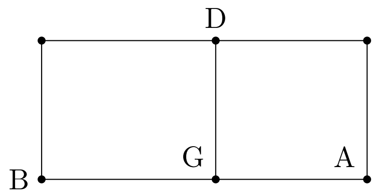
[108] He does not use the algebraic Equal here because he is thinking in terms of his diagram.

roots. But the known number with unknown square AE is equal to ten roots. Thus MNS is equal to the known number. Therefore we cut it off from square AG, leaving square EG known. We take its root, which is DB, and we subtract it from AB or we add it to it, to get the remainder or the sum AD, which is a root of the sought-after *māl*.



[4.2.19] As for the figure for equality, it is when the number is equal to a square of half the multitude of roots. We know that the figure for the number <is a square>, and also, since the sum of the number and the *māl* Equals known roots, we know that the figure for that sum is a surface contained by two lines, one of them necessarily the root and the other a number equal to the multitude of roots. But that surface is divided into two square halves, one of them a *māl* and the other (the) number, and the sum of their two sides is equal to the multitude of roots. Therefore, half the number of roots is a root of the unknown *māl*.

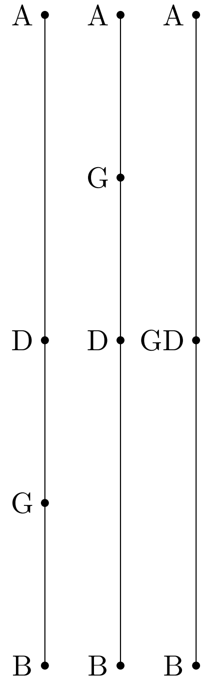
[4.2.20] We decided to place line AB equal to the multitude of unknown roots, and we halve it at G and we construct on each of AG, GB squares AD, BD, to get that one of the lines AG, GB is a root of the unknown *māl*, which is equal to half the multitude of roots.



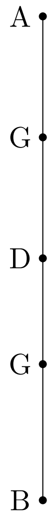
[4.2.21] Construction of this problem by geometry, and a proof of it and of the reason for halving the roots in it by lines.

[4.2.22] If we want to find a side of the unknown *māl*, we put down line AB in three places, for addition, subtraction, and equality, equal to the multitude of roots, and we apply to it a right-angled surface equal to the known number falling short in its completion by a square, as explained in the twenty-eighth proposition (of the sixth book) of the book of *Elements*. And let the applied surface be surface AG by GB, and a side of the lacking square be BG. So I say that GB is a side of the unknown *māl*.

[4.2.23] Its proof is that we halve line AB at point D. Point D falls in one of the figures, which is for subtraction, between points A, G on line AG, and in the next figure, which is for addition, between points G, B on line GB, and in the third figure, which is for equality, precisely at point G. Because line AB is divided into two halves at D and into two unequal[109] parts at G, the multiplication of AG by GB is known, since it is equal to the known number, and the square on BD is known since BD is equal to half the multitude of roots. Therefore, we subtract the known number, which is surface AG by GB, from the square of half the multitude of roots, that is, the square on BD, and we take a root of what remains, that is, DG.[110] We subtract it from half the multitude of roots or we add it to it, to get that the remainder or the sum is a side of the unknown *māl*, which is BG.



[4.2.24] And for impossibility, we put line AB equal to the multitude of roots, and we make BG belonging to it a side of the unknown *māl*. Then the multiplication of AB by BG is equal to the roots.[111] But the roots[112] are equal to the known number and the unknown *māl*, and the surface AB by BG is equal to the multiplication of AG by GB with the square on GB, and we made GB a side of the unknown *māl*. So surface AG by GB is equal to the known number. If we halve line AB, the halving occurs either on part AG or on part GB of the line AB. In both cases the multiplication of AG by GB with the square on GD is equal to the square on DB. But the multiplication of AG by GB, which is the number assumed to be greater than the square on DB, and that is not possible.



[4.2.25] Finding a side of the *māl* in the third combined (Equation).

[4.2.26] As for the procedure for finding a side of the *māl* in the third combined (Equation), which is roots and a number Equal *māl*s, after returning the *māl*s to one of them if it is smaller or greater, it is that we multiply half the multitude of

[109]It works also for the equal parts in the third diagram.

[110]Euclid's *Elements* Proposition II.5.

[111]The MS mistakenly has "the multitude of roots".

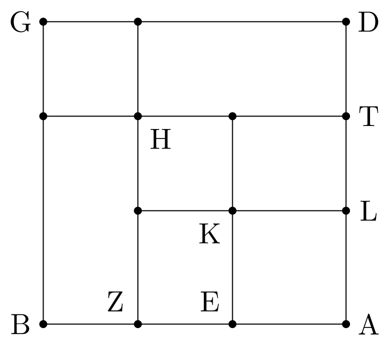
[112]The MS again mistakenly has "the multitude of roots".

roots by its same, and adding that to the number and taking a root of the outcome and adding (that) to half the roots, what comes from that is a root of the unknown *māl*.

[4.2.27] For example, three roots and four in number Equal a *māl*. If we want to find a side of the *māl*, we multiply half the multitude of roots, which is one and a half, by its same to get two and a fourth. We add that to the number, which is four, to get six and a fourth, and we take its root, which is two and a half. We add it to the multitude of half the roots, which is one and a half, to get four, which is a root of the unknown *māl*.

[4.2.28] Figure for this procedure. Since the sum of the number and a square of half the multitude of roots is a square, we know that the figure for the number is a gnomon (added) to a square of half the multitude of roots. And since a side of this whole square added to half the multitude of roots is a root of the unknown *māl*, we decided to put the figure for the unknown *māl* as a square of equal sides and angles ABGD, and let BE from side AB be equal to the multitude of roots, and we halve EB at point Z and we construct on AZ square AZHT, and we extend lines ZH, HT in a straight line to sides DG, BG, and we construct on AE square AEKL and we extend the two lines EK, LK in a straight line to sides TH, ZH.

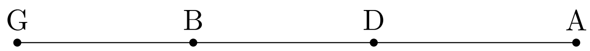
[4.2.29] Because DG is a root and DT is one and a half, since it is equal to half the multitude of roots, surface TG is a root and half a root, and likewise, it is also clear that surface GZ is a root and half a root. So surface DH once and HG twice and HB once are equal to three roots. But surface KH equals surface HG, leaving the surfaces TK, KA, KZ, that is, the gnomon, equal to the known number. Therefore we add the number to a square of half the multitude of roots so that we get square AH. We take its root, which is AZ, we add it to half the multitude of roots, which is ZB, to get the outcome AB, which is a root of the unknown *māl*, and that is what we wanted to show.



[4.2.30] Construction of this problem by geometry, and a proof of it and of the reason for halving the roots in it by lines.

[4.2.31] If we want to find a side of the unknown *māl* in this problem, we put down line AB equal to the known multitude of roots and we apply to it a right-angled surface equal to the number and exceeding its completion by a square,[113] and let the added surface be surface AG by GB, and a side of the added square be BG. I say that AG is a side of the unknown *māl*.

[4.2.32] Its proof is that we divide known line AB into two halves at point D, so line AB is divided into two halves at point D and is extended by an extension, which is BG. Therefore the multiplication of AG by GB with the square on DB equals the square on DG.[114] But the multiplication of AG by GB is known, since it equals the known number, and the square on DB is known, since DB equals the multitude of roots. Thus, the square on DG is known. Therefore, a square of half the roots, that is, the square on DB, is added to the known number, which is surface AG by GD, and we take a root of the outcome, which is a root of the square on DG, that is, DG. We add to it the multitude of half the roots, that is, AD, to get the outcome AG, which is a root of the *māl*, and that is what we wanted to show.



[4.2.33] It is clear by what we gave above that the method by which the sides of unknown *māl*s were extracted in each of these three combined (Equations) is the direction that Euclid presented toward the end of the sixth book of his book on the *Elements*, which treat the application of a surface of parallel sides to a known line that exceeds its completion or falls short of it by a square, and that is that a side of the exceeding square is a side of the unknown *māl* in the first combined (Equation), and in the second combined (Equation) it is a side of the lacking square, and in the third combined (Equation) it is the sum of the line to which the surface is added and a side of the exceeding square, and that is what we wanted to show.

[4.2.34] As for combinations involving Equality that arise between three elements that are not in proportion, and again for those with more (elements) whether proportional or not proportional, such as those which can arise in the two domains of triples, of which one of them is cubes, *māl*s, and number, and the other is cubes, root, and number, consisting of six combined (Equations),[115] or in the single domain of quadruples, which are cubes, *māl*s, roots, and number, consisting of seven combined (Equations),[116] or of any others which are used in what arises above these places,

[113] Euclid's *Elements* Proposition VI.29.

[114] Euclid's *Elements* Proposition II.6.

[115] In modern notation these equations are $ax^3 + bx^2 = c$; $ax^3 + c = bx$; $bx + c = ax^3$; $ax^3 + bx = c$; $ax^3 + c = bx$; and $bx + c = ax^3$.

[116] In modern notation these equations are $ax^3 = bx^2 + cx + d$; $bx^2 = ax^3 + cx + d$; $cx = ax^3 + bx^2 + c$; $d = ax^3 + bx^2 + cx$; $ax^3 + bx^2 = cx + d$; $ax^3 + cx = bx^2 + d$; and $ax^3 + d = bx^2 + cx$.

these can hardly be pursued by the arithmetical rules we gave above, though (they can be pursued) with regard to the determination geometrical method of magnitude determination by introducing conic sections.

[4.2.35] As for the two domains of triples that we mentioned, this is due to the departure in what each of them contains of the three species from the domain of continuous proportion, and that is, the ratio of the cube to the *māl* is not as the ratio of the *māl* to the number, since there is one place between the *māl* and the number, which is the place of the root; and the ratio of the cube to the root is not as the ratio of the root to the number, since there is one place between the cube and the root, which is the place of the *māl*.

[4.2.36] None of these six combined (Equations) will yield to the arithmetical rules we gave above, and that is because the unknown which one needs to solve for and to find in each of these combined (Equations) is a side of the cube cited in it. The analysis of it leads to an application of a known body with parallel surfaces (i.e., a parallelepipedal solid) to a known line exceeding its completion or falling short of it by a square. But that can only be constructed by using conic sections. And for the domain of quadruples, it is due to the species contained in it being more than three. Although it satisfies the condition of continuous proportion, the seven combined (Equations) are not subject to the general rules, since the unknown that one is required to find is a side of the cited cube, which can hardly be extracted by the arithmetical rules we gave above, except by what we mentioned of conic sections.

[4.2.37] These are the principles of algebra and the methods of the single and combined Equations on which are based types of arithmetical problem that are subject to the correct general rules. We have explained them with the clearest explanation and the most correct proofs, and we have completed their establishment with regard to division (into cases), organization, correction, and simplification. As for the problems leading to them, we do not pay attention to them to gain something from them, since they fall outside what we intended and proposed, and because they are types of practical applications that stem from the (theoretical) elements that we have indicated.

[4.2.38] Let us finish the discussion with the praise of God, Lord of the Worlds, and blessings on Muḥammad and his Family, the good ones. Composed in the year 395 of the Hijra.

[4.2.39] Its edition was completed on Friday, the twelfth of Rabīʿ al-Ākhir of the year five hundred eighty-one. May God forgive its scribe and the people who read it. May God only reward us; verily, He is the Helper.

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